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Relationship Between Hyper MV-algebras and Hyperlattices

R. A. Borzooei, Akefe Radfar, and Sogol Niazian

Abstract. Sh. Ghorbani, et al. [9], generalized the concept of MV-algebras and defined the notion of hyper MV-algebras. Now, in this paper, we try to prove that any hyper MV-algebra is a hyperlattice. First we prove that any hyper MV-algebra that satisfies the semi negation property is a hyperlattice. Then with a computer program, we show that any hyper MV-algebra of order less than 6, is a hyperlattice. Finally, we claim that this result is correct for any hyper MV-algebra.

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1 Introduction

The first studies regarding multiple-valued logics were conducted by J. Lukasiewicz and E. Post when they introduced a three-valued logical system in 1920 [14]. The latter built a different n-valued logical system in 1921 [17]. Then Lukasiewicz and Tarski developed in 1930 [15] a logic for which the truth values are the rationales in $[0, 1]$. In 1940, Gr.C. Moisil introduced the three-valued Lukasiewicz algebras as algebraic models for the corresponding logic of Lukasiewicz. In 1941, Moisil also defined n-valued Lukasiewicz algebras. Then, in 1956, A. Rose showed that for a number of truth values greater than 5 the Lukasiewicz algebras are no longer the algebras of Lukasiewicz logic. In fact, by defining the n-valued Lukasiewicz algebras, Moisil invented a distinct logical system. In 1958, C.C. Chang defined MV-algebras as models for the infinitely valued Lukasiewicz-Tarski logic [5]. In 1977, R. Grigolia introduced MV_n-algebras to model the n-valued Lukasiewicz logic [10].

The study of hyperstructures, started in 1934 by Marty's paper at the 8th Congress of Scandinavian Mathematicians [16] where hypergroups were introduced. Sh. Ghorbani et al. [9] applied the hyperstructure to MV-algebras and introduce the concept of hyper MV-algebras which is a generalization of MV-algebras and investigated some results. They also discussed quotient structure and category of hyper MV-algebras ([8], [7]). Specially, they clarified the relation between the class of hyper MV-algebras and hyper K-algebras [2]. R. A. Borzooei et al. [1] proved that these relations are not true, which unfortunately is used to prove some important results of several hyper MV-algebras paper. L. Torkzadeh et al [18] discussed hyper MV-ideals and define some hyperoperations on it. Then they get some results and give a problem which want to prove or disprove the hyperoperations $\vee$ and $\wedge$ are associative. As another hyper algebraic structures the notions of (weak) hyper MV-deductive systems and (weak) implicative hyper MV-deductive systems are introduced in [12]. Then the relation among them are discussed. Also, as a continue, new types of hyper MV-deductive systems are introduced in [?newded]. Now, in this paper, we try to find a relationship between hyper MV-algebras and hyperlattices.

2 Preliminary

In this section we give some definitions and properties of MV-algebras and hyper MV-algebras which we need in the next section.

Definition 2.1. [5] *An MV-algebra is an algebra $(A, \oplus, *, 0)$ of type $(2, 2, 0)$ that satisfying the following axioms:*

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $x^{**} = x$,
(MV5) $x \oplus 0^* = 0^*$,
(MV6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Let A be an MV-algebra. We define the operations $\odot$ and $\ominus$ on A by, $x \odot y =: (x^* \oplus y^*)^*$ and $x \ominus y =: x \odot y^*$, for any $x, y \in A$ and we consider $1 =: 0^*$. Moreover, the relation $x \leq y$ on A is defined by $x \leq y$ if and only if $x^* \oplus y = 1$, for any $x, y \in A$. The relation $\leq$ is a partial order on A which is called the natural order of A. This natural order determines a lattice structure $(A, \vee, \wedge)$, where $x \vee y =: (x \odot y^*) \oplus y$ and $x \wedge y =: (x^* \vee y^*)^*$, for any $x, y \in A$. As a first example of nontrivial MV-algebra, consider the real unit interval $[0,1]$ with $x \oplus y = min\{x+y, 1\}$ and $x^* = 1 - x$. It is easy to see that $([0,1], \oplus, *, 0)$ is an MV-algebra.

Proposition 2.1. [5] *Let A be an MV-algebra and $x, y \in A$. Then the following is hold:*

(i) $1^* = 0$,

(ii) $x \oplus y = (x^* \odot y^*)^*$,

(iii) $x \oplus 1 = 1$,

(iv) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$,

(v) $x \oplus x^* = 1$,

(vi) $x \leq y$ *if and only if* $y^* \leq x^*$,

(vii) *if* $x \leq y$, *then for each* $z \in A$, $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,

(viii) $x \odot y \leq z$ *if and only if* $x \leq y^* \oplus z$,

(ix) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(x) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

Definition 2.2. [6] *A hyperoperation on a nonempty set H is a map $\circ : H \times H \to P^\star(H) = P(H) - \{\emptyset\}$. In this case, $(H, \circ)$ is called a hypergroupoid. Let $(H, \circ)$ be a hypergroupoid. Then an element $a \in H$ is called scalar if $|a \odot x| = 1$, for any $x \in H$. Moreover, if A and B are two non-empty subsets of H, then we define $A \circ B$, $a \circ B$ and $A \circ b$ as follows, for any $a \in A$ and $b \in B$:*

$$A \circ B = \bigcup_{a \in A, b \in B} (a \circ b), \qquad a \circ B = \{a\} \circ B, \qquad A \circ b = A \circ \{b\}.$$

Definition 2.3. [9] *A hyper MV-algebra is a nonempty set M endowed with a hyperoperation "$\oplus$", a unary operation "$*$" and a constant "0" satisfying the following axioms, for all $x, y, z \in M$, :*

(HMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(HMV2) $x \oplus y = y \oplus x$,

(HMV3) $(x^*)^* = x$,
(HMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(HMV5) $0^* \in x \oplus 0^*$,
(HMV6) $0^* \in x \oplus x^\star$,
(HMV7) $x \ll y,\ y \ll x \ \Rightarrow\ x = y$
where $x \ll y$ *is defined by* $0^* \in x^\star \oplus y$. *For any* $A, B \subseteq M$, *we define* $A \ll B$ *if and only if there exist* $a \in A$ *and* $b \in B$ *such that* $a \ll b$. *We define* $0^* := 1$ *and* $A^* = \{a^* : a \in A\}$.

Proposition 2.2. [9] *Let* $(M, \oplus, *, 0)$ *be a hyper* MV*-algebra. Then for all* $x, y, z \in M$ *and for all nonempty subsets* A, B *and* C *of* M *the following hold:*

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$,

(ii) $0 \ll x$, $x \ll 1$, $x \ll x$ *and* $A \ll A$,

(iii) If $x \ll y$, *then* $y^* \ll x^*$ *and* $A \ll B$ *implies* $B^* \ll A^*$,

(iv) $(A^*)^* = A$,

(v) $0 \oplus 0 = \{0\}$ *and* $x \in x \oplus 0$,

(vi) If $y \in x \oplus 0$, *then* $y \ll x$.

