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Relevant Classes of Weakly Picard Operators

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Abstract. In this paper we consider the following problems: (1) Which weakly Picard operators satisfy a retraction-displacement condition? (2) For which weakly Picard operators the fixed point problem is well posed? (3) Which weakly Picard operators have Ostrowski property?
Some applications and open problems are also presented.

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0 Introduction and preliminaries

Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. By definition, f is a weakly Picard operator (WPO) if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and its limit (which may depend on x) is a fixed point of f.

If f is a WPO, then we consider the operator $f^\infty : X \to X$, defined by, $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. We remark that the operator f^∞ is a set retraction of X on the fixed point set of f, F_f.

Each WPO generates a partition of X in the following way. Let $x^* \in F_f$ and $X_{x^*} := \{x \in X \mid \lim_{n\to\infty} f^n(x) = x^*\}$. Then, $X = \bigcup_{x^*\in F_f} X_{x^*}$ is a partition of X. Moreover, $f(X_{x^*}) \subset X_{x^*}$, and $X_{x^*} \cap F_f = \{x^*\}$.

If f is a WPO and $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

If (X,d) is a metric space, $f : X \to X$ is a WPO and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, then by definition f is a ψ-WPO iff,

(a) ψ is increasing, continuous at 0 and $\psi(0) = 0$;

(b) $d(x, f^\infty(x)) \leq \psi(d(x, f(x)))$, $\forall\ x \in X$.

We call the condition (b) a retraction-displacement condition.

Remark 0.1. The condition (b) can be presented (in terms of the partition, $X = \bigcup_{x^*\in F_f} X_{x^*}$) as follows

(b) $d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall\ x \in X_{x^*}$, $\forall\ x^* \in F_f$.

The following notion is useful in our paper.

Definition 0.1. *Let* (X,d) *be a metric space,* $f : X \to X$ *be a* WPO *and* $0 \leq l < 1$. *By definition,* f *is an* l*-quasicontraction iff*

$$d(f(x), f^\infty(x)) \leq l d(x, f^\infty(x)),\ \forall\ x \in X.$$

In the terms of the partition, $X = \bigcup_{x^*\in F_f} X_{x^*}$, the above condition takes the following form,

$$d(f(x), x^*) \leq l d(x, x^*),\ \forall\ x \in X_{x^*},\ \forall\ x^* \in F_f.$$

Remark 0.2. There are some relevant metric conditions which appear in the WPO theory. Here are some of them:

(1) $d(f^2(x), f(x)) \leq l d(x, f(x))$, $\forall\ x \in X$, with some $0 \leq l < 1$.

(2) $d(f(x), f(y)) \leq l d(x,y) + L d(y, f(x))$, $\forall\ x, y \in X$, where $0 \leq l < 1$, $L \geq 0$.

(3) $\min\{d(f(x), f(y)), d(x, f(x)), d(y, f(y))\} - \min\{d(x, f(y)), d(y, f(x))\} \leq l d(x,y)$, for all $x, y \in X$, with some $0 \leq l < 1$.

(4) There exists a function $\varphi : X \to \mathbb{R}_+$ such that, $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, $\forall\, x \in X$.

Remark 0.3. For more considerations on PO and WPO see: [38], [46], [3], [9], [13], [33], [35], [37], [39], [44], [45], [47], [48], ...

In the paper [44] we have considered the following problems:

Problem 0.1. *Which Picard operators satisfy a retraction-displacement condition ?*

Problem 0.2. *For which Picard operators the fixed point problem is well posed ?*

Problem 0.3. *Which Picard operators have the Ostrowski property ?*

In this paper we shall consider these problems in the case of weakly Picard operators. The structure of the paper is the following:

1. **Main results**
2. **Graphic contractions**
3. **Berinde operators**
4. **Caristi operators**
5. **Applications**
6. **Other research directions**

Throughout this paper the notations and terminologies in [44], [48] and [33] are used.

