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On the Growth of Wronskians Using their Relative Orders, Relative Types and Relative Weak Types

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Abstract. In this paper the comparative growth properties of composition of entire and meromorphic functions on the basis of their relative orders (relative lower orders), relative types and relative weak types of Wronskians generated by entire and meromorphic functions have been investigated.

AMS Subject Classification (2000). 30D20; 30D30; 30D35
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1 Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function relating to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. If f is non-constant then it has the following property:

Property (A) ([2]) : A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For exapmles of functions with or without the Property (A), one may see [2].

However, for any two entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The order (lower order) of an entire function f which is usually applied in computational purpose is defined in the following way in terms of the growth of f with respect to the $\exp z$ function:

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}\right).$$

Whenever f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic. In this case one can define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f in the following manner which perform the same role as maximum modulus function:

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r\right),$$

in addition we symbolize by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞ -point is a pole of f. In many situations, $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\bar{N}_f(r)$ respectively. Also the function $m_f(r,\infty)$ alternatively symbolized by $m_f(r)$ known as the proximity function of f is defined in the following way:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0,$$

and some times one may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

When f is entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as follows:

$$T_f(r) = m_f(r) \ .$$

If f is a non-constant entire function then $T_f(r)$ is rigorously increasing and continuous function of r and its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist where $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is known as growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g. Further in case of meromorphic functions, the growth markers such as order and lower order which are traditional in complex analysis are defined in terms of their growth with respect to the $\exp z$ function in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right),$$

and the growth of functions is said to be regular if their lower order coincides with their order.

In this connection the following two definitions are also well known:

Definition 1.1. *The* type σ_f *and* lower type $\overline{\sigma}_f$ *of a meromorphic function* f *are defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \ \text{and} \ \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty \ .$$

If f *is entire then*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \ \text{and} \ \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}, \ 0 < \rho_f < \infty \ .$$

Definition 1.2. [6] *The* weak type τ_f *and the growth indicator* $\overline{\tau}_f$ *of a meromorphic function* f *of finite positive lower order* λ_f *are defined by*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} \ \text{and} \ \tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty \ .$$

When f *is entire then*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \ \text{and} \ \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty \ .$$

However, extending the thought of relative order of entire functions as initiated by Bernal {[1], [2]} , Lahiri and Banerjee [12] introduced the definition of relative order of a meromorphic function f with respect to another entire function g, symbolized by $\rho_g(f)$ to avoid comparing growth just with $\exp z$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large } r\}\\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one if $g(z) = \exp z$ {cf. [12] }.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite *relative order* with respect to another entire function, Roy [13] introduced the notion of *relative type* of two entire functions in the following way:

Definition 1.3. [13] *Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the* relative type $\sigma_g(f)$ *of f with respect to g is defined as :*

$$\begin{aligned}&\sigma_g(f)\\ =\ &\inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \textit{ for all sufficiently large values of } r\right\}\\ =\ &\limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}} .\end{aligned}$$

Likewise, one can define the relative lower type *of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :*

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}},\ 0 < \rho_g(f) < \infty .$$

Analogusly, to determine the relative growth of two entire functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [7] introduced the definition of *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way:

Definition 1.4. [7] *The* relative weak type $\tau_g(f)$ *of an entire function* f *with respect to another entire function* g *having finite positive* relative lower order $\lambda_g(f)$ *is defined as:*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} \ .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ *of an entire function* f *with respect to an entire function* g *in the following way :*

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}}, \ 0 < \lambda_g(f) < \infty \ .$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas also [7] gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 1.5. [7] *The* relative type $\sigma_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *are defined as*

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \textit{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the lower relative type $\overline{\sigma}_g(f)$ *in the following way:*

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \textit{where } 0 < \rho_g(f) < \infty.$$

Definition 1.6. [7] *The* relative weak type $\tau_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g(f)$ *is defined by*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g(f)$ *as*

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 1.3 , Definition 1.4, Definition 1.5 and Definition 1.6 coincide with the classical definitons of type (lower type) and weak type of entire ane meromorphic functions respectively.

