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# Characterization of Dini Lipschitz Functions in Terms of Their Helgason Transform

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**Abstract.** In this paper, using a generalized translation operator, we obtain an analog of Younis Theorem 5.2 in [6] for the Helgason Fourier transform of a set of functions satisfying the Dini Lipschitz condition in the space $L^2$ for functions on noncompact rank one Riemannian symmetric spaces.

**AMS Subject Classification (2000).** 42B37
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## 1 Introduction

Younis Theorem 5.2 [6] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have:

**Theorem 1.1.** *([6]) Let $f \in L^2(\mathbb{R})$. Then the following are equivalents*
$(i)$ $\quad \|f(x+t) - f(x)\| = O\left(\frac{t^\delta}{(\log \frac{1}{t})^\gamma}\right),$ *as* $\quad t \to 0, 0 < \delta < 1, \gamma \geq 0,$
$(ii)$ $\quad \int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right),$ *as* $\quad r \to \infty,$
*where $\widehat{f}$ stands for the Fourier transform of $f$.*

In this paper, for rank one symmetric spaces, we prove the generalization of Theorem 1.1 for the Helgason Fourier transform of a class of functions satisfying the Dini Lipschitz condition in the space $L^2$. For this purpose, we use the generalized translation operator. We point out that similar results have been established in the context of non compact rank one Riemannian symetric spaces [9].

## 2 Helgason Fourier Transformation on Symmetric Spaces

Here we collect the necessary facts about the Fourier transformation on symmetric spaces and the spherical Fourier transformation (see [1,2]). For the required properties of semisimple Lie groups and symmetric spaces, we refer the reader, e.g., to [3,4]. An arbitrary Riemannian symmetric space $X$ of noncompact type can be represented as the factor space $G/K$, where $G$ is a connected noncompact semisimple Lie group with finite center, and $K$ is a maximal Compact subgroup of $G$. ON $X = G/K$ the group $G$ acts transitively by left shifts, and $K$ coincides with the stabilizer of the point $o = eK$ (e is the unity of G). Let $G = NAK$ be an Iwasawa decomposition for $G$, and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of the groups $G$, $K$, $A$, $N$, respectively. We denote by $M$ we mean the centralizer of the subgroup $A$ in $K$ and put $B = K/M$. Let $dx$ be $a$ G-invariant measure on $X$; the symbols $db$ and $dk$ will denote the normalized $K$-invariant measures on $B$ and $K$, respectively. We denote by $\mathfrak{a}^*$ the real space dual to $\mathfrak{a}$, and by $W$ the finite Weyl group acting on $\mathfrak{a}^*$. Let $\sum$ be the set of restricted roots ($\sum \subset \mathfrak{a}^*$), Let $\sum^+$ be the set of restricted positive roots, and let

$$\mathfrak{a}^+ = \left\{ h \in \mathfrak{a} : \alpha(h) > 0, \alpha \in \sum^+ \right\},$$

be the positive Weyl chamber. If $\rho$ is the half-sum of the positive roots (with multiplicity), then $\rho \in \mathfrak{a}^*$. Let $\langle , \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$. This form is positive definite on $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let $H_\lambda$ denote a vector in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu\rangle := \langle H_\lambda, H_\mu\rangle$. The correspondence $\lambda \mapsto H_\lambda$ enables us to identify $\mathfrak{a}^*$ and $\mathfrak{a}$. Via this identification, the action of the Weyl group $W$ can be transferred to $\mathfrak{a}$. Let

$$\mathfrak{a}^*_+ = \left\{ \lambda \in \mathfrak{a}^* : H_\lambda \in \mathfrak{a}^+ \right\}.$$

If $X$ is a symmetric space of rank 1, then $\dim \mathfrak{a}^* = 1$, and the set $\sum^+$ consists of the roots $\alpha$ and $2\alpha$ with some multiplicities $a$ and $b$ depending on