1 Main results

The following results are fundamental to study the problems stated in §0.

Theorem 1.1. *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose that:*

(a) *f is a WPO*

(b) *There exists $c > 1$ such that:*

$$W_{d,f}(x) := \sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq cd(x, f(x)),\ \forall\, x \in X.$$

Then we have:

(1) *$d(x, f^\infty(x)) \leq cd(x, f(x))$, $\forall\ x \in X$, i.e., f is a c-WPO.*

(2) *$x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0 \Rightarrow y_n \to x^*$, i.e., the fixed point problem for f is well posed.*

If in addition, $1 < c < \frac{3}{2}$, then we have:

(3) *$d(f(x), f^\infty(x)) \leq \frac{c-1}{2-c} d(x, f^\infty(x))$, $\forall\ x \in X$, i.e., f is a quasicontraction.*

(4) *$x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0 \Rightarrow y_n \to x^*$, i.e., f has the Ostrowski property.*

(5) *Let $Y \subset X$ be a bounded subset with, $f(Y) \subset Y$ and $F_f \subset Y$. Then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

Proof. (1). $d(x, f^\infty(x)) \leq \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) + d(f^{n+1}(x), f^\infty(x))$, $\forall\ x \in X$, $\forall\ n \in \mathbb{N}$. For $n \to \infty$ we have that

$$d(x, f^\infty(x)) \leq W_{d,f}(x),\ \forall\ x \in X.$$

From the condition (b) we have (1).

(2). It follows from (1).

(3).

$$\begin{aligned} d(f(x), f^\infty(x)) &\leq W_{d,f}(f(x)) = W_{d,f}(x) - d(x, f(x)) \leq (c-1)d(x, f(x)) \leq \\ &\leq (c-1)d(x, f^\infty(x)) + (c-1)d(f(x), f^\infty(x)). \end{aligned}$$

So, $d(f(x), f^\infty(x)) \leq \frac{c-1}{2-c} d(x, f^\infty(x))$, $\forall\ x \in X$.

(4).

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \leq \\ &\leq d(y_{n+1}, f(y_n)) + \frac{c-1}{2-c} d(y_n, x^*) \leq \\ &\leq d(y_{n+1}, f(y_n)) + \frac{c-1}{2-c} d(y_n, f(y_{n-1}) + \\ &\quad + \ldots + \left(\frac{c-1}{2-c}\right)^n d(y_0, f(y_0)) + \left(\frac{c-1}{2-c}\right)^n d(y_0, x^*). \end{aligned}$$

Now, the proof follows from a Cauchy (or a Toeplitz) lemma.

(5). Since f is a WPO we have, $X = \bigcup\limits_{x^* \in F_f} X_{x^*}$. Let $y \in X_{x^*}$. From (3) we have

$$d(f(y), x^*) \leq \frac{c-1}{2-c} d(y, x^*).$$

From this inequality we have that

$$\delta(f^n(Y \cap X_{x^*}), \{x^*\}) \leq \left(\frac{c-1}{2-c}\right)^n \delta(Y \cap X_{x^*}, \{x^*\}) \to 0 \text{ as } n \to \infty.$$

So, $\bigcap\limits_{n \in \mathbb{N}} f^n(Y \cap X_{x^*}) = \{x^*\}$, and $\bigcap\limits_{n \in \mathbb{N}} f^n(Y) = F_f$. □

Theorem 1.2. *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

- (a) $f : (X, \rho) \to (X, \rho)$ *is* WPO.
- (b) *There exists $c > 1$ such that*

$$\sum_{n \in \mathbb{N}} \rho(f^n(x), f^{n+1}(x)) \leq c\rho(x, f(x)), \ \forall \ x \in X.$$

- (c) *There exists $c_1, c_2 > 0$ such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y), \ \forall \ x, y \in X.$$

Then we have:

- (1) $d(x, f^\infty(x)) \leq \frac{c_2}{c_1} c d(x, f(x))$, $\forall \ x \in X$.
- (2) *The fixed point problem for f is well posed in (X, d).*

If in addition $1 < c < \frac{3}{2}$, then we have:

- (4) *f has the Ostrowski property in (X, d).*
- (5) *Let $Y \subset X$ be a bounded subset in (X, d), with $f(Y) \subset Y$ and $F_f \subset Y$. Then,*

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = F_f.$$

Proof. From (c) we have that $f : (X, d) \to (X, d)$ is a WPO with (1).