The following definitions are also well known:

Definition 1.7. *A meromorphic function* $a \equiv a(z)$ *is called small with respect to* f *if* $T(r,a) = S(r,f)$ *where* $S(r,f) = o\{T(r,f)\}$ *i.e.,* $\frac{S(r,f)}{T(r,f)} \to 0$ *as* $r \to \infty$.

Definition 1.8. *Let* $a_1, a_2, ...a_k$ *be linearly independent meromorphic functions and small with respect to* f *.We denote by* $L(f) = W(a_1, a_2, ...a_k, f)$ *, the Wronskian determinant of* $a_1, a_2, ..., a_k,\ f$ *i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

Definition 1.9. *If* $a \in \mathbb{C} \cup \{\infty\}$*,the quantity*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$$

is called the Nevanlinna's deficiency of the value "a".

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (cf [10],.*p*.43). If in particular, $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative type* as well as *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysists and they are not aware of the technical advantages of using the relative growth indicators of the functions. In this paper we wish to prove some newly developed results based on the growth properties of *relative order, relative type* and *relative weak type* of wronskians generated by entire and meromorphic functions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [10] and [14].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) \ .$$

Lemma 2.2. [4] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f(\exp(r^{\mu})) \ .$$

Lemma 2.3. [11] *Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) > T_g(\exp(r^{\mu})) \, ,$$

where $0 < \mu < \rho_g$.

Lemma 2.4. [5] *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f(\exp(r^{\mu})) \ .$$

Lemma 2.5. [5] *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_g(\exp(r^{\mu})) \ .$$

Lemma 2.6. [9] *Let f be an entire function which satisfy the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

Lemma 2.7. [8] *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$, then the relative order and relative lower order of $L(f)$ with respect to $L(g)$ are same as those of f with respect to g i.e.,*

$$\rho_{L[g]}(L[f]) = \rho_g(f) \text{ and } \lambda_{L[g]}(L[f]) = \lambda_g(f)\ .$$

Lemma 2.8. [8] *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$. Then the relative type and relative lower type of $L(f)$ with respect to $L(g)$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite where $L(f) = W(a_1, a_2, \ldots a_{k_1}; f)$ and $L(g) = W(a_1, a_2, \ldots a_{k_2}; g)$.*

Lemma 2.9. [8] *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$. Then $\tau_{L(g)}(L(f))$ and $\overline{\tau}_{L(g)}(L(f))$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g where $L(f) = W(a_1, a_2, \ldots a_{k_1}; f)$ and $L(g) = W(a_1, a_2, \ldots a_{k_2}; g)$.*

3 Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite order and $\sum_{a \neq \infty} \delta(a; h) + \delta(\infty; h) = 2$ such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\sigma_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \sigma_g \cdot \rho_h(f)}{\lambda_h(f)}\ .$$

Proof. Let us suppose that $\alpha > 2$.
Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 2.1, Lemma 2.6 and the inequality $T_g(r) \leq \log M_g(r)$ $\{cf.$ [10]$\}$ for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1}T_{f\circ g}(r) &\leqslant \alpha\left[T_h^{-1}T_f(M_g(r))\right]^{\delta} \\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\leqslant \delta\log T_h^{-1}T_f(M_g(r)) + O(1) \end{aligned} \tag{3.1}$$

$$\begin{aligned} &i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \\ &\leq \frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} = \frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log M_g(r)}\cdot \\ &\qquad \frac{\log M_g(r)}{r^{\rho_g}}\cdot\frac{\log\exp r^{\rho_g}}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \end{aligned} \tag{3.2}$$

$$\begin{aligned} &i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \\ &\leq \limsup_{r\to\infty}\frac{\delta\log T_h^{-1}T_f(M_g(r))+O(1)}{\log M_g(r)}\cdot\limsup_{r\to\infty}\frac{\log M_g(r)}{r^{\rho_g}}\cdot \\ &\qquad \limsup_{r\to\infty}\frac{\log\exp r^{\rho_g}}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \end{aligned}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \leq \delta\cdot\rho_h(f)\cdot\sigma_g\cdot\frac{1}{\lambda_{L[h]}(L[f])}.$$

Therefore in view of Lemma 2.7 we get from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\rho_g})} \leq \frac{\delta\cdot\sigma_g\cdot\rho_h(f)}{\lambda_h(f)}.$$

Thus the theorem is established. □

In the line of Theorem 3.1 the following theorem can be proved :

Theorem 3.2. *Let f be a meromorphic function, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function of regular growth having non zero finite order with with the maximum* *deficiency sum such that* $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\sigma_g < \infty$ *and* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \sigma_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Using the notion of l*ower type*, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 3.1 and Theorem 3.2 respectively.

