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Convergence Analysis of a Three Step Newton-like Method for Nonlinear Equations in Banach Space under Weak Conditions

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Abstract.
In the present paper, we study the local convergence analysis of a fifth convergence order method considered by Sharma and Guha in [15] to solve equations in Banach space. Using our idea of restricted convergence domains we extend the applicability of this method. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1 Introduction

Recently Sharma and Guha, in [15] studied a three step Newton-like method defined by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - 5F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= y_n - \frac{9}{5}F'(x_n)^{-1}F(y_n) - \frac{1}{5}F'(x_n)^{-1}F(z_n), \end{aligned} \tag{1.1}$$

where $x_0 \in D$ an initial point, with convergence order five for solving systems of nonlinear equations, where $F : D \subset \mathbb{R}^i \longrightarrow \mathbb{R}^i, i$ a natural integer. This method was shown to be simple and efficient.

In this study we present the local convergence analysis of method (1.1) for approximating the solution of a nonlinear equation

$$F(x) = 0, \tag{1.2}$$

but, where $F : \Omega \subseteq B_1 \longrightarrow B_2$ is a continuously Fréchet-differentiable operator and Ω is a convex subset of the Banach space B_1. Due to the wide applications, finding solution for the equation (1.2) is an important problem in mathematics. Many authors considered higher order methods for solving (1.2) [1–16]. In [15] the existence of the Fréchet derivative of F of order up to five was used for the convergence analysis. This assumption on the higher order Fréchet derivatives of the operator F restricts the applicability of method (1.1). For example consider the following;

EXAMPLE 1.1. Let $X = C[0,1]$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 6–9, 12] defined by

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.2), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2} + x(t))dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|).$$

One can see that, higher order derivatives of F do not exist in this example.

Our goal is to weaken the assumptions in [15], so that the applicability of the method (1.1) can be extended. Notice that the same technique can be used to extend the applicability of other iterative methods that have appeared in [1–16].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the last section.

2 Local convergence

The following scalar functions and parameters are used for the convergence analysis of method (1.1). Let $w_0 : [0, +\infty) \longrightarrow (0, +\infty)$ be a continuous nondecreasing function with $w_0(0) = 0$. Define the parameter r_0 by

$$r_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let also $w : [0, r_0) \longrightarrow [0, +\infty), v : [0, r_0) \longrightarrow [0, +\infty)$ be continuous nondecreasing functions with $w(0) = 0$. Moreover define functions $g_i, h_i, i = 1, 2, 3$ on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)},$$

$$g_2(t) = \left(1 + \frac{5\int_0^1 v(\theta g_1(t)t)d\theta}{1 - w_0(t)}\right) g_1(t),$$

$$g_3(t) = \left(1 + \frac{9\int_0^1 v(\theta g_1(t)t)d\theta}{5(1 - w_0(t))} + \frac{\int_0^1 v(\theta g_2(t)t)d\theta}{5(1 - w_0(t))}\right) g_1(t)$$

and

$$h_i(t) = g_i(t) - 1.$$

We have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_0^-$. It then follows from the intermediate value theorem that function h_1 has zeros in the interval $(0, r_0)$. Denote by r_1 the smallest such zero. We also have that $h_2(0) = -1 < 0$ and $h_2(r_1) = \frac{5\int_0^1 v(\theta r_1)d\theta}{1 - w_0(r_1)}$, since $g_1(r_1) = 1$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, r_1)$. We obtain that $h_3(0) = -1 < 0$ and

$h_3(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. Denote by r_3 the smallest zero of function h_3 on the interval $(0, r_0)$. Define the radius of convergence r by

$$r = \min\{r_2, r_3\}. \tag{2.2}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq g_i(t) < 1. \tag{2.3}$$

Let $U(x, \rho), \bar{U}(x, \rho)$ stand respectively for the open and closed balls in B_1 with center $x \in B_1$ and of radius $\rho > 0$. Now, we will state and prove the main result of this section using the preceding notations.

THEOREM 2.1. Let $F : D \subset B_1 \to B_2$ be a continuously Fréchet-differentiable operator. Suppose: there exist $x^* \in D$, and a function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous, nondecreasing with $w_0(0) = 0$ such that for each $x \in D$

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(B_2, B_1), \tag{2.4}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \leq w_0(\|x - x^*\|); \tag{2.5}$$

there exist functions $w : [0, r_0) \longrightarrow [0, +\infty), v : [0, r_0) \longrightarrow [0, +\infty)$, continuous, nondecreasing with $w(0) = 0$ such that for each $x, y \in D_0 = D \cap U(x^*, r_0)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|), \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|), \tag{2.7}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.8}$$

where the radius of convergence r is given by (2.2). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.1) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.9}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.11}$$

where the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists $R \geq r$ such that

$$\int_0^1 w_0(\theta R) d\theta < 1, \tag{2.12}$$

then, the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, R)$.