(2), (4) and (5) follow from the invariance of these properties with respect to a strongly equivalent metric (see [33]). □

2 Graphic contractions

One relevant class of WPO is that of orbitally continuous graphic contractions (see [24], [38], [46], [3], [16], [19], [37], [39], ...). Let (X, d) be a metric space, $f : X \to X$ be an operator and $0 \leq l < 1$. Then by definition f is a graphic l-contraction iff,

$$d(f^2(x), f(x)) \leq l d(x, f(x)),\ \forall\ x \in X.$$

For this class of operators we have,

Theorem 2.1 (Saturated principle of graphic contraction)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic l-contraction. Then we have:*

(*i*) *$\{f^n(x)\}_{n\in\mathbb{N}}$ converges, $\forall\ x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, $\forall\ x \in X$.*

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, $\forall\ x \in X$, then,

(*ii*) *$F_f = F_{f^n} \neq \emptyset$, $\forall\ n \in \mathbb{N}$.*

(*iii*) *f is a WPO.*

(*iv*) *$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.*

(*v*) *The fixed point problem for f is well posed.*

(*vi*) *If $l < \frac{1}{3}$, then*

$$d(f(x), f^\infty(x)) \leq \frac{l}{1-2l} d(x, f^\infty(x)),\ \forall\ x \in X.$$

(*vii*) *If $l < \frac{1}{3}$, then the operator f has the Ostrowski property.*

(*viii*) *Let $l < \frac{1}{3}$. If $Y \in P_b(X)$, $f(Y) \subset Y$ and $F_f \subset Y$, then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

Proof. (*i*). $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \left(\sum_{n\in\mathbb{N}} l^n\right) d(x, f(x)) = \frac{1}{1-l} d(x, f(x)).$

(*ii*)-(*iii*). This is the graphic contraction principle.

(*iv*)-(*viii*). Follow from Theorem 1.1. $\square$

Remark 2.1. In some variants of graphic contraction principle the operator f is continuous, or with closed graph, or orbitally continuous.

Remark 2.2. In Turinici [51], an operator $f : (X, d) \to (X, d)$ is called Picard operator if the sequence $\{f^n(x)\}$ is convergent, for all $x \in X$.

Remark 2.3. In Osilike [25], an operator $f : (X, d) \to (X, d)$ which satisfies the condition,

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x))) < +\infty, \ \forall\ x \in X$$

is called a good operator. In our paper we call such an operator, Weierstrass operator and we use the notation,

$$W_{d,f}(x) := \sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)).$$

From Theorem 2.1 and Theorem 1.1, we have:

Theorem 2.2 (Saturated principle of quasicontraction)**.** *Let (X, d) be a metric space and $f : X \to X$ be an operator such that:*

(1) *f is WPO.*

(2) *f is an l-quasicontraction.*

Then we have:

(*iv*) *$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.*

(*v*) *The fixed point problem for f is well posed.*

(*vii*) *If $l < \frac{1}{3}$, then the operator f has the Ostrowski property.*

(*viii*) *Let $l < \frac{1}{3}$. If $Y \in P_b(X)$, $f(Y) \subset Y$ and $F_f \subset Y$, then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

From the Theorem 1.2 and Theorem 2.1, we have:

Theorem 2.3. *Let X be a nonempty set, d and ρ be two metrics on X. We suppose that:*