Theorem 2.3. [1] *Let* M *be a finite hyper* MV*-algebra such that* $0 \oplus x = \{x\}$, *for all* $x \in M$. *Then* M *is an* MV*-algebra.*

Proposition 2.4. [18] *Let* $(M, \oplus, *, 0)$ *be a hyper* MV*-algebra. Define the following hyperopoerations on* M *as follows:*

$$x \vee y = (x^* \oplus y)^* \oplus y, \quad x \wedge y = (x^* \vee y^*)^*$$

Then for all $x, y, z \in M$:

(i) $x \in (x \wedge x) \cap (x \vee x)$,

(ii) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$,

(iii) $x \in (x \wedge (x \vee y)) \cap (x \vee (x \wedge y))$,

(iv) *if* $x \ll y$, *then* $y \in x \vee y$ *and* $x \in x \wedge y$,

(vi) $x, y \ll x \vee y$ *and* $x \wedge y \ll x, y$.

3 Relationship between hyper MV-algebras and hyperlattices

In this section, we try to show that any finite hyper MV-algebra is a hyperlattice.

Definition 3.1. *If $x^* = x$, for any $x \in M - \{0, 1\}$, then we say that M satisfied the Semi Negation Property (or (SNP), for short).*

Example 3.1. Let $M = \{0, a, b, 1\}$ and hyperoperation $\oplus$ and unary operation $*$ on M are defined as follows;

$\oplus$	0	a	b	1
0	{0}	{0,a}	{0,b}	A
a	{0,a}	A	{0,a,b}	A
b	{0,b}	{0,a,b}	A	A
1	A	A	A	A

$*$	0	a	b	1
	1	a	b	0

Then it is easy to see that $(M, \oplus, *, 0)$ is a hyper MV-algebra that satisfying the (SNP).

Note: Throughout this section, we let M be a hyper MV-algebra and satisfies the (SNP), unless otherwise stated.

Lemma 3.1. *For all $x, y \in M - \{0, 1\}$:*

(i) $x \ll y$*, implies* $x = y$*,*

(ii) *if* $0 \oplus x = \{x\}$*, then* $y \notin 1 \oplus x$*.*

Proof. (i) If $x \ll y$, then by Proposition 2.2(iii), $y^* \ll x^*$ and so $y \ll x$. Hence, by (HMV7), $x = y$.
(ii) On the contrary, let $y \in 1 \oplus x$, for $y \in M - \{0, 1\}$. By (HMV4), we get

$$y \oplus x = y^* \oplus x \subseteq (1 \oplus x)^* \oplus x = (x \oplus 0)^* \oplus 0 = x^* \oplus 0 = x \oplus 0 = \{x\}.$$

Thus $y \oplus x = \{x\} = y^* \oplus x$. Now, by (HMV4),

$$\begin{aligned} x \oplus x &= x^* \oplus x = (y^* \oplus x)^* \oplus x = (x^* \oplus y)^* \oplus y \\ &= (x \oplus y)^* \oplus y = x^* \oplus y = x \oplus y = \{x\}. \end{aligned}$$

Hence, $x \oplus x = \{x\}$. Also, by (HMV6), $1 = 0^* \in x^* \oplus x = x \oplus x$ which is a contradiction. Therefore, $y \notin 1 \oplus x$. □

Lemma 3.2. *For any $x, y \in M$ and $A \subseteq M$,* *(i)* *if* $1 \in x \oplus y$*, then* $x = y$ *or* $x = 1$ *or* $y = 1$*;*

(ii) *if* $x \notin 0 \oplus 1$*, then* $x \in 0 \oplus A$ *implies* $x \in A$*;*

(iii) *if* $\{0, 1\} \subseteq A$ *or* $0, 1 \notin A$*, then* $A^* = A$*.*

Proof. (i) Let $1 = 0^* \in x \oplus y$. If $x, y \in M - \{0, 1\}$, then $x \ll y$ and so by Lemma 3.1(i), $x = y$. If $x \neq 0, 1$ and $y = 0$, then $0^* = 1 \in x \oplus 0 = x^* \oplus 0$ and implies that $x \ll 0$. Hence $x = 0$, which is a contradiction. Therefore, if $x \neq 0, 1$, then $y = 1$ and similarly, $y \neq 0, 1$ implies that $x = 1$. If $x, y \in \{0, 1\}$ and $x = y = 0$, then $1 \in x \oplus y = 0 \oplus 0 = \{0\}$, which is a contradiction. So, $x = 1$ or $y = 1$.
(ii) If $x \in 0 \oplus A$, then there is $a \in A$ such that $x \in 0 \oplus a$. By Proposition 2.2(vi), $x \ll a$. By (i), $a = 1$, $x = 1$ or $x = a$. Since $1 \in 0 \oplus 1$, by Proposition 2.2(v), we get $x \neq 1$. Also, $a = 1$ means that $x \in 0 \oplus 1$ which against the assumption. Thus $x = a \in A$.
(iii) We know $A^* = \{x^* : \ x \in A\}$. If $\{0, 1\} \subseteq A$, then for any $x \in A$, $x = 0$ or $x = 1$ or $x \in M - \{0, 1\}$ and so $x^* = 1$ or $x^* = 0$ or $x^* = x$. Hence, $x^* \in A$ i.e. $A^* = A$. Now, let $0, 1 \notin A$. Then $A \subseteq M - \{0, 1\}$ and since M satisfies the (SNP), we get $A^* = A$. □

Theorem 3.3. [1] *Let* M *be a hyper* MV*-algebra and* x *be an element of* M *such that* $0 \oplus x = \{x\}$ *and* $x^* = x$*. Then* $0, x \notin 1 \oplus x$*.*

Lemma 3.4. *Let* x *be an element of* $M - \{0, 1\}$ *such that* $0 \oplus x = \{x\}$*. Then we get*

(i) $1 \oplus x = \{1\}$, $0 \oplus 1 = \{1\}$*;*

(ii) $x \oplus x = \{1\}$*;*

(iii) $0 \oplus y = \{y\}$*, for all* $y \in M - \{0, 1\}$*.*

Proof. (i) By Theorem 3.3, $0, x \notin 1 \oplus x$ and by Lemma 3.1(ii), $y \notin 1 \oplus x$, for all $y \in M - \{0, 1\}$. Thus $1 \oplus x = \{1\}$. Also, we get $0 \oplus 1 = 0 \oplus (1 \oplus x) = 1 \oplus (0 \oplus x) = 1 \oplus x = \{1\}$.
(ii) By part (i) and (HMV4), we get

$$\begin{aligned} x \oplus x &= x^* \oplus x = (0 \oplus x)^* \oplus x = (x^* \oplus 1)^* \oplus 1 \\ &= (x \oplus 1)^* \oplus 1 = 1^* \oplus 1 = 0 \oplus 1 = \{1\}. \end{aligned}$$

(iii) Let $y \in M - \{0, 1\}$ and $y \neq x$. By Proposition 2.2(v), $y \in 0 \oplus y$. Now, by the contrary, let $0 \oplus y \neq \{y\}$. Then there exists $z \in M$ such that $z \neq y$ and $z \in 0 \oplus y$.