$X$ (see [1 ]). In this case we identify the set $\mathfrak{a}^*$ with $\mathbb{R}$ via the correspondence $\lambda \leftrightarrow \lambda\alpha$, $\lambda \in \mathbb{R}$. Upon this identification positive numbers correspond to the set $\mathfrak{a}^*_+$. The numbers $m_\alpha$ and $m_{2\alpha}$ are frequent in various formulas for rank 1 symmetric spaces. For example, the area of a sphere of radius $t$ on $X$ is equal to

$$S(t) = c(\sinh t)^{m_\alpha}(\sinh 2t)^{m_{2\alpha}},$$

where $c$ is some constant; the dimension of $X$ is equal to

$$\dim X = m_\alpha + m_{2\alpha} + 1.$$

We return to the case in which $X = G/K$ is an arbitrary symmetric space. Given $g \in G$, denote by $A(g) \in \mathfrak{a}$ the unique element satisfying

$$g = n. \exp A(g).u,$$

where $u \in K$ and $n \in N$. For $x = gK \in X$ and $b = kM \in B = K/M$, we put

$$A(x, b) := A(k^{-1}g).$$

We denote by $\mathcal{D}(X)$ and $\mathcal{D}(G)$ the sets of infinitely differentiable compactly-supported functions on $X$ and $G$. Let $dg$ be the element of the Haar measure on $G$. We assume that the Haar measure on $G$ is normed so that

$$\int_X f(x)dx = \int_G f(go)dg, \quad f \in \mathcal{D}(X).$$

For a function $f(x) \in \mathcal{D}(X)$, the Helgason Fourier transform on $X$ was introduced by S. Helgason (see [2] or [5]) and is defined by the formula

$$\widehat{f}(\lambda, b) := \int_X f(x)e^{(i\lambda+\rho)(A(x,b))}dx, \lambda \in \mathfrak{a}^*, b \in B = K/M.$$

We can norm the measure on $X$ so that the inverse Fourier transform on $X$ would have the form

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} \widehat{f}(\lambda, b)e^{(i\lambda+\rho)(A(x,b))}|c(\lambda)|^{-2}d\lambda db,$$

where $|W|$ is the order of the Weyl group, $d\lambda$ is the element of the Euclidean measure on $\mathfrak{a}^*$, and $c(\lambda)$ is the Harish-Chandra function. Henceforth, for brevity, we use the notation

$$d\mu(\lambda) := |c(\lambda)|^{-2}d\lambda.$$

Also, the Plancherel formula is valid:

$$\|f\|_2^2 := \int_X |f(x)|^2 dx = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} |\widehat{f}(\lambda,b)|^2 d\mu(\lambda)db = \int_{\mathfrak{a}^*_+\times B} |\widehat{f}(\lambda,b)|^2 d\mu(\lambda)db.$$

By continuity, the mapping $f(x) \mapsto \widehat{f}(\lambda,b)$ extends from $\mathcal{D}(X)$ to an isomorphism of the Hilbert space $L^2(X) = L^2(X,dx)$ onto the Hilbert space $L^2(\mathfrak{a}^*_+ \times B, d\mu(\lambda)db)$.
Introduce the translation operator on $X$. Let $n = \dim X$. Denote by $d(x,y)$ the distance between points $x, y \in X$ and let

$$\sigma(x;t) = \{y \in X : d(x,y) = t\},$$

be the sphere of radius $t > 0$ on $X$ centered at $x$. Let $d\sigma_x(y)$ be the $(n-1)-$dimensional area element of the sphere $\sigma(x;t)$ and let $|\sigma(t)|$ be the area of the whole sphere $\sigma(x;t)$ (it is independent of the point $x$). We denote by $C_0(X)$ the set of all continuous compactly-supported functions on $X$. Given $f(x) \in C_0(X)$, define the generalized translation operator $S^h$ by the formula

$$(S^t f)(x) = \frac{1}{|\sigma(t)|}\int_{\sigma(x;t)} f(y)d\sigma_x(y), t > 0;$$

i.e., $(S^t f)(x)$ is the average of $f$ over $\sigma(x;t)$.