(1) *f is a graphic l-contraction with respect to the metric ρ.*

(2) *(X,ρ) is a complete metric space.*

(3) *There exists $c_1, c_2 > 0$ such that:*

$$c_1 d(x,y) \le \rho(x,y) \le c_2 d(x,y),\ \forall\ x,y \in X.$$

Then we have:

(*i*) *f is a Weierstrass operator in (X,d).*

If in addition, in the metric space (X,ρ) we have

$$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)),\ \forall\ x \in X,$$

then,

(*ii*) *$F_f = F_{f^n} \neq \emptyset$, $\forall\ n \in \mathbb{N}^*$.*

(*iii*) *$f^n(x) \xrightarrow{d} f^\infty(x) \in F_f$.*

(*iv*) *$d(x, f^\infty(x)) \le \frac{c_2}{c_1(1-l)} d(x, f(x))$, $\forall\ x \in X$.*

(*v*) *In (X,d), the fixed point problem for f is well posed.*

(*vii*) *If $l < \frac{1}{3}$, then f has the Ostrowski property in (X,d).*

(*viii*) *Let $l < \frac{1}{3}$. If $Y \in P_b(X,d)$, $f(Y) \subset Y$ and $F_f \subset Y$, then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

3 Berinde operators

In [2], V. Berinde considered the following class of operators.

Let (X,d) be a metric space, $f : X \to X$ be an operator, $0 \le l < 1$ and $L \ge 0$. The operator f is called (l,L)-almost contraction iff,

$$d(f(x), f(y)) \le l d(x,y) + L d(y, f(x)),\ \forall\ x,y \in X.$$

In this paper we call such an operator, (l,L)-Berinde operator. For more considerations of this class of operators see also: [7], [3], ...

In our paper we need the following properties of a Berinde operator.

Lemma 3.1. *If f is an (l,L)-Berinde operator then f is a graphic l-contraction. So, the Picard iterations are convergent and f is a Weierstrass operator.*

Lemma 3.2. *For a Berinde operator f we have that:*

$$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)).$$

From Theorem 2.1 and the above remarks, we have the following results:

Theorem 3.1 (Saturated theorem of Berinde)**.** *Let (X,d) be a complete metric space and $f : X \to X$ be an (l,L)-Berinde operator. Then we have:*

(*i*) *f is a Weierstrass operator.*

(*ii*) *$F_f = F_{f^n} \neq \emptyset$, $\forall\ n \in \mathbb{N}$.*

(*iii*) *$f^n(x) \to f^\infty(x) \in F_f$, $\forall\ x \in X$, i.e., f is a WPO.*

(*iv*) *f is a $\frac{1}{1-l}$-WPO.*

(*v*) *The fixed point problem for f is well posed.*

(*vi*) *If $l < \frac{1}{3}$, then f is an $\frac{l}{1-2l}$-quasicontraction.*

(*vii*) *If $l < \frac{1}{3}$, then f has the Ostrowski property.*

(*viii*) *Let $l < \frac{1}{3}$. If $Y \in P_b(X)$, $f(Y) \subset Y$ and $F_f \subset Y$, then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

Theorem 3.2 (Saturated theorem of Berinde, with respect to a strongly equivalent metric)**.** *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(*a*) *(X,ρ) is a complete metric space.*

(*b*) *There exist $c_1, c_2 > 0$ such that,*

$$c_1 d(x,y) \leq \rho(x,y) \leq c_2 d(x,y),\ \forall\ x,y \in X.$$

(*c*) *f is an (l,L)-Berinde operator with respect to the metric ρ.*

Then we have the conclusions (i)-(v) and (vii)-(viii) in Theorem 3.1, with respect to the metric d and with *(iv)* f *is a* $\frac{c_2}{c_1(1-l)}$*-WPO in* (X,d).