If $z \neq 0, 1$, then by Proposition 2.2(vi), $z \ll y$ and so by Lemma 3.1(i) we get $z = y$, which is a contradiction.
If $z = 1$, then $1 \in 0 \oplus y$ and so by Proposition 2.2$(vi), 1 \ll y$. Hence $y = 1$ which is a contradiction by $y \neq 0, 1$.
If $z = 0$, since $x \in 0 \oplus x \subseteq (0 \oplus y) \oplus x = (0 \oplus x) \oplus y = x \oplus y$, we get $x \in x \oplus y$. So, by (HMV4),

$$1 \in x^* \oplus x \subseteq (x \oplus y)^* \oplus x = (y^* \oplus x)^* \oplus x = (x^* \oplus y)^* \oplus y = (x \oplus y)^* \oplus y.$$

Hence there is $t \in x \oplus y$ such that $1 \in t^* \oplus y$. By Lemma 3.2(i), $t^* = 1$ or $t^* = y$ and so $t = 0$ or $t = y$. It means that $0 \in x \oplus y$ or $y \in x \oplus y$. If $0 \in x \oplus y$, then by (HMV1) and (ii),

$$x \in 0 \oplus x \subseteq (x \oplus y) \oplus x = (x \oplus x) \oplus y = 1 \oplus y.$$

Hence, by (HMV4), we get $x \in x \oplus y = x^* \oplus y \subseteq (1 \oplus y)^* \oplus y = (y \oplus 0)^* \oplus 0$. Since $x \notin 0 \oplus 1 = \{1\}$, by Lemma 3.2(ii), we get $x \in (y \oplus 0)^*$. So, $x = x^* \in ((y \oplus 0)^*)^* = y \oplus 0$. Thus $x \ll y$ and so by Lemma 3.1(i), $x = y$ which is a contradiction. Similarly, for the case $y \in x \oplus y$, we get a contradiction. Therefore, $0 \oplus y = \{y\}$, for all $y \neq 0, 1$. □

Theorem 3.5. *If M is finite and x be an element of $M - \{0, 1\}$ such that $0 \oplus x = \{x\}$, then M is an MV-algebra.*

Proof. Let M be finite and $x \in M - \{0, 1\}$ such that $0 \oplus x = \{x\}$. Then by Lemma 3.4(iii), $0 \oplus y = \{y\}$, for all $y \in M - \{0, 1\}$. Moreover, by Lemma 3.4(i), $0 \oplus 1 = \{1\}$ and by Proposition 2.2(v), $0 \oplus 0 = \{0\}$. Hence $0 \oplus y = \{y\}$ for all $y \in M$ and so by Theorem 2.3, M is an MV-algebra. □

Proposition 3.6. *Let M be finite and proper. Then for all distinct elements $x, y, z \in M - \{0, 1\}$,*

(P_1) $0 \oplus x = \{0, x\}$,

(P_2) *if* $x \in 0 \oplus 1$, *then* $0 \in 0 \oplus 1$,

(P_3) $x, y \in x \oplus y$,

(P_4) $x \in 1 \oplus x$ *or* $0 \in 1 \oplus x$. *Indeed,* $1 \oplus x \neq \{1\}$,

(P_5) *if* $y \notin 0 \oplus 1$, *then* $y \notin 1 \oplus x$,

(P_6) *if* $x \notin 0 \oplus 1$, *then* $x \in 1 \oplus x$,

(P_7) $1 \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$,

(P_8) $1 \oplus 1 \subseteq 0 \oplus 1$,

(P_9) *if* $y \in x \oplus x$, *then* $y \in 0 \oplus 1$. *Indeed,* $x \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$,

(P_{10}) $0 \oplus 1 \setminus \{0, x\} \subseteq x \oplus x$,

(P_{11}) $0 \oplus 1 \subseteq (x \oplus x) \cup (1 \oplus x)$ *and* $\{0, x\} \subseteq (x \oplus x) \cup (1 \oplus x)$,

(P_{12}) $(x \oplus x) \cup (1 \oplus x) = (0 \oplus 1) \cup \{0, x\}$,

(P_{13}) $z \in x \oplus y$ *implies* $x \in y \oplus z$,

(P_{14}) *if* $z \in x \oplus y$, *then* $x, y, z \in 0 \oplus 1$ *or* $x, y, z \notin 0 \oplus 1$,

(P_{15}) $(0 \oplus 1) \cup (x \oplus y) \cup \{0\} \subseteq x \vee y$,

(P_{16}) *if* $x, y \notin 0 \oplus 1$, *then* $x \oplus (x \oplus y) \cup \{0\} = x \oplus y \cup \{0\}$.

Proof. (P_1) : Let there exists $x \in M - \{0, 1\}$ such that $0 \oplus x = \{x\}$, by the contrary. Then by Theorem 3.5, M is an MV-algebra and so it is not proper which is a contradiction. Hence $0 \oplus x \neq \{x\}$, for all $x \in M - \{0, 1\}$. Thus there is $y \in 0 \oplus x$ and $y \neq x$. By Lemma 3.1 (i), we imply that $y \in \{0, 1\}$. Thus $y = 0$ or $y = 1$. If $y = 1$, then $1 \in 0 \oplus x$ and so $1 \ll x$. Hence, by Proposition 2.2 (vi), $x = 1$ which is a contradiction with $x \in M - \{0, 1\}$. Thus, $y = 0 \in 0 \oplus x$, for all $x \in M - \{0, 1\}$. Therefore, $0 \oplus x = \{0, x\}$, for all $x \in M - \{0, 1\}$.

(P_2) : Let $x \in 0 \oplus 1$. Then by (P_1), $0 \in 0 \oplus x \subseteq 0 \oplus (0 \oplus 1) = (0 \oplus 0) \oplus 1 = 0 \oplus 1$. Hence, $0 \in 0 \oplus 1$.

(P_3) : Let $x, y \in M - \{0, 1\}$ be two distinct elements. By (P_1) and (HMV1),

$$x \in 0 \oplus x \subseteq (0 \oplus y) \oplus x = 0 \oplus (x \oplus y),$$

Then there exists $t \in x \oplus y$ such that $x \in 0 \oplus t$. Thus $x \ll t$ and so $1 \in x \oplus t$. Now, by Lemma 3.2(i), we get $x \in x \oplus y$ or $1 \in x \oplus y$. But $1 \in x \oplus y = x^* \oplus y$ implies $x = y$, which is a contradiction. Therefore, $x \in x \oplus y$. By the similar way, $y \in x \oplus y$.

(P_4) : Since $0 \ll 0$, $1 \in 0^* \oplus 0$. By (HMV_4) and (P_1), $1 \in 0^* \oplus 0 \subseteq (0 \oplus x)^* \oplus 0 = (1 \oplus x)^* \oplus x$. By Lemma 3.2(i), $1 \in (1 \oplus x)^*$ or $x \in (1 \oplus x)^*$. Therefore, $0 \in 1 \oplus x$ or $x \in 1 \oplus x$. So $1 \oplus x \neq \{1\}$.

(P_5) : Let $y \notin 0 \oplus 1$. On the contrary, if $y \in 1 \oplus x$, then by (P_1), (P_3) and (HMV4), $y \in y \oplus x = y^* \oplus x \subseteq (1 \oplus x)^* \oplus x = (x \oplus 0)^* \oplus 0 = \{0, x\}^* \oplus 0 =$

$(1 \oplus 0) \cup (x \oplus 0) = (0 \oplus 1) \cup \{0, x\}$.Thus $y \in 0 \oplus 1$, which is a contradiction. Hence $y \notin 1 \oplus x$.

(P_6) : Let $x \notin 0 \oplus 1$. If $0 \oplus 1 \neq \{1\}$, then by (P_2), $0 \in 0 \oplus 1$ and so by (HMV1) and (P_1),

$$\{0, x\} = 0 \oplus x \subseteq (1 \oplus 0) \oplus x = 1 \oplus (0 \oplus x) = 1 \oplus \{0, x\} = (1 \oplus 0) \cup (1 \oplus x).$$

Now, since $x \notin 0 \oplus 1$, hence $x \in 1 \oplus x$. If $0 \oplus 1 = \{1\}$, then by routine calculations, we get $1 \oplus x = \{0, x, 1\} \ni x$.