Proof. We shall base our proof on mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1) and (2.5), we have in turn that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le w_0(\|x_0 - x^*\|) \le w_0(r) < 1. \tag{2.13}$$

It follows from (2.13) and the Banach Lemma on invertible operators [2, 13] that $F'(x)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.14}$$

We also have that y_0, z_0, x_1 well defined by method (1.1) for $n = 0$. Using the identity

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0), \tag{2.15}$$

(2.2), (2.3) (for $i = 1$), (2.6) and (2.14), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \left\| \int_0^1 F'(x^*)^{-1}(F'(x_0 + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta \right\| \\
&\le \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta \|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned} \tag{2.16}$$

which shows (2.9) for $n = 0$ and $y_0 \in U(x^*, r)$. We can write by (2.4) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))d\theta. \tag{2.17}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. Using (2.7) and (2.17) we get

$$\|F'(x^*)^{-1}F(x_0)\| \le \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|. \tag{2.18}$$

Similarly to (2.18) (for $x_0 = y_0$) and also using (2.16), we get that

$$\begin{aligned}
\|F'(x^*)^{-1}F(y_0)\| &\le \int_0^1 v(\theta\|y_0 - x^*\|)d\theta\|y_0 - x^*\| \\
&\le \int_0^1 v(\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|.
\end{aligned} \tag{2.19}$$

In view of the second substep of method (1.1) (for $n = 0$), (2.2), (2.3) (for $i = 2$), (2.14), (2.16), (2.18) and (2.19), we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|y_0 - x^*\| + 5\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\
&\leq \left(1 + 5\frac{\int_0^1 v(\theta\|y_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)}\right)\|y_0 - x^*\| \\
&\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.20)
\end{aligned}$$

which shows (2.10) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, by the last substep of method (1.1) for $n = 0$, (2.2), (2.3) (for $i = 3$), (2.14), (2.18) (for $x_0 = z_0$), (2.19) and (2.20), we obtain in turn that

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|y_0 - x^*\| + \frac{9}{5}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\
&\quad + \frac{1}{5}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\
&\leq \|y_0 - x^*\| + \frac{9}{5}\frac{\int_0^1 v(\theta\|y_0 - x^*\|)d\theta\|y_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&\quad + \frac{1}{5}\frac{\int_0^1 v(\theta\|z_0 - x^*\|)d\theta\|z_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.21)
\end{aligned}$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates, we arrive at estimates (2.9)–(2.11). Then, from (2.11), we have the estimate

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| < r, \qquad (2.22)$$

where $c = g_3(\|x_0 - x^*\|) \in [0, 1)$, so we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $y^* \in D_1$ with $F(y^*) = 0$. Define $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Then, using (2.5) and (2.12) we get that

$$\begin{aligned}
\|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\
&\leq \int_0^1 w_0(\theta R)d\theta < 1,
\end{aligned} \qquad (2.23)$$

so $Q^{-1} \in L(B_2, B_1)$. Then, from the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. (1) The local convergence analysis of method (1.1) was studied in [15] based on Taylor expansions and hypotheses reaching up to the fifth Fréchet derivative of F. Moreover, no computable error bounds were given nor the radius of convergence. We have addressed these problems in Theorem 2.1.

(2) Let $w_0(t) = L_0 t$, $w(t) = Lt$, $v(t) = M$ for some $L_0 > 0, L > 0$ and $M \geq 1$. In this special case, the results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

(3) The radius r_1 was shown by us to be the convergence radius of Newton's method [5, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.24}$$

under the conditions (2.4)–(2.6). It follows from the definition of r that the convergence radius r of the method (1.1) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.24). As already noted in [2] r_1 is at least as large as the convergence ball given by Rheinboldt [13]

$$r_R = \frac{2}{3L}. \tag{2.25}$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \ as \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [16].

(4) It is worth noticing that method (1.1) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [15]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

(5) Using (2.5) we see that condition (2.7) can be dropped, if we define function v by $v(t) = 1 + w_0(t)$ or $v(t) = 1 + w_0(r_0)$ for each $t \in [0, r_0]$, since

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &\leq \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| + \|I\| \\ &\leq 1 + w_0(\|x - x^*\|) \leq 1 + w_0(t) \end{aligned}$$

for $\|x - x^*\| \leq t \leq r_0$.

3 Numerical Examples

We present two examples in this section.

EXAMPLE 3.1. Let $B_1 = B_2 = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using (2.5)–(2.7), we can choose $w_0(t) = L_0 t, w(t) = e^{\frac{1}{L_0}} t, v(t) = e^{\frac{1}{L_0}}, L_0 = e - 1$.

Then, the radius of convergence r is given by

$$r_2 = 0.0836, r_3 = 0.0221 = r.$$

EXAMPLE 3.2. Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (5) for function v) $w_0(t) = w(t) = \frac{1}{8}(\frac{3}{2}\sqrt{t} + t)$ and $v(t) = 1 + w_0(r_0), r_0 \backsimeq 4.7354$. Then, the radius of convergence r is given by

$$r_2 = 0.3295, r_3 = 0.2500 = r.$$

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