**Lemma 2.1.** *([8]) The following inequality is valid for every function $f \in L^2(X)$ and every $t \in \mathbb{R}_+ = [0;+\infty)$:*

$$\|S^t f\|_2 \leq \|f\|_2.$$

The polar decomposition of $G$ takes the form $G = KA^+K$, where $A^+ = \{a_t = \exp(A(g)) : t \geq 0\}$. Following standard practice, functions $f$ on $X$ are identified with right $K$-invariant functions on $G$ and write $f(x) = f(g)$, where $x = gK$. In terms of this decomposition, the invariant measure $dx$ on $X$ has the form

$$dx = \Delta(t)dtdk,$$

where $\Delta(t) = \Delta_{(\alpha,\beta)}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}, \alpha = (m_\alpha + m_{2\alpha} - 1)/2$ and $\beta = (m_{2\alpha} - 1)/2$, and $dk$ is normalized Haar measure on $K$. The Laplacian on $X$ is denoted $\Lambda$ and its radial part is given by

$$\Lambda_r = \frac{d^2}{dt^2} + \frac{\Delta'(t)}{\Delta(t)}\frac{d}{dt}.$$

The spherical function on $X$ is the unique radial solution to the equation

$$\Lambda u = -(\lambda^2 + \rho^2)u$$

which is one at the origin of $X$.
An important role in harmonic analysis on symmetric spaces is played by spherical functions (see [1]). The Harish-Chandra formula for the spherical function is

$$\varphi_\lambda(x) = \int_B e^{(i\lambda+\rho)A(x,b)}db,$$

where $db$ is normalized measure on $B$. If we write $x = ka_tK$, then it is well known that $\varphi_\lambda(x) = \varphi_\lambda^{(\alpha,\beta)}(t)$, where $\varphi_\lambda^{(\alpha,\beta)}(t)$ is Jacobi function of the first kind (see [10]).

**Lemma 2.2.** *([8]) If $f \in L^2(X)$, then*

$$\widehat{S^t f}(\lambda, b) = \varphi_\lambda(t)\widehat{f}(\lambda, b), \quad \lambda, t \in \mathbb{R}_+ = [0; +\infty).$$

**Lemma 2.3.** *([7]) The following inequalities are valid for a spherical function $\varphi_\lambda(t)$ $(\lambda, t \in \mathbb{R}_+)$*
*(i)* $|\varphi_\lambda(t)| \leq 1,$
*(ii)* $1 - \varphi_\lambda(t) \leq t^2(\lambda^2 + \rho^2),$
*(iii) there is a constant $c > 0$ such that*

$$1 - \varphi_\lambda(t) \geq c,$$

*for $\lambda t \geq 1$.*

**Lemma 2.4.** *([8]) Suppose that $f(x)$ and $\Lambda f(x)$ belong to $L^2(X)$. Then*

$$\widehat{\Lambda f}(\lambda, b) = -(\lambda^2 + \rho^2)\widehat{f}(\lambda, b).$$

For $\alpha > \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind $j_\alpha$ defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad x \in \mathbb{R}.$$

Moreover, we see that

$$\lim_{x\to 0} \frac{j_\alpha(x) - 1}{x^2} \neq 0. \tag{2.1}$$

**Lemma 2.5.** *([9, Lemma 9]) Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$ and let $t_0 > 0$. Then for $|\nu| \leq \rho$, there exists a positive constant $c_0$ such that*

$$|1 - \varphi_{\lambda+i\nu}^{(\alpha,\beta)}(t)| \geq c_0|1 - j_\alpha(\lambda t)|,$$