(P_7) : Let $y \in 1 \oplus x$ and $y \neq \{0, x\}$. Then $y = 1$ or $y \neq 0, 1$ and $y \neq x$. If $y = 1$, then by Proposition 2.2(v), $y = 1 \in 0 \oplus 1$. Now, let $x, y \in M - \{0, 1\}$ be distinct and $y \notin 0 \oplus 1$, by the contrary. Then by (P_5), $y \notin 1 \oplus x$, which is a contradiction. Hence $y \in 0 \oplus 1$ i.e. $(1 \oplus x) \setminus \{0, x\} \subseteq 0 \oplus 1$. So $1 \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$.

(P_8) : Let $x \in 1 \oplus 1$. Then by (P_1) and (HMV1), $x \in 0 \oplus x \subseteq 0 \oplus (1 \oplus 1) = (0 \oplus 1) \oplus 1$. Thus there is $t \in 0 \oplus 1$ such that $x \in t \oplus 1$. By (P_7), $x \in 1 \oplus t \subseteq (0 \oplus 1) \cup \{0, t\}$. Thus $x \in 0 \oplus 1$ or $x = t$. We note that $x = t$ means $x = t \in 0 \oplus 1$. Hence $x \in 0 \oplus 1$ and so $1 \oplus 1 \subseteq 0 \oplus 1$.

(P_9) : Let $y \in x \oplus x$ and $y \neq x$. Then by $(HMV4)$ and (P_1),

$$y \in (x \oplus x) \cup (1 \oplus x) = \{x, 1\} \oplus x = \{0, x\}^* \oplus x = (0 \oplus x)^* \oplus x = (x \oplus 1)^* \oplus 1.$$

Hence there is $t \in 1 \oplus x$ such that $y \in 1 \oplus t^*$. We note that $t = 1$ or $t = 0$ or $t \in M - \{0, 1\}$.

If $t = 1$, then $y \in 1 \oplus 1^* = 1 \oplus 0$. If $t = 0$, then $y \in 1 \oplus 1$ and so by (P_8), $y \in 1 \oplus 1 \subseteq 0 \oplus 1$.

If $t \in M - \{0, 1\}$, then by (P_7), $y \in 1 \oplus t \subseteq (0 \oplus 1) \cup \{0, t\}$. Thus $y \in 0 \oplus 1$ or $y = t$. If $y = t$, then $y = t \in 1 \oplus x$. Again by (P_7), $y \in 1 \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$. Since x and y are distinct and $y \neq 0$, we get $y \in 0 \oplus 1$ in all cases. Therefore, $x \oplus x \setminus \{0, x, 1\} \subseteq 0 \oplus 1$ and so, by $1 \in 0 \oplus 1$, $x \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$.

(P_{10}) : Since $x \ll x$, we get $1 \in x \oplus x$. By (HMV1), we get

$$0 \oplus 1 \subseteq 0 \oplus (x \oplus x) = (0 \oplus x) \oplus x = \{0, x\} \cup (x \oplus x).$$

Hence $0 \oplus 1 \setminus \{0, x\} \subseteq x \oplus x$.

(P_{11}) : Since $0 \ll x$, we conclude that $0^* \in 1 \oplus x$. By (HMV4),

$$0 \oplus 1 \subseteq (x \oplus 1)^* \oplus 1 = (0 \oplus x)^* \oplus x = \{0, x\}^* \oplus x = (x \oplus x) \cup (1 \oplus x) \qquad (1).$$

Also, if $x \in 0 \oplus 1$, then by (P_2) we get $0 \in 0 \oplus 1$. Thus $\{0, x\} \subseteq 0 \oplus 1 \subseteq (x \oplus x) \cup (1 \oplus x)$. If $x \notin 0 \oplus 1$, then by (P_6), $x \in 1 \oplus x$. Hence, by (P_1), (HMV1) and (1), $\{0, x\} = 0 \oplus x \subseteq 0 \oplus (1 \oplus x) = (0 \oplus x) \oplus 1 = \{0, x\} \oplus 1 = (0 \oplus 1) \cup (x \oplus 1) \subseteq (x \oplus x) \cup (x \oplus 1)$.

(P_{12}) : By (P_7), (P_9) and (P_{11}), the proof is clear.

(P_{13}) : Let $z \in x \oplus y$. Then $1 \in z \oplus z^* = z \oplus z \subseteq (x \oplus y) \oplus z = x \oplus (y \oplus z)$. By Lemma 3.2(i), $1 \in y \oplus z$ or $x \in y \oplus z$. If $1 \in y \oplus z$, then by Lemma 3.2(i), $y = z$, that is a contradiction. Therefore, $x \in y \oplus z$.

(P_{14}) : Let $z \in x \oplus y$, $z \in 0 \oplus 1$ and $x \notin 0 \oplus 1$, by the contrary. By (P_{13}), $z \in x \oplus y$, implies $x \in y \oplus z$. Since $z \in 0 \oplus 1$, we get $x \in z \oplus y \subseteq (0 \oplus 1) \oplus y = 0 \oplus (1 \oplus y)$. Since $x \notin 0 \oplus 1$, by Lemma 3.2(ii), $x \in 1 \oplus y$. Also, by (P_5), $x \notin 1 \oplus y$ which is a contradiction. So, $x \in 0 \oplus 1$. Similarly, $y \in 0 \oplus 1$. Therefore, $x, y, z \in 0 \oplus 1$.

Now, let $z \notin 0 \oplus 1$ and $x \in 0 \oplus 1$, by the contrary. Then $z \in x \oplus y \subseteq (0 \oplus 1) \oplus y = 0 \oplus (1 \oplus y)$ and so by Lemma 3.2(ii), $z \in 1 \oplus y$. Also, by (P_5), $z \notin 1 \oplus y$, which is a contradiction. So $x \notin 0 \oplus 1$. Similarly, $y \notin 0 \oplus 1$. Therefore, $x, y, z \notin 0 \oplus 1$.

(P_{15}) : By (P_3), $x, y \in x \oplus y$ and so $x \oplus y = x^* \oplus y \subseteq (x \oplus y)^* \oplus y = (x^* \oplus y)^* \oplus y = x \vee y$. Also by (P_{14})

$$(0 \oplus 1) \setminus \{0, y\} \subseteq y \oplus y = y^* \oplus y \subseteq (x \oplus y)^* \oplus y = (x^* \oplus y)^* \oplus y = x \vee y.$$

Since $y \in x \oplus y \subseteq x \vee y$, it is enough to show that $0 \in x \vee y$:
By (P_{12}), $(x \oplus x) \cup (1 \oplus x) = (0 \oplus 1) \cup \{0, x\}$. Thus $0 \in x \oplus x$ or $0 \in 1 \oplus x$. If $0 \in x \oplus x$, then $0 \in x \oplus x = x^* \oplus x \subseteq (y \oplus x)^* \oplus x = (y^* \oplus x)^* \oplus x = x \vee y$. So $0 \in x \vee y$ and the proof is complete.
If $0 \in 1 \oplus x$, then by $1 \in y \oplus y$, $0 \in x \oplus 1 \subseteq x \oplus (y \oplus y) = (x \oplus y) \oplus y$. It means that there is $t \in x \oplus y$ such that $0 \in t \oplus y$. We note that by Lemma 3.1(i), $1 \notin x \oplus y$ and so $t \neq 1$. If $t = 0$, then $0 \in x \oplus y \subseteq x \vee y$ and the proof is complete. Otherwise, $t \in M - \{0, 1\}$ and $0 \in t \oplus y = t^* \oplus y \subseteq (x \oplus y)^* \oplus y = x \vee y$. So, $0 \in x \vee y$ in all cases. Now, we get

$$(x \oplus y) \cup (0 \oplus 1) \cup \{0\} \subseteq x \vee y.$$

(P_{16}) : Let $x, y \notin 0 \oplus 1$. Since by (P_3), $y \in x \oplus y$, we conclude that $x \oplus y \subseteq x \oplus (x \oplus y)$. Now, let $t \in x \oplus (x \oplus y)$ be arbitrary. Then there is $u \in x \oplus y$ such that $t \in x \oplus u$. Since $x, y \notin 0 \oplus 1$ by (P_{14}), we get $u \notin 0 \oplus 1$.