Proof. From Theorem 2.1 we have the conclusions (i)-$(viii)$ in (X,ρ). From the results given in Petruşel-Rus-Şerban [33], we have the proof. □

Remark 3.1. From the above considerations the following questions rise:

Problem 3.1. *Which metric conditions imply the graphic contraction condition ?*

Problem 3.2. *Which metric conditions which imply the convergence of Picard iteration imply the condition*

$$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)) \ ?$$

Example 3.1. In [13], L.B. Ćirić gives the following result (see also [27]).

Theorem 3.3. *Let* (X,d) *be a metric space and* $f : X \to X$ *be an orbitally continuous operator. We suppose that:*

(1) (X,d) *is* f*-orbitally complete, i.e., every Cauchy sequence of the form* $\{f^{n_i}(x)\}$, $x \in X$, *converges in* X.

(2) *There exists* $0 < l < 1$ *such that:*
$\min\{d(f(x),f(y)), d(x,f(x)), d(y,f(y))\} - \min\{d(x,f(y)), d(y,f(x))\} \leq ld(x,y)$, *for all* $x,y \in X$.

Then, $\{f^n(x)\}$ *converges to a fixed point of* f.

Now, the proof of Ćirić theorem reads as follows.

Ćirić metric condition implies that f is a graphic l-contraction. This implies the convergence of the Picard iterations. The orbitally continuity of f implies that the limits are fixed points of f. Moreover we have also the conclusions (iv)-$(viii)$ in Theorem 2.1.

4 Caristi operators

An operator $f : (X,d) \to (X,d)$ is a Caristi operator if there exists a functional $\varphi : X \to \mathbb{R}_+$ such that,

$$d(x,f(x)) \leq \varphi(x) - \varphi(f(x)),\ \forall\, x \in X.$$

The following result is well known (see for example, [10]).

Lemma 4.1. *An operator $f : (X, d) \to (X, d)$ is a Caristi operator iff f is a Weierstrass operator, i.e.,*

$$W_{d,f}(x) := \sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty, \ \forall\ x \in X.$$

If f is a φ-Caristi operator, then,

$$W_{d,f}(x) \leq \varphi(x), \ \forall\ x \in X.$$

From Lemma 4.1 it follows that the graphic contractions, Berinde operators and Ćirić operators are Caristi operators.

For the class of Caristi operators we have:

Theorem 4.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a φ-Caristi operator. Then we have that:*

(i) *$\{f^n(x)\}$ is convergent for all $x \in X$ and f is a Weierstrass operator with, $W_{d,f}(x) \leq \varphi(x)$.*

If in addition, $\lim\limits_{n\to\infty} f(f^n(x)) = f(\lim\limits_{n\to\infty} f^n(x))$, $\forall\ x \in X$, then:

(ii) *$F_f = F_{f^n} \neq \emptyset$, $\forall\ n \in \mathbb{N}^*$.*

(iii) *$f^n(x) \to f^\infty(x) \in F_f$, $\forall\ x \in X$.*

(iv) *If $\varphi(x) \leq cd(x, f(x))$, $\forall\ x \in X$, for some $c > 1$, then f is ψ-WPO, with $\psi(t) = ct$, $t \geq 0$.*

(v) *If $\varphi(x) \leq cd(x, f(x))$, $\forall\ x \in X$, for some $c > 1$, then, the fixed point problem for f is well posed.*

(vi) *If $\varphi(x) \leq cd(x, f(x))$, $\forall\ x \in X$, for some $c \in]1, \frac{3}{2}[$ then f is a $\frac{c-1}{2-c}$-quasicontraction.*

(vii) *If φ is as in (vi), then the operator f has the Ostrowski property.*

(viii) *Let φ be as in (vi). If $Y \subset X$ is bounded, $f(Y) \subset Y$ and $F_f \subset Y$, then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = F_f.$$

Proof. Follows from Theorem 1.1. □

Remark 4.1. Let X be a nonempty set, d and ρ be two strongly equivalent metrics and $f : X \to X$ be an operator. Then the following statements are equivalent:

(1) f is a Caristi operator in (X, d).