*for all $0 \leq t \leq t_0$.*

For $f \in L^2(X)$, we define the finite differences of first and higher order as follows:

$$\begin{aligned}\Delta_t^1 f &= \Delta_t f = (I - S^t)f, \\ \Delta_t^k f &= \Delta_t(\Delta_t^{k-1} f) = (I - S^t)^k f, \quad k = 2, 3, ...,\end{aligned}$$

where $I$ is the unit operator in the space $L^2(X)$.
We denote by $W^{2,k}$, $k \in \mathbb{N}^*$, the Sobolev space constructed by the operator $\Lambda$, i.e.,

$$W^{2,k} = \{f \in L^2(X); \Lambda^j f \in L^2(X), j = 0, 1, 2, ..., k\},$$

where $\Lambda^0 f = f$, $\Lambda^1 f = \Lambda f$, $\Lambda^m f = \Lambda(\Lambda^{m-1} f)$, $m = 2, 3, ....$

## 3 Dini Lipschitz Condition

**Definition 3.1.** *Let $f \in W^{2,k}$, and define*

$$\|\Delta_t^k \Lambda^m f\|_2 \leq C \frac{t^\eta}{(\log \frac{1}{t})^\gamma}, \quad \eta > 0, \gamma \geq 0,$$

*i.e.,*

$$\|\Delta_t^k \Lambda^m f\|_2 = O\left(\frac{t^\eta}{(\log \frac{1}{t})^\gamma}\right),$$

*for all $m = 0, 1, ..., k$, and for all sufficiently small $t$, $C$ being a positive constant. Then we say that $f$ satisfies a Dini Lipschitz of order $\eta$, or $f$ belongs to $Lip(\eta, \gamma, 2)$.*

**Definition 3.2.** *If however*

$$\frac{\|\Delta_t^k \Lambda^m f\|_2}{\frac{t^\eta}{(\log \frac{1}{t})^\gamma}} \to 0, \quad as \quad t \to 0,$$

*i.e.,*

$$\|\Delta_t^k \Lambda^m f\|_2 = O\left(\frac{t^\eta}{(\log \frac{1}{t})^\gamma}\right),$$

*then $f$ is said to be belong to the little Dini Lipschitz class $lip(\eta, \gamma, 2)$.*

**Remark 3.1.** It follows immediately from these definitions that

$$lip(\eta, \gamma, 2) \subset Lip(\eta, \gamma, 2).$$

**Theorem 3.1.** *Let $\eta > 1$. If $f \in Lip(\eta, \gamma, 2)$, then $f \in lip(1, \gamma, 2)$.*

*Proof.* For $x \in X$, $t$ small and $f \in Lip(\eta, \gamma, 2)$ we have

$$\|\Delta_t^k \Lambda^m f\|_2 \leq C \frac{t^\eta}{(\log \frac{1}{t})^\gamma}.$$

Then

$$(\log \frac{1}{t})^\gamma \|\Delta_t^k \Lambda^m f\|_2 \leq C t^\eta.$$

Therefore

$$\frac{(\log \frac{1}{t})^\gamma}{t} \|\Delta_t^k \Lambda^m f\|_2 \leq C t^{\eta - 1},$$

which tends to zero with $t \to 0$. Thus

$$\frac{(\log \frac{1}{t})^\gamma}{t} \|\Delta_t^k \Lambda^m f\|_2 \to 0, \quad t \to 0.$$

Then $f \in lip(1, \gamma, 2)$. □

**Theorem 3.2.** *If $\eta < \nu$, then $Lip(\eta, 0, 2) \supset Lip(\nu, 0, 2)$ and $lip(\eta, 0, 2) \supset lip(\nu, 0, 2)$.*

*Proof.* We have $0 \leq t \leq 1$ and $\eta < \nu$, then $t^\nu \leq t^\eta$.
Then the proof of theorem is immediate. □

# 4 New Results On Dini Lipschitz Class

**Lemma 4.1.** *For $f \in W^{2,k}$, then*

$$\|\Delta_t^k \Lambda^m f\|_2^2 = \int_0^{+\infty} \int_B (\lambda^2 + \rho^2)^{2m} |1 - \varphi_\lambda(t)|^{2k} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db,$$