Again since $x, u \notin 0 \oplus 1$, we conclude that $t \notin 0 \oplus 1$. Also, we have

$$\begin{aligned}
x \oplus (x \oplus y) &= (x \oplus x) \oplus y \subseteq ((0 \oplus 1) \cup \{0, x\}) \oplus y \quad \text{by } (P_9) \\
&= ((0 \oplus 1) \oplus y) \cup \{0, y\} \cup (x \oplus y) \quad \text{by } (P_1) \\
&= (0 \oplus (1 \oplus y)) \cup \{0, y\} \cup (x \oplus y) \quad \text{by } (HMV1) \\
&\subseteq (0 \oplus ((0 \oplus 1) \cup \{0, y\})) \cup \{0, y\} \cup (x \oplus y) \quad \text{by } (P_7) \\
&= (0 \oplus 0 \oplus 1) \cup (0 \oplus 0) \cup (0 \oplus y) \cup \{0, y\} \cup (x \oplus y) \\
&= (0 \oplus 1) \cup \{0, y\} \cup (x \oplus y) \quad \text{by Proposition 2.2(v)} \\
&= (0 \oplus 1) \cup \{0\} \cup (x \oplus y) \quad \text{by } (P_3)
\end{aligned}$$

Since $t \notin 0 \oplus 1$, we get $t \in (x \oplus y) \cup \{0\}$ and so $x \oplus (x \oplus y) \subseteq (x \oplus y) \cup \{0\}$. Therefore, $x \oplus (x \oplus y) \cup \{0\} = (x \oplus y) \cup \{0\}$. □

Lemma 3.7. *For all distinct elements $x, y \in M - \{0, 1\}$,*

(*i*) $x \vee y = (x \oplus y) \cup (0 \oplus 1) \cup \{0\}$,

(*ii*) $0 \vee x = 1 \vee x = (0 \oplus 1) \cup \{0, x\}$,

(*iii*) $1 \vee 1 = 0 \oplus 1$ *and* $0 \vee 0 = \begin{cases} \{0\} & \text{if } 0 \oplus 1 = \{1\}, \\ 0 \oplus 1 & \text{if } 0 \oplus 1 \neq \{1\}, \end{cases}$

(*iv*) $x \vee x = \begin{cases} (0 \oplus 1 \setminus \{1\}) \cup \{0, x\} & \text{if } 0, x \notin x \oplus x, \\ (0 \oplus 1) \cup \{0, x\} & \text{otherwise}, \end{cases}$

(*v*) $0 \vee 1 = 0 \oplus 1$.

Proof. (i) By (P_{15}), $(0 \oplus 1) \cup (x \oplus y) \cup \{0\} \subseteq x \vee y$. We note that by Lemma 3.1(i), $1 \notin x \oplus y$ and so $0 \in x \oplus y$ or $x \oplus y \subseteq M - \{0, 1\}$. Thus $(x \oplus y)^* \subseteq (x \oplus y \setminus \{0\}) \cup \{1\}$. Now, we get

$$\begin{aligned}
& x \vee y \\
= \ & (x \oplus y)^* \oplus y \subseteq ((x \oplus y \setminus \{0\}) \cup \{1\}) \oplus y \\
\subseteq \ & ((x \oplus y) \oplus y) \cup (1 \oplus y) = (x \oplus (y \oplus y)) \cup (1 \oplus y) \text{ , by (HMV1)} \\
\subseteq \ & x \oplus ((0 \oplus 1) \cup \{0, y\}) \cup (1 \oplus y) \text{ , by } (P_9) \\
= \ & (x \oplus (0 \oplus 1)) \cup (x \oplus 0) \cup (x \oplus y) \cup (1 \oplus y) \\
\subseteq \ & ((x \oplus 0) \oplus 1) \cup \{0, x\} \cup (x \oplus y) \cup (0 \oplus 1) \cup \{0, y\}, \text{by } (P_1) \text{ and } (P_7) \\
= \ & (0 \oplus 1) \cup (x \oplus 1) \cup \{0, x\} \cup (x \oplus y) \cup \{0, y\} \\
\subseteq \ & (0 \oplus 1) \cup \{0, x, y\} \cup (x \oplus y) \quad \text{by } (P_7) \\
= \ & (0 \oplus 1) \cup \{0\} \cup (x \oplus y) \quad \text{by } (P_3).
\end{aligned}$$

Therefore, $x \vee y = (0 \oplus 1) \cup (x \oplus y) \cup \{0\}$.

(ii) By (P_1), $0 \vee x = (x \oplus 0)^* \oplus 0 = \{0, x\}^* \oplus 0 = \{1, x\} \oplus 0 = (1 \oplus 0) \cup (x \oplus 0) = (0 \oplus 1) \cup \{0, x\}$. Also, by (P_{12}), $1 \vee x = (0 \oplus x)^* \oplus x = \{1, x\} \oplus x = (1 \oplus x) \cup (x \oplus x) = (0 \oplus 1) \cup \{0, x\} = 0 \vee x$.

(iii) If $0 \oplus 1 = \{1\}$, then $0 \vee 0 = (1 \oplus 0)^* \oplus 0 = 1^* \oplus 0 = 0 \oplus 0 = \{0\}$ and $1 \vee 1 = (0 \oplus 1)^* \oplus 1 = 0 \oplus 1$. If $0 \oplus 1 \neq \{1\}$, then by (P_2), $0 \in 0 \oplus 1$ and so by Lemma 3.2(iii), $(0 \oplus 1)^* = 0 \oplus 1$. Thus

$$0 \vee 0 = (1 \oplus 0)^* \oplus 0 = (1 \oplus 0) \oplus 0 = (0 \oplus 0) \oplus 1 = 0 \oplus 1.$$

Also, by (P_8), we get

$$1 \vee 1 = (0 \oplus 1)^* \oplus 1 = (0 \oplus 1) \oplus 1 = 0 \oplus (1 \oplus 1) \subseteq 0 \oplus (0 \oplus 1) = (0 \oplus 0) \oplus 1 = 0 \oplus 1.$$

Since $0^* = 1 \in 0 \oplus 1$, we have $0 \in (0 \oplus 1)^*$ and so $0 \oplus 1 \subseteq (0 \oplus 1)^* \oplus 1 = 1 \vee 1$. Therefore, in the two cases, $1 \vee 1 = 0 \oplus 1$.