Theorem 3.3. *Let* f *be a transcendental meromorphic function with the maximum deficiency sum,* g *be entire and* h *be a transcendental entire function of regular growth having non zero finite order and* $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ *such that* $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ *and* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \overline{\sigma}_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.4. *Let* f *be a meromorphic function,* g *be a transcendental entire function with* $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ *and* h *be a transcendental entire function of regular growth having non zero finite order with with the maximum deficiency sum such that* $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ *and* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \overline{\sigma}_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Using the concept of the growth indicators τ_g and $\overline{\tau}_g$ of an entire function g, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.5. *Let* f *be a transcendental meromorphic function with the maximum deficiency sum,* g *be entire and* h *be a transcendental entire function of regular growth having non zero finite order and* $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ *such that* $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\tau}_g < \infty$ *and* h *satisfy the Property (A). Then for any* $\delta > 1$,

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \overline{\tau}_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.6. *Let f be a meromorphic function, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function of regular growth having non zero finite order with with the maximum deficiency sum such that $\lambda_h(g)>0$, $\rho_h(f)<\infty$, $\overline{\tau}_g<\infty$ and h satisfy the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[g]}(\exp r^{\lambda_g})}\leq\frac{\delta\cdot\overline{\tau}_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Theorem 3.7. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $0<\lambda_h(f)\leq\rho_h(f)<\infty$, $\tau_g<\infty$ and h satisfy the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\lambda_g})}\leq\frac{\delta\cdot\tau_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Theorem 3.8. *Let f be a meromorphic function, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function of regular growth having non zero finite order with with the maximum deficiency sum such that $\lambda_h(g)>0$, $\rho_h(f)<\infty$, $\tau_g<\infty$ and h satisfy the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[g]}(\exp r^{\lambda_g})}\leq\frac{\delta\cdot\tau_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Theorem 3.9. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that (i) $0<\rho_h(f)<\infty$, (ii) $\rho_h(f)=\rho_g$, (iii) $\sigma_g<\infty$, (iv) $0<\sigma_h(f)<\infty$ and h satisfy the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Proof. From (3.1), we get for all sufficiently large values of r that

$$\log T_h^{-1}T_{f\circ g}(r)\leqslant\delta(\rho_h(f)+\varepsilon)\log M_g(r)+O(1)\ . \tag{3.3}$$

Using Definition 1.1, we obtain from (3.3) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f \circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_g} + O(1) . \tag{3.4}$$

Now in view of condition (ii), we obtain from (3.4) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f \circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_h(f)} + O(1) . \tag{3.5}$$

Again in view of Definition 1.5 and with the help of Lemma 2.7 and Lemma 2.8, we get for a sequence of values of r tending to infinity that

$$T_{L[h]}^{-1} T_{L[f]}(r) \geq \left(\sigma_{L[h]}(L[f]) - \varepsilon\right) r^{\rho_{L[h]}(L[f])}$$

$$i.e.,\ T_{L[h]}^{-1} T_{L[f]}(r) \geq \left(\left(\frac{1 + k_1 - k_1 \delta(\infty; f)}{1 + k_2 - k_2 \delta(\infty; h)} \right)^{\frac{1}{\rho_h}} \cdot \sigma_h(f) - \varepsilon \right) r^{\rho_h(f)} . \tag{3.6}$$