*where $m = 0, 1, ..., k$.*

*Proof.* Using Lemma 2.2 we get

$$\begin{aligned} \widehat{\Delta_t^1 f}(\lambda, b)) &= \widehat{f}(\lambda, b) - \widehat{S^t f}(\lambda, b) \\ &= (1 - \varphi_\lambda(t)) \widehat{f}(\lambda, b), \end{aligned}$$

and

$$\widehat{\Delta_t^k f}(\lambda, b) = (1 - \varphi_\lambda(t))^k \widehat{f}(\lambda, b). \tag{4.1}$$

Furthermore, we obtain by the Lemma 2.4

$$\widehat{\Lambda^m f}(\lambda, b) = (-1)^m(\lambda^2+\rho^2)^m \widehat{f}(\lambda, b). \tag{4.2}$$

Using the formulas (4.1) and (4.2) we get

$$\widehat{\Delta_t^k \Lambda^m f}(\lambda, b) = (-1)^m(\lambda^2+\rho^2)^m(1-\varphi_\lambda(t))^k \widehat{f}(\lambda, b).$$

Now by Plancherel formula, we have the result. □

**Theorem 4.2.** *Let $\eta > 2k$. If $f$ belong to the Dini Lipschitz class, i.e.,*

$$f \in Lip(\eta, \gamma, 2), \quad \eta > 2k, \gamma \geq 0.$$

*Then $f$ is the null function.*

*Proof.* Assume that $f \in Lip(\eta, \gamma, 2)$. Then

$$\|\Delta_t^k \Lambda^m f\|_2 \leq C\frac{t^\eta}{(\log\frac{1}{t})^\gamma}, \quad \gamma \geq 0.$$

From Lemma 4.1, we have

$$\int_0^{+\infty}\int_B (\lambda^2+\rho^2)^{2m}|1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db \leq C^2\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}.$$

In view of Lemma 2.5, we conclude that

$$\int_0^{+\infty}\int_B (\lambda^2+\rho^2)^{2m}|1-j_\alpha(\lambda t)|^{2k}|\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db \leq \left(\frac{C}{c_0}\right)^2 \frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}.$$

Then

$$\frac{\int_0^{+\infty}\int_B (\lambda^2+\rho^2)^{2m}|1-j_\alpha(\lambda t)|^{2k}|\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db}{t^{4k}} \leq \left(\frac{C}{c_0}\right)^2 \frac{t^{2\eta-4k}}{(\log\frac{1}{t})^{2\gamma}},$$

Since $\eta > 2k$ we have

$$\lim_{t\to 0}\frac{t^{2\eta-4k}}{(\log\frac{1}{t})^{2\gamma}} = 0.$$

Then

$$\lim_{t\to 0}\int_0^{+\infty}\int_B \left(\frac{|1-j_\alpha(\lambda t)|}{\lambda^2 t^2}\right)^{2k} \lambda^{4k}(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db = 0.$$

and also from the formula (2.1) and Fatou Theorem, we obtain

$$\int_0^{+\infty}\int_B \lambda^{4k}(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db = 0.$$

and so $f$ is the null function. □

Analog of the Theorem 4.2, we obtain this Theorem.

**Theorem 4.3.** *Let $f \in L^2(X)$. If $f$ belong to $lip(2,0,2)$. i.e.,*

$$\|\Delta_t^k \Lambda^m f\|_2 = O(t^2), \quad as \quad t \to 0.$$

*Then $f$ is the null function.*

Now, we give another main result of this paper analog of Theorem 1.1.

**Theorem 4.4.** *Let $f \in W^{2,k}$. Then the following are equivalents*

*(a)* $f \in Lip(\eta, \gamma, 2)$, $\eta \in (0,1)$,

*(b)* $\displaystyle\int_r^{+\infty} \int_B (\lambda^2 + \rho^2)^{2m} |\widehat{f}(\lambda, b)|^2 d\lambda db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty,$

*where* $m = 0, 1, ..., k$.