(iv) At the first, we prove that $1 \in x \vee x$ if and only if $0 \in x \oplus x$ or $x \in x \oplus x$. If $1 \in x \vee x = (x \oplus x)^* \oplus x$, then there is $z \in x \oplus x$, such that $1 \in z^* \oplus x$. By Lemma 3.2(i), $z^* = 1$ or $z^* = x$. Thus $z = 0 \in x \oplus x$ or $z = x \in x \oplus x$. Conversely, if $0 \in x \oplus x$ or $x \in x \oplus x$, then since $x \ll 1$ and $x \ll x$, we conclude that $1 \in x \oplus 1$ and $1 \in x \oplus x$. So

$$\begin{aligned} 1 &\in x \oplus 1 = x \oplus 0^* \subseteq x \oplus (x \oplus x)^* = x \vee x, \\ 1 &\in x \oplus x = x^* \oplus x \subseteq (x \oplus x)^* \oplus x = x \vee x. \end{aligned}$$

Thus $1 \in x \vee x$, for two cases.
Now, let $t \in x \vee x = (x \oplus x)^* \oplus x$. Then there is $u \in x \oplus x$ such that $t \in u^* \oplus x$. If $u = 0$, then by (P_7), $t \in 1 \oplus x \subseteq (0 \oplus 1) \cup \{0, x\}$. If $u = 1$, then by (P_1), $t \in 0 \oplus x = \{0, x\} \subseteq (0 \oplus 1) \cup \{0, x\}$. If $u \in M - \{0, 1\}$, then we get

$$\begin{aligned} t \in u \oplus x &\subseteq (x \oplus x) \oplus x \subseteq ((0 \oplus 1) \cup \{0, x\}) \oplus x && \text{by } (P_9) \\ &= ((0 \oplus 1) \oplus x) \cup (0 \oplus x) \cup (x \oplus x) \\ &= ((0 \oplus x) \oplus 1) \cup \{0, x\} \cup (x \oplus x) && \text{by (HMV1) and } (P_1) \\ &\subseteq (0 \oplus 1) \cup (x \oplus 1) \cup \{0, x\} && \text{by } (P_9) \\ &\subseteq (0 \oplus 1) \cup \{0, x\} && \text{by } (P_7). \end{aligned}$$

Hence $t \in (0 \oplus 1) \cup \{0, x\}$ in all cases. Therefore, $x \vee x \subseteq (0 \oplus 1) \cup \{0, x\}$. Conversely, let $t \in (0 \oplus 1) \setminus \{0, x, 1\}$. Then by (P_{10}), $t \in x \oplus x$. By (P_3), $t \in t \oplus x = t^* \oplus x \subseteq (x \oplus x)^* \oplus x = x \vee x$. Thus $(0 \oplus 1) \setminus \{0, x, 1\} \subseteq x \vee x$. On the other hand, $\{0, x\} = 0 \oplus x = 1^* \oplus x \subseteq (x \oplus x)^* \oplus x = x \vee x$. Hence,

$(0 \oplus 1 \setminus \{1\}) \cup \{0, x\} \subseteq x \vee x$.

(v) $0 \vee 1 = (1^* \oplus 0)^* \oplus 0 = (0 \oplus 0)^* \oplus 0 = 0^* \oplus 0 = 1 \oplus 0$. □

Lemma 3.8. *For all distinct element $x, y, z \in M - \{0, 1\}$,*

(i) if $x, y \in 0 \oplus 1$, then $0 \vee x = 1 \vee x = x \vee y = 0 \oplus 1$ and $x \vee x = 0 \oplus 1 \setminus \{1\}$ or $0 \oplus 1$,

(ii) if $x \in 0 \oplus 1$ and $y \notin 0 \oplus 1$, then $x \vee y = (0 \oplus 1) \cup \{y\}$ and $y \vee y = (0 \oplus 1) \cup \{0, y\}$,

(iii) $(x \oplus y) \vee z = x \vee (y \oplus z) = (0 \oplus 1) \cup \{0\} \cup ((x \oplus y) \oplus z)$.

Proof. (i) Let $x, y \in 0 \oplus 1$. Since, by (P_2), $0 \in 0 \oplus 1$, by Lemma 3.7, $0 \vee x = 1 \vee x = 0 \oplus 1$ and $x \vee x = 0 \oplus 1 \setminus \{1\}$ or $0 \oplus 1$. Moreover, since, by (P_{14}), $x \oplus y \subseteq 0 \oplus 1$, by Lemma 3.7, $x \vee y = (x \oplus y) \cup (0 \oplus 1) \cup \{0\} = 0 \oplus 1$.

(ii) Let $x \in 0 \oplus 1$ and $y \notin 0 \oplus 1$. Then by (P_2), $0 \in 0 \oplus 1$. Also, by Lemma 3.2(i) and assumption we get $1 \notin x \oplus y$. By (P_{14}), we imply that $z \notin x \oplus y$ for all distinct elements $x, y, z \in M - \{0, 1\}$. Hence, by (P_3), $x \oplus y = \{x, y\}$ or $x \oplus y = \{0, x, y\}$. So, by Lemma 3.7(i), $x \vee y = (x \oplus y) \cup (0 \oplus 1) \cup \{0\} = (0 \oplus 1) \cup \{y\}$.

(iii) By Lemma 3.7(i) and (HMV1) we get

$$\begin{aligned}
(x \oplus y) \vee z &= \bigcup_{t \in x \oplus y} t \vee z = \bigcup_{t \in x \oplus y} (0 \oplus 1) \cup \{0\} \cup (t \oplus z) \\
&= (0 \oplus 1) \cup \{0\} \cup ((x \oplus y) \oplus z) \\
&= (0 \oplus 1) \cup \{0\} \cup (x \oplus (y \oplus z)) \\
&= \bigcup_{u \in y \oplus z} (0 \oplus 1) \cup \{0\} \cup (x \oplus u) \\
&= \bigcup_{u \in y \oplus z} x \vee u = x \vee (y \oplus z).
\end{aligned}$$

We note that, since by (P_3), $x, y \in x \oplus y$ and by Lemma 3.7, $z \vee z \subseteq 0 \vee z = 1 \vee z \subseteq x \vee z$, we can suppose that $z \neq t \in M - \{0, 1\}$, without loss of generality (similarly, $x \neq u \in M - \{0, 1\}$). □

Lemma 3.9. *For all distinct elements $x, y, z \in M - \{0, 1\}$ we have*

(i) $0 \vee (0 \oplus 1) = 0 \oplus 1 = 1 \vee (0 \oplus 1)$;

(ii) $(0 \oplus 1) \vee x = 0 \vee x = 1 \vee x = \begin{cases} (0 \oplus 1), & \text{if } x \in 0 \oplus 1, \\ (0 \oplus 1) \cup \{0, x\}, & \text{otherwise.} \end{cases}$

(iii) $x \vee (x \vee z) = (0 \oplus 1) \cup \{0\} \cup (x \oplus (x \oplus z))$;

(iv) $0 \vee (y \oplus z) = (0 \oplus 1) \cup \{0\} \cup (y \oplus z)$.