Therefore from (3.5) and (3.6), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} \leq \frac{\delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_h(f)} + O(1)}{\left(\left(\frac{1+k_1 - k_1 \delta(\infty; f)}{1+k_2 - k_2 \delta(\infty; h)} \right)^{\frac{1}{\rho_h}} \cdot \sigma_h(f) - \varepsilon \right) r^{\rho_h(f)}} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} \leq \left(\frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\sigma_h(f)} \right) \left(\frac{1 + k_2 - k_2 \delta(\infty; h)}{1 + k_1 - k_1 \delta(\infty; f)} \right)^{\frac{1}{\rho_h}} .$$

Hence the theorem follows. □

Using the notion of *lower type* and *relative lower type*, we may state the following theorem without its proof as it can be carried out in the line of Theorem 3.9 :

Theorem 3.10. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a \neq \infty} \delta(a; h) + \delta(\infty; h) = 2$ *such that (i) $0 < \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\overline{\sigma}_g < \infty$, (iv) $0 < \overline{\sigma}_h(f) < \infty$ and h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} \leq \left(\frac{\delta \cdot \rho_h(f) \cdot \overline{\sigma}_g}{\overline{\sigma}_h(f)} \right) \left(\frac{1 + k_2 - k_2 \delta(\infty; h)}{1 + k_1 - k_1 \delta(\infty; f)} \right)^{\frac{1}{\rho_h}}$$

where $L(f) = W(a_1, a_2, ...a_{k_1}; f)$ *and* $L(h) = W(a_1, a_2, ...a_{k_2}; h)$.

Similarly using the notion of *type* and *relative lower type,* one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 3.9 :

Theorem 3.11. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that (i)* $0<\lambda_h(f)\leq\rho_h(f)<\infty$, *(ii)* $\rho_h(f)=\rho_g$, *(iii)* $\sigma_g<\infty$, *(iv)* $0<\overline{\sigma}_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\left(\frac{\delta\cdot\lambda_h(f)\cdot\sigma_g}{\overline{\sigma}_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.12. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that (i)* $0<\rho_h(f)<\infty$, *(ii)* $\rho_h(f)=\rho_g$, *(iii)* $\sigma_g<\infty$, *(iv)* $0<\overline{\sigma}_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\overline{\sigma}_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Similarly, using the concept of *weak type* and *relative weak type*, we may state next four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.9, Theorem 3.10, Theorem 3.11 and Theorem 3.12 respectively.

Theorem 3.13. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that (i)* $0<\lambda_h(f)\leq\rho_h(f)<\infty$, *(ii)* $\lambda_h(f)=\lambda_g$, *(iii)* $\overline{\tau}_g<\infty$, *(iv)* $0<\overline{\tau}_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\overline{\tau}_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.14. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that (i)* $0<\lambda_h(f)\le\rho_h(f)<\infty$, *(ii)* $\lambda_h(f)=\lambda_g$, *(iii)* $\tau_g<\infty$, *(iv)* $0<\tau_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\le\left(\frac{\delta\cdot\rho_h(f)\cdot\tau_g}{\tau_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.15. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such thatt (i)* $0<\lambda_h(f)<\infty$, *(ii)* $\lambda_h(f)=\lambda_g$, *(iii)* $\overline{\tau}_g<\infty$, *(iv)* $0<\tau_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\le\left(\frac{\delta\cdot\lambda_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.16. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that (i)* $0<\lambda_h(f)\le\rho_h(f)<\infty$, *(ii)* $\lambda_h(f)=\lambda_g$, *(iii)* $\overline{\tau}_g<\infty$, *(iv)* $0<\tau_h(f)<\infty$ *and h satisfies the Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\le\left(\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\right)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.17. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(f)\le\rho_h(f)<\rho_g\le\infty$ *and* $\sigma_h(f)<\infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\ge\frac{\lambda_h(f)}{\sigma_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Proof. Since $\rho_h(f) < \rho_g$ and $T_h^{-1}(r)$ is a increasing function of r, we get from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &\geq \log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r &\geq \left(\lambda_h(f) - \varepsilon\right) \cdot r^{\mu} \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r) &\geq \left(\lambda_h(f) - \varepsilon\right) \cdot r^{\rho_h(f)} . \end{aligned} \tag{3.7}$$