*Proof.* $(a) \Rightarrow (b)$ Let $f \in Lip(\eta, \gamma, 2)$. Then we have

$$\|\Delta_t^k \Lambda^m f\|_2^2 = O\left(\frac{t^\eta}{(\log \frac{1}{t})^\gamma}\right) \quad \text{as} \quad t \to 0.$$

From Lemma 3.2, we have

$$\|\Delta_t^k \Lambda^m f\|_2^2 = \int_0^{+\infty} \int_B (\lambda^2 + \rho^2)^{2m} |1 - \varphi_\lambda(t)|^{2k} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db.$$

If $\lambda \in [\frac{1}{t}, \frac{2}{t}]$, then $\lambda t \geq 1$ and $(iii)$ of Lemma 2.3 implies that

$$1 \leq \frac{1}{c^{2k}} |1 - \varphi_\lambda(t)|^{2k}.$$

Then

$$\begin{aligned}
&\int_{\frac{1}{t}}^{\frac{2}{t}} \int_B (\lambda^2 + \rho^2)^{2m} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db \\
&\leq \frac{1}{c^{2k}} \int_{\frac{1}{t}}^{\frac{2}{t}} \int_B (\lambda^2 + \rho^2)^{2m} |1 - \varphi_\lambda(t)|^{2k} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db \\
&\leq \frac{1}{c^{2k}} \int_0^{+\infty} \int_B (\lambda^2 + \rho^2)^{2m} |1 - \varphi_\lambda(t)|^{2k} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db \\
&\leq \frac{1}{c^{2k}} \|\Delta_t^k \Lambda^m f\|_2^2 \\
&= O\left(\frac{t^{2\eta}}{(\log \frac{1}{t})^{2\gamma}}\right).
\end{aligned}$$

From [8], we have

$$|c(\lambda)|^{-2} \asymp \lambda^{n-1}, \quad n = \dim X,$$

then

$$\int_{\frac{1}{t}}^{\frac{2}{h}} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 \lambda^{n-1} d\lambda db = O\left(\frac{t^{2\eta}}{(\log \frac{1}{t})^{2\gamma}}\right),$$

or, equivalently,

$$\int_r^{2r} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db \leq C\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}, \quad r \to \infty,$$

where $C$ is a positive constant. Now,

$$\begin{aligned}
&\int_r^{+\infty} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db \\
&= \sum_{i=0}^{\infty} \int_{2^i r}^{2^{i+1} r} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db \\
&\leq C\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}} + \frac{(2r)^{-2\eta-n+1}}{(\log 2r)^{2\gamma}} + \frac{(4r)^{-2\eta-n+1}}{(\log 4r)^{2\gamma}} + \cdots\right) \\
&\leq C\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}} \left(1 + 2^{-2\eta-n+1} + (2^{-2\eta-n+1})^2 + (2^{-2\eta-n+1})^3 + \cdots\right) \\
&\leq K_{\eta,n}\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}},
\end{aligned}$$

where $K_{\eta,n} = C(1-2^{-2\eta-n+1})^{-1}$ since $2^{-2\eta-n+1} < 1$.
Consequently

$$\int_r^{+\infty} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$$

$(b) \Rightarrow (a)$. Suppose now that

$$\int_r^{+\infty} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty,$$

then

$$\int_r^{2r} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right),$$

whence

$$\begin{aligned}
\int_r^{2r} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 \lambda^{n-1} d\lambda db &\leq 2^{n-1} r^{n-1} \int_r^{2r} \int_B (\lambda^2+\rho^2)^{2m} |\widehat{f}(\lambda,b)|^2 d\lambda db \\
&\leq C'\frac{r^{-2\eta}}{(\log r)^{2\gamma}}.
\end{aligned}$$