Proof. (i) If $0 \oplus 1 = \{1\}$, then by Lemma 3.7(v), $0 \vee (0 \oplus 1) = 0 \vee 1 = 0 \oplus 1$. If $0 \oplus 1 = \{0, 1\}$, then by Lemma 3.7(iii), $0 \vee (0 \oplus 1) = (0 \vee 0) \cup (0 \vee 1) = (0 \oplus 1) \cup (0 \oplus 1) = 0 \oplus 1$. Otherwise, there exists $x \in M - \{0, 1\}$ such that $x \in 0 \oplus 1$ and so by (P_2), $0 \in 0 \oplus 1$. Thus

$$\begin{aligned} 0 \vee (0 \oplus 1) &= \bigcup_{t \in 0 \oplus 1} (0 \vee t) = (0 \vee 0) \cup (0 \vee 1) \cup \bigcup_{\acute{t} \in (0 \oplus 1) - \{0,1\}} (0 \vee \acute{t}) \\ &= (0 \oplus 1) \cup (0 \oplus 1) \cup (0 \oplus 1) \cup \{0, \acute{t}\}, \text{by Lemma 3.7(ii),(iii)} \\ &= 0 \oplus 1 \qquad \text{Since } 0, \acute{t} \in 0 \oplus 1. \end{aligned}$$

Therefore, $0 \vee (0 \oplus 1) = 0 \oplus 1$, for all cases. Similarly, we can prove $1 \vee (0 \oplus 1) = 0 \oplus 1$.

(ii) If $0 \oplus 1 = \{1\}$ or $0 \oplus 1 = \{0, 1\}$, then by the similar way of (i) and using Lemma 3.7(ii), we get $(0 \oplus 1) \vee x = (0 \oplus 1) \cup \{0, x\}$. Let there exists $s \in M - \{0, 1\}$ such that $s \in 0 \oplus 1$. Then by (P_2), $0 \in 0 \oplus 1$. Now, if $x \notin 0 \oplus 1$, then we get

$$\begin{aligned} (0 \oplus 1) \vee x &= \bigcup_{t \in 0 \oplus 1} (t \vee x) = (0 \vee x) \cup (1 \vee x) \cup \bigcup_{x \neq t' \in (0 \oplus 1) - \{0,1\}} (t' \vee x) \\ &= (0 \oplus 1) \cup \{0, x\} \cup (0 \oplus 1) \cup \{x\}, \text{by Lemmas 3.7(ii), 3.8(ii)} \\ &= (0 \oplus 1) \cup \{0, x\} = 0 \vee x = 1 \vee x. \end{aligned}$$

For the case $x \in 0 \oplus 1$, the proof is similar. Therefore, $(0 \oplus 1) \vee x = (0 \oplus 1) \cup \{0, x\}$, for all cases.
The proof of (iii) and (iv) is routine. □

Theorem 3.10. *Let M be finite. Then for all $x, y, z \in M$,*

$$(x \vee y) \vee z = x \vee (y \vee z) \qquad \textit{and} \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z).$$

Proof. Case 1: Let $x, y, z \in M$ such that $x = z$. Then by commutativity of " $\vee$ ", we get

$$x \vee (y \vee x) = (y \vee x) \vee x = (x \vee y) \vee x.$$

Case 2: Let $x, y, z \in M - \{0, 1\}$ be distinct elements and $x \notin 0 \oplus 1$ (for $x \in 0 \oplus 1$ the proof is similar). Then

$$\begin{aligned} x \vee (y \vee z) &= x \vee ((0 \oplus 1) \cup \{0\} \cup (y \oplus z) \\ &= \underbrace{(x \vee (0 \oplus 1)) \cup (x \vee 0)} \cup (x \vee (y \oplus z)), \text{by Lemma 3.7(i)} \\ &= (0 \oplus 1) \cup \{0, x\} \cup (x \vee (y \oplus z)), \text{by Lemmas3.9(ii), 3.7(ii)} \\ &= (0 \oplus 1) \cup \{0\} \cup (x \oplus y \oplus z), \text{by Lemma 3.8(iii) and } (P_3). \end{aligned}$$

By the similar way, we get $(x \vee y) \vee z = (0 \oplus 1) \cup \{0\} \cup ((x \oplus y) \oplus z)$. Therefore, $x \vee (y \vee z) = (x \vee y) \vee z$.

Case 3: Let $x, y, z \in M - \{0, 1\}$ such that $x = y$. Then
(i) if $x, z \in 0 \oplus 1$, then by Lemma 3.8(i) and Lemma 3.9(ii), $x \vee (x \vee z) = x \vee (0 \oplus 1) = 0 \oplus 1$. Also, it is routine to see that $(x \vee x) \vee z = 0 \oplus 1$. Therefore, $x \vee (x \vee z) = 0 \oplus 1 = (x \vee x) \vee z$.

(ii) If $x \in 0 \oplus 1$ and $z \notin 0 \oplus 1$, then by Lemma 3.8(ii) and Lemma 3.9(ii), $x \vee (x \vee z) = x \vee ((0 \oplus 1) \cup \{z\}) = (x \vee (0 \oplus 1)) \cup (x \vee z) = (0 \oplus 1) \cup (0 \oplus 1) \cup \{z\} = (0 \oplus 1) \cup \{z\}$. Since by Lemma 3.8(i), $x \vee x = (0 \oplus 1) \setminus \{1\}$ or $0 \oplus 1$, we get $(x \vee x) \vee z = (0 \oplus 1) \cup \{z\} = x \vee (x \vee z)$ in both cases. Thus $x \vee (x \vee z) = (x \vee x) \vee z$.

(iii) If $x, z \notin 0 \oplus 1$, then

$$\begin{aligned} & x \vee (x \vee z) \\ = \; & x \vee ((0 \oplus 1) \cup \{0\} \cup (x \oplus z)) \\ = \; & \underbrace{(x \vee (0 \oplus 1)) \cup (x \vee 0)} \cup (x \vee (x \oplus z)), \text{by Lemma 3.7(i)} \\ = \; & (0 \oplus 1) \cup \{0, x\} \cup (x \vee (x \oplus z)), \text{by Lemma 3.9(ii), 3.7(ii)} \\ = \; & (0 \oplus 1) \cup \{0, x\} \cup (0 \oplus 1) \cup \{0\} \cup (x \oplus (x \oplus z), \text{by Lemma 3.9(iii)} \\ = \; & (0 \oplus 1) \cup \{0\} \cup (x \oplus (x \oplus z) \\ = \; & (0 \oplus 1) \cup (x \oplus z) \cup \{0\}, \text{by } (P_3) \text{ and } (P_{16}). \end{aligned}$$

Also, we have,

$$\begin{aligned} & (x \vee x) \vee z \\ = \; & ((0 \oplus 1) \cup \{0, x\}) \vee z \\ = \; & \underbrace{((0 \oplus 1) \vee z) \cup (0 \vee z)} \cup (x \vee z), \text{ by Lemma 3.8(ii)} \\ = \; & (0 \oplus 1) \cup \{0, z\} \cup (0 \oplus 1) \cup \{0\} \cup (x \oplus z), \text{ by Lemmas 3.9(ii), 3.7(ii)} \\ = \; & (0 \oplus 1) \cup \{0\} \cup (x \oplus z), \text{ by } P_3. \end{aligned}$$

Thus $x \vee (x \vee z) = (x \vee x) \vee z$. For the case $x, y, z \in M - \{0, 1\}$ such that $y = z$, the proof is similar.