Again in view of Definition 1.5 and with the help of Lemma 2.7 and Lemma 2.8, we get for all sufficiently large values of r that

$$T_{L[h]}^{-1} T_{L[f]}(r) \leq \left(\sigma_{L[h]}\left(L[f]\right) + \varepsilon\right) r^{\rho_{L[h]}(L[f])}$$

$$i.e., \ T_{L[h]}^{-1} T_{L[f]}(r) \leq \left(\sigma_h(f)\left(\frac{1 + k_1 - k_1\delta(\infty; f)}{1 + k_2 - k_2\delta(\infty; h)}\right)^{\frac{1}{\rho_h}} + \varepsilon\right) r^{\rho_h(f)} . \tag{3.8}$$

Now from (3.7) and (3.8), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} \geq \frac{\left(\lambda_h(f) - \varepsilon\right) r^{\rho_h(f)}}{\left(\sigma_h(f)\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} + \varepsilon\right) r^{\rho_h(f)}} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} \geq \frac{\lambda_h(f)}{\sigma_h(f)}\left(\frac{1 + k_2 - k_2\delta(\infty; h)}{1 + k_1 - k_1\delta(\infty; f)}\right)^{\frac{1}{\rho_h}} .$$

Thus the theorem follows. □

In the line of Theorem 3.17, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.18. *Let f be meromorphic, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a; h) + \delta(\infty; h) = 2$ such that $0 < \lambda_h(f)$, $0 < \rho_h(g) < \rho_g \leq \infty$ and $\sigma_h(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[g]}(r)} \geq \frac{\lambda_h(f)}{\sigma_h(g)}\left(\frac{1 + k_2 - k_2\delta(\infty; h)}{1 + k_1 - k_1\delta(\infty; g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g) = W(a_1, a_2, ...a_{k_1}; g)$ and $L(h) = W(a_1, a_2, ...a_{k_2}; h)$.

The following two theorems can also be proved in the line of Theorem 3.17 and Theorem 3.18 respectively and with help of Lemma 2.3. Hence their proofs are omitted.

Theorem 3.19. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(g)$, $0<\lambda_f$, $0<\rho_h(f)<\rho_g<\infty$ *and* $\sigma_h(f)<\infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\geq\frac{\lambda_h(g)}{\sigma_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.20. *Let f be meromorphic, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(g)$, $0<\lambda_f$, $0<\rho_h(g)<\rho_g<\infty$ *and* $\sigma_h(g)<\infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\geq\frac{\lambda_h(g)}{\sigma_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Now we state the following four theorems without their proofs as those can be carried out in the line of Theorem 3.17, Theorem 3.18, Theorem 3.19 and Theorem 3.20 and with the help of Definition 1.6 and Lemma 2.9 :

Theorem 3.21. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(f)<\rho_g\leq\infty$ *and* $\overline{\tau}_h(f)<\infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\geq\frac{\lambda_h(f)}{\overline{\tau}_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.22. *Let f be meromorphic, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function* *of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(f)$, $0<\lambda_h(g)<\rho_g\leq\infty$ *and* $\overline{\tau}_h(g)<\infty$*. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\geq\frac{\lambda_h(f)}{\overline{\tau}_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.23. *Let f be a transcendental meromorphic function with the maximum deficiency sum, g be entire and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(g)<\rho_g<\infty$, $0<\lambda_f$ *and* $\overline{\tau}_h(f)<\infty$*. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\geq\frac{\lambda_h(g)}{\overline{\tau}_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.24. *Let f be meromorphic, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_h(g)<\rho_g<\infty$, $0<\lambda_f$ *and* $\overline{\tau}_h(g)<\infty$*. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\geq\frac{\lambda_h(g)}{\overline{\tau}_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.25. *Let f be a transcendental meromorphic function with the maximum deficiency sum having non zero finite order and lower order. Also let g be entire function and h be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *such that* $0<\lambda_g<\rho_h(f)<\infty$ *and* $\overline{\sigma}_h(f)>0$*. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\rho_h(f)}{\overline{\sigma}_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ *and* $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Proof. As $\lambda_g < \rho_h(f)$ and $T_h^{-1}(r)$ is a increasing function of r, it follows from Lemma 2.4 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &< \log T_h^{-1} T_f(\exp(r^{\mu})) \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r &< (\rho_h(f)+\varepsilon)\cdot r^{\mu} \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r) &< (\rho_h(f)+\varepsilon)\cdot r^{\rho_h(f)} \ . \end{aligned} \tag{3.9}$$