Now,

$$\begin{aligned}\int_r^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2\lambda^{n-1}d\lambda db &\leq \sum_{k=0}^{\infty}\int_{2^kr}^{2^{k+1}r}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2\lambda^{n-1}d\lambda\\ &\leq C'\sum_{k=0}^{\infty}2^{-2k\eta}\frac{r^{-2\eta}}{(\log r)^{2\gamma}}.\end{aligned}$$

Consequently,

$$\int_r^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2\lambda^{n-1}d\lambda db = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right),$$

and, by $|c(\lambda)|^{-2}\asymp\lambda^{n-1}$,

$$\int_r^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right).$$

Write

$$\|\Delta_t^k\Lambda^m f\|_2^2 = I_1+I_2,$$

where

$$I_1=\int_0^{\frac{1}{t}}\int_B(\lambda^2+\rho^2)^{2m}|1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db,$$

and

$$I_2=\int_{\frac{1}{t}}^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db.$$

Firstly, it follows from the inequality $|\varphi_\lambda(t)|\leq 1$ that

$$I_2\leq 2^{2k}\int_{\frac{1}{t}}^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db = O\left(\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}\right),\quad as\quad t\to 0.$$

To estimate $I_1$, we use the inequalities $(i)$ and $(ii)$ of Lemma 2.3

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{t}}\int_B(\lambda^2+\rho^2)^{2m}|1-\varphi_\lambda(t)|^{2k-1}|1-\varphi_\lambda(t)||\widehat{f}(\lambda,b)|^2d\mu(\lambda)db\\ &\leq 2^{2k-1}\int_0^{\frac{1}{t}}\int_B(\lambda^2+\rho^2)^{2m}|1-\varphi_\lambda(t)||\widehat{f}(\lambda,b)|^2d\mu(\lambda)db\\ &\leq 2^{2k-1}t^2\int_0^{\frac{1}{t}}\int_B(\lambda^2+\rho^2)^{2m+1}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db.\end{aligned}$$

Now, we apply integration by parts for a function

$$\phi(r)=\int_r^{+\infty}\int_B(\lambda^2+\rho^2)^{2m}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db,$$

to get

$$\begin{aligned}
I_1 &\leq 2^{2k-1}t^2\int_0^{1/t}-(r^2+\rho^2)\phi'(r)dr\\
&\leq 2^{2k-1}t^2\int_0^{1/t}-r^2\phi'(r)dr\\
&\leq 2^{2k-1}t^2\left(-\frac{1}{t^2}\phi(\frac{1}{t})+2\int_0^{1/t}r\phi(r)dr\right)\\
&\leq -2^{2k-1}\phi(\frac{1}{t})+2^{2k}t^2\int_0^{1/t}r\phi(r)dr\\
&\leq 2^{2k}t^2\int_0^{1/t}r\phi(r)dr.
\end{aligned}$$

Since $\phi(r)=O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, we have $r\phi(r)=O\left(\frac{r^{1-2\eta}}{(\log r)^{2\gamma}}\right)$ and

$$\begin{aligned}
\int_0^{1/t}r\phi(r)dr &= O\left(\int_0^{1/t}\frac{r^{1-2\eta}}{(\log r)^{2\gamma}}dr\right)\\
&= O\left(\frac{t^{2\eta-1}}{(\log\frac{1}{t})^{2\gamma}}\int_0^{1/t}dr\right)\\
&= O\left(\frac{t^{2\eta-2}}{(\log\frac{1}{t})^{2\gamma}}\right),
\end{aligned}$$

so that

$$I_1=O\left(\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}\right).$$

Combining the estimates for $I_1$ and $I_2$ gives

$$\|\Delta_t^k\Lambda^m f\|_2=O\left(\frac{t^{\eta}}{(\log\frac{1}{t})^{\gamma}}\right)\quad\text{as}\quad t\to 0,$$

and this ends the proof of the theorem. $\square$

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