Case 4: Let $x \in \{0, 1\}$ and $y, z \in M - \{0, 1\}$ be distinct elements. We suppose $x = 0$ (for $x = 1$ the proof is similar)

$$\begin{aligned}
0 \vee (y \vee z) &= 0 \vee ((0 \oplus 1) \cup \{0\} \cup (y \oplus z)), \text{Lemma 3.7(i)} \\
&= (0 \vee (0 \oplus 1)) \cup (0 \vee 0) \cup (0 \vee (y \oplus z)) \\
&= (0 \oplus 1) \cup \{0\} \cup (0 \vee (y \oplus z)), \text{by Lemma 3.9(i)} \\
&= (0 \oplus 1) \cup \{0\} \cup (y \oplus z), \text{by Lemma 3.9(iv)}.
\end{aligned}$$

On the other hand,
(i) If $z \notin 0 \oplus 1$, then

$$\begin{aligned}
& (0 \vee y) \vee z \\
= \;& ((0 \oplus 1) \cup \{0, y\}) \vee z \\
= \;& \underbrace{((0 \oplus 1) \vee z) \cup (0 \vee z)} \cup (y \vee z), \text{by Lemma 3.7(ii)} \\
= \;& (0 \oplus 1) \cup \{0, z\} \cup (0 \oplus 1) \cup \{0\} \cup (y \oplus z), \text{by Lemmas 3.9(ii), 3.7(ii),(i)} \\
= \;& (0 \oplus 1) \cup \{0\} \cup (y \oplus z), \text{by Lemma 3.7(i), (ii) and } (P_3).
\end{aligned}$$

(ii) If $z \in 0 \oplus 1$, then by Lemma 3.9(ii) and Lemma 3.8(i), this is routine to see that $(0 \vee y) \vee z = (0 \oplus 1) \cup \{0\} \cup (y \oplus z)$. Hence, $(0 \vee y) \vee z = 0 \vee (y \vee z)$ for any cases. If $y \in \{0, 1\}$ and $x, z \in M - \{0, 1\}$ or $z \in \{0, 1\}$ and $x, y \in M - \{0, 1\}$, then we can prove by the similar way.

Case 5: Let $x \in \{0, 1\}$ and $y, z \in M - \{0, 1\}$ such that $y = z$. Suppose that $x = 0$ (for $x = 1$ the proof is similar).
(i) If $y \notin 0 \oplus 1$, then

$$\begin{aligned}
(0 \vee y) \vee y &= ((0 \oplus 1) \cup \{0, y\}) \vee y \\
&= \underbrace{((0 \oplus 1) \vee y) \cup (0 \vee y)} \cup (y \vee y), \text{by Lemma 3.7(ii)} \\
&= (0 \oplus 1) \cup \{0, y\} \cup (y \vee y) \qquad \text{by Lemmas 3.9(ii), 3.7(ii)} \\
&= (0 \oplus 1) \cup \{0, y\}, \text{by Lemma 3.8(ii)}.
\end{aligned}$$

Moreover, by Lemma 3.9(i) and Lemma 3.8(ii),

$$\begin{aligned}
0 \vee (y \vee y) &= 0 \vee ((0 \oplus 1) \cup \{0, y\} = (0 \vee (0 \oplus 1)) \cup (0 \vee 0) \cup (0 \vee y) \\
&= (0 \oplus 1) \cup \{0, y\} = (0 \vee y) \vee y.
\end{aligned}$$

(ii) If $y \in 0 \oplus 1$, then it is routine to see that $0 \vee (y \vee y) = 0 \oplus 1 = (0 \vee y) \vee y$. Similarly, $x = y \in M - \{0, 1\},\ z \in \{0, 1\}$ or $x = z \in M - \{0, 1\},\ y \in \{0, 1\}$

can be proved.

Case 6: Let $x, y \in \{0,1\}$ and $z \in M - \{0,1\}$. Suppose $x = y = 0$ (for $x = 1$ or $y = 1$ the proof is similar).

$$\begin{aligned}
0 \vee (0 \vee z) &= 0 \vee ((0 \oplus 1) \cup \{0, z\}) \\
&= (0 \vee (0 \oplus 1)) \cup (0 \vee 0) \cup (0 \vee z), \text{by Lemma 3.7(ii)} \\
&= (0 \oplus 1) \cup \{0\} \cup (0 \oplus 1) \cup \{0, z\} \\
&= (0 \oplus 1) \cup \{0, z\}, \text{by Lemma 3.9(i)} \\
&= 0 \vee x = (0 \vee 0) \vee z.
\end{aligned}$$

Similarly, $x, z \in \{0,1\},\ y \in M - \{0,1\}$ or $y, z \in \{0,1\},\ x \in M - \{0,1\}$ can be proved.

Case 7: Let $x, y, z \in \{0,1\}$. Suppose $x = y = 0$, $z = 1$ and $0 \oplus 1 = \{1\}$. Then by Lemas 3.7(v) and 3.9(i), we get $0 \vee (0 \vee 1) = 0 \vee (0 \oplus 1) = 0 \oplus 1 = 0 \vee 1 = (0 \vee 0) \vee 1$. For other cases the proof is similar.

Finally, by definition of $\wedge$, it can easily prove that $x \wedge (y \wedge z) = (x \wedge y) \wedge z$. □

Definition 3.2. [11] *Let L be a nonempty set endowed with hyperoperations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a hyperlattice if for any $a, b, c \in L$, the following conditions are satisfied:*
(i) $a \in a \wedge a$, $a \in a \vee a$;
(ii) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;
(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$;
(iv) $a \in a \wedge (a \vee b)$, $a \in a \vee (a \wedge b)$.

Corollary 3.11. *If M is a finite hyper MV-algebra that satisfies the (SNP), then M is a hyperlattice.*

Proof. By Theorem 3.10 and Proposition 2.4, the proof is clear. □

Corollary 3.12. *Any finite hyper MV-algebra of the orders 2 and 3, satisfies the (SNP), and so is a hyperlattice.*

Computer Check: All hyper MV-algebras of orders 4, 5 and 6, are hyperlattices.

Example 3.2. (i) Let $M = \{0, a, b, 1\}$ and hyperoperation $\oplus$ and unary operation $*$ on M are defined as follows;

$\oplus$	0	a	b	1
0	{0}	{0,a}	{b}	{b,1}
a	{0,a}	{0,a}	{b,1}	{b,1}
b	{b}	{b,1}	{b,1}	{b,1}
1	{b,1}	{b,1}	{b,1}	{b,1}

$*$	0	a	b	1
	1	b	a	0

Then by routine calculations $(M, \oplus, *, 0)$ is a hyper MV-algebra, which dose not satisfies the (SNP.) But $(M, \vee, \wedge)$ is a hyperlattice.
(ii) Let $M = [0, 1]$. We define unary operetion "$*$" and hyperoperation "$\oplus$" on M by $x^* = 1 - x$ and $x \oplus y = [0,\ min\{1, x + y\}]$. Then $(M, \oplus, *, 0)$ is a hyper MV-algebra. It is easy to see that $x^* \neq x$ for any $x \neq \frac{1}{2}$ i.e. M is not satisfied (SNP). But by routine calculation we get $x \vee y = [0, 1]$ and so $x \vee (y \vee z) = [0, 1] = (x \vee y) \vee z$ for all $x, y, z \in M$. By the similar way, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, for all $x, y, z \in M$. Hence $(M, \vee, \wedge)$ is a hyperlattice.

Note: In Corollary 3.11, the condition "finite with (SNP)" is sufficient but it is not necessary. Indeed, we have not found any finite or infinite hyper MV-algebra, which is not a hyperlattice.

Open problem: Any hyper MV-algebra is a hyperlattice.

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R. A. Borzooei

Department of Mathematics,
Shahid Beheshti University
Tehran, Iran

E-mail: borzooei@sbu.ac.ir

Akefe Radfar

Department of Mathematics,
Payame Noor University
p.o.box 19395-3697
Tehran, Iran

E-mail: Radfar@pnu.ac.ir

Sogol Niazian
Tehran Medical Sciences Branch,
Islamic Azad University
Tehran, Iran
E-mail: s.niazian@iautmu.ac.ir

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