(2) f is a Caristi operator in (X, ρ).

5 Applications

5.1 Data dependence of fixed points under operator perturbation

Let (X, d) be a complete metric space, $f, g : X \to X$ be two operators. We suppose that $F_f \neq \emptyset$, $F_g \neq \emptyset$ and there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, $\forall\ x \in X$. The problem is to estimate, $H_d(F_f, F_g)$, where H_d is the functional of Pompeiu-Hausdorff. In terms of WPO theory a result for this problem is the following (see [37], [38], [46], [48], [3], ...).

Theorem 5.1. *Let (X, d) be a metric space and $f, g : X \to X$ be two operators. We suppose that:*

(a) *f and g are φ-WPO.*

(b) *There exists $\eta > 0$ such that,*

$$d(f(x), g(x)) \leq \eta,\ \forall\ x \in X.$$

Then, $H_d(F_f, F_g) \leq \psi(\eta)$.

From this general result we have some results on data dependence in the case of graphic contractions, Berinde operators, Ćirić operators and Caristi operators, for example.

5.2 Ulam stability of fixed point equations

Let $f : (X, d) \to (X, d)$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{5.1}$$

is Ulam-Hyers stable if there exists a constant $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution $y^* \in X$ of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{5.2}$$

there exists a solution x^* of the equation (5.1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

For this notion of stability we have:

Theorem 5.2. *Let (X, d) be a metric space. If $f : X \to X$ is a c-WPO, then the equation* (5.1) *is Ulam-Hyers stable.*

For this result and for other Ulam stabilities such as: generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, see [43], [33], [48], ...

5.3 Abstract Gronwall lemmas

One relevant application of WPO is in the abstract Gronwall lemma theory. For example we have:

Lemma 5.1. *Let $(X, \leq, \to)$ be an ordered L-space and $f : X \to X$ be an operator. We suppose that:*

(a) *f is a WPO.*

(b) *f is increasing.*

Then we have that:

(1) $x \leq f(x) \Rightarrow x \leq f^\infty(x)$.

(2) $x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

In terms of the partition of X, $X = \bigcup\limits_{x^* \in F_f} X_{x^*}$, generated by f, the conclusions (1) *and* (2) *take the following form:*

(1) $x \in X_{x^*}$, $x \leq f(x) \Rightarrow x \leq x^*$, $x^* \in F_f$;

(2) $x \in X_{x^*}$, $x \geq f(x) \Rightarrow x \geq x^*$, $x^* \in F_f$.

For abstract Gronwall lemmas and for concrete Gronwall lemmas in terms of WPO see: [40], [38], [46], [20], [22], [48], ...

5.4 Differential, integral, functional differential and functional integral equations

As basic references in this direction we mention the following: [41], [54], [1], [14], [15], [17], [20], [22], [23], [26], [37], [38], [45], [49], [52], [53], ...

6 Other research directions

6.1. What does it mean Saturated principle of fiber WPO ?

References: [49], [50], [46], [48], ...

6.2. What does it mean nonself WPO ?

Take as examples: nonself Berinde operators and nonself Caristi operators.

References: [6], [9], [11], [16], [24], ...

6.3. To extend the results of our paper to the case of multivalued operators.

References: [30], [31], [16], [19], [34], ...

6.4. To study similar problems in the case of generalized metric spaces: $d(x,y) \in \mathbb{R}^m_+$; $d(x,y) \in s(\mathbb{R}_+)$; $d(x,y) \in \mathbb{K}$, where $\mathbb{K}$ is a cone in an ordered Banach space, ...

References: [4], [15], [18], [35], [36], [46], ...

6.5. To study similar problems with respect to a convergent iterative algorithm.

References: [3], [5], [8], [12], [21], [32], [42], ...

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