Further in view of Definition 1.5, we obtain for all sufficiently large values of r that

$$T_{L[h]}^{-1} T_{L[f]}(r) \geq \left(\overline{\sigma}_{L[h]}(L[f]) - \varepsilon\right) r^{\rho_{L[h]}(L[f])} \ .$$

Therefore in view of Lemma 2.7 and Lemma 2.8, we get from above for all sufficiently large values of r that

$$T_{L[h]}^{-1} T_{L[f]}(r) \geq \left(\overline{\sigma}_h(f)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)} \ . \tag{3.10}$$

Since $\varepsilon\,(>0)$ is arbitrary, therefore from (3.9) and (3.10) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} &\leq \frac{(\rho_h(f)+\varepsilon)\cdot r^{\rho_h(f)}}{\left(\overline{\sigma}_h(f)\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}} \\ i.e., \ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} &\leq \frac{\rho_h(f)}{\overline{\sigma}_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}} \ . \end{aligned}$$

Hence the theorem is established. □

In the line of Theorem 3.25, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.26. *Let f be meromorphic with non zero finite order and lower order, g be a transcendental entire function with the maximum deficiency sum and h bbe a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ with $\rho_h(f) < \infty$, $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[g]}(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g) = W(a_1, a_2, ...a_{k_1}; g)$ and $L(h) = W(a_1, a_2, ...a_{k_2}; h)$.

Moreover, the following two theorems can also be deduced in the line of Theorem 3.17 and Theorem 3.18 respectively and with help of Lemma 2.5 and therefore their proofs are omitted.

Theorem 3.27. *Let f be a transcendental meromorphic function with the maximum deficiency sum having non zero finite order. Also let g be entire function and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $\rho_h(g)<\infty$, $0<\lambda_g<\rho_h(f)<\infty$ and $\overline{\sigma}_h(f)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\rho_h(g)}{\overline{\sigma}_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.28. *Let f be meromorphic with finite order, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ with $0<\lambda_g<\rho_h(g)<\infty$ and $\overline{\sigma}_h(g)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\leq\frac{\rho_h(g)}{\overline{\sigma}_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Finally we state the following four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.25, Theorem 3.26, Theorem 3.27 and Theorem 3.28 using the concept of *relative weak type:*

Theorem 3.29. *Let f be a transcendental meromorphic function with the maximum deficiency sum having non zero finite order and lower order. Also let g be entire function and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $0<\lambda_g<\lambda_h(f)\leq\rho_h(f)<\infty$ and $\tau_h(f)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\rho_h(f)}{\tau_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.30. *Let f be meromorphic with non zero finite order and lower order, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $\rho_h(f)<\infty$, $0<\lambda_g<\lambda_h(g)<\infty$ and $\tau_h(g)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\leq\frac{\rho_h(f)}{\tau_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.31. *Let f be a transcendental meromorphic function with the maximum deficiency sum having finite order. Also let g be entire function and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $\rho_h(g)<\infty$, $0<\lambda_g<\lambda_h(f)<\infty$ and $\tau_h(f)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\rho_h(g)}{\tau_h(f)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;f)}\right)^{\frac{1}{\rho_h}}$$

where $L(f)=W(a_1,a_2,....a_{k_1};f)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

Theorem 3.32. *Let f be meromorphic with finite order, g be a transcendental entire function with the maximum deficiency sum and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ such that $0<\lambda_g<\lambda_h(f)\leq\rho_h(g)<\infty$ and $\tau_h(g)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}\leq\frac{\rho_h(g)}{\tau_h(g)}\left(\frac{1+k_2-k_2\delta(\infty;h)}{1+k_1-k_1\delta(\infty;g)}\right)^{\frac{1}{\rho_h}}$$

where $L(g)=W(a_1,a_2,....a_{k_1};g)$ and $L(h)=W(a_1,a_2,....a_{k_2};h)$.

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