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On the Ψ-Conditional Exponential Asymptotic Stability of Nonlinear Lyapunov Matrix Differential Equations

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Abstract. It is proved (necessary and) sufficient conditions for $\Psi-$ conditional exponential asymptotic stability of the trivial solution of nonlinear Lyapunov matrix differential equations

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1 Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis.

Recent works for $\Psi-$ boundedness, $\Psi-$ stability, $\Psi-$ instability, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See, for example, [6 - 13, 15 - 17] and the references cited therein.

The purpose of present paper is to prove (necessary and) sufficient conditions for $\Psi-$ conditional exponential asymptotic stability of trivial solution of the nonlinear Lyapunov matrix differential equation

$$Z' = A(t)Z + ZB(t) + F(t, Z) \tag{1}$$

and the linear Lyapunov matrix differential equation

$$Z' = [A(t) + A_1(t)]Z + Z[B(t) + B_1(t)], \tag{2}$$

which can be seen as a perturbed equations of the linear equation

$$Z' = A(t)Z + ZB(t). \tag{3}$$

We investigate conditions on the fundamental matrices of the equations

$$Z' = A(t)Z, \tag{4}$$

$$Z' = ZB(t) \tag{5}$$

and on the functions A_1, B_1 and F under which the trivial solutions of the equations (1) – (5) are $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. Here, Ψ is a matrix function whose introduction permits us obtaining a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices, which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the above cited papers and the references cited therein.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let $\mathbb{R}^d$ be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $\parallel x \parallel = \max\{\mid x_1\mid, \mid x_2\mid, \mid x_3\mid, ..., \mid x_d\mid\}$ be the norm of x (here, ^{T} denotes transpose).

Let $\mathbb{M}_{d\times d}$ be the linear space of all $d \times d$ real valued matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $\mid A \mid$ by formula $\mid A \mid = \sup\limits_{\parallel x\parallel\leq 1} \parallel Ax \parallel$. It is well-known that $\mid A \mid = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} \mid a_{ij}\mid\}$.

By a solution of the equation (1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1) for all $t \in \mathbb{R}_+$.

In equation (3), we assume that A and B are continuous $d\times d$ matrices on $\mathbb{R}_+ = [0, \infty)$. It is well-known that continuity of A and B ensure the existence and uniqueness on $\mathbb{R}_+$ of a solution of (3) passing through any given point $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d\times d}$.

In addition, in equation (1), we assume that $F : \mathbb{R}_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ is continuous such that $F(t, O_d) = O_d$ (null matrix of order $d \times d$).

It is well-known that these conditions ensure the local existence of a solution passing through any given point $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d\times d}$, but it not guarantee that the solution is unique or that it can be continued for large values of t.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag } [\Psi_1, \Psi_2, \cdots \Psi_d].$$

Definition 2.1. *([7], [12], [13]). The solution z(t) of the differential equation $z' = f(t,z)$ (where $z \in R^d$ and f is a continuous d vector function) is said to be $\Psi-$ stable on R_+ if for every $\varepsilon > 0$ and every $t_0 \in R_+$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that, any solution $\widetilde{z}(t)$ of the equation which satisfies the inequality $\| \Psi(t_0)(\widetilde{z}(t_0) - z(t_0)) \| < \delta$, exists and satisfies the inequality $\| \Psi(t)(\widetilde{z}(t) - z(t)) \| < \varepsilon$ for all $t \geq t_0$.*

Otherwise, is said that the solution z(t) is $\Psi-$ unstable on R_+.

Definition 2.2. *([3]). A function $\varphi : R_+ \longrightarrow R^d$ is said to be $\Psi-$ bounded on R_+ if $\Psi(t)\varphi(t)$ is bounded on R_+.*

Otherwise, is said that the function φ is *$\Psi-$ unbounded on R_+.*

Definition 2.3. *([3], [5], [12], [13]). The solution z(t) of the differential equation $z' = f(t,z)$ is said to be $\Psi-$ conditionally stable on R_+ if it is not $\Psi-$ stable on R_+ but there exists a sequence $(z_n(t))$ of solutions of the equation defined for all $t \in R_+$ such that*

$$\lim_{n\to\infty} \Psi(t)z_n(t) = \Psi(t)z(t), \ \textit{uniformly on } R_+.$$

In addition:

If the sequence $(z_n(t))$ can be chosen so that

$$\lim_{t\to\infty} \Psi(t)(z_n(t) - z(t)) = 0, \ \textit{for } n = 1,2,...$$

then $z(t)$ is said to be $\Psi-$ conditionally asymptotically stable on R_+.

If there exist the constants $N, \lambda > 0$ such that

$$\| \Psi(t)(z_n(t) - z(t)) \| \leq Ne^{-\lambda t}, \ \text{for all } t \in \mathbb{R}_+ \text{ and } n \in N,$$

then $z(t)$ is said to be $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Remark 2.1. These definitions generalize the classical definitions of various types of stability or conditional stability (see [2]).

Remark 2.2. 1. It is easy to see that if the solution z(t) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$, then it is $\Psi-$ conditionally asymptotically stable on $\mathbb{R}_+$.

2. It is easy to see that if a solution z(t) of the linear equation $z' = A(t)z$ is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$, then so are all solutions of this equation. In this case, we can speak about $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of this linear differential equation.

Now, we extend these definitions for a matrix differential equation $Z' = F(t, Z)$, where $Z \in \mathbb{M}_{d\times d}$ and $F : \mathbb{R}_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ is a continuous function.

Definition 2.4. *([7], [11]). The solution Z(t) of the matrix differential equation* $Z' = F(t, Z)$ is said to be $\Psi-$ *stable on* R_+ *if for every* $\varepsilon > 0$ *and every* $t_0 \in \mathbb{R}_+$, *there exists a* $\delta = \delta(\varepsilon, t_0) > 0$ *such that, any solution* $\widetilde{Z}(t)$ *of the equation which satisfies the inequality* $\mid \Psi(t_0)(\widetilde{Z}(t_0) - Z(t_0)) \mid < \delta$, *exists and satisfies the inequality* $\mid \Psi(t)(\widetilde{Z}(t) - Z(t)) \mid < \varepsilon$ *for all* $t \geq t_0$.

Otherwise, is said that the solution Z(t) is $\Psi-$ *unstable on* R_+.

Definition 2.5. *([11], [12]) A matrix function* $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{d\times d}$ *is said to be* $\Psi-$ *bounded on* R_+ *if the matrix function* $\Psi(t)M(t)$ *is bounded on* R_+ (i.e. there exists $m > 0$ such that $\mid \Psi(t)M(t) \mid \leq m$, for all $t \in R_+$).

Otherwise, is said that the matrix function M is $\Psi-$ *unbounded on* R_+.

Definition 2.6. *([11], [12]). The solution* $Z(t)$ *of the matrix differential equation* $Z' = F(t, Z)$ *is said to be* $\Psi-$ *conditionally stable on* R_+ *if it is not* $\Psi-$ *stable on* R_+ *but there exists a sequence* $(Z_n(t))$ *of solutions of the equation defined on* R_+ *such that*

$$\lim_{n\to\infty} \Psi(t)Z_n(t) = \Psi(t)Z(t), \text{ uniformly on } R_+.$$

In addition:

If the sequence $(Z_n(t))$ *can be chosen so that*

$$\lim_{t\to\infty} \Psi(t)(Z_n(t) - Z(t)) = 0, \text{ for } n = 1,2,...$$

then Z(t) is said to be $\Psi-$ *conditionally asymptotically stable on* R_+.

If there exist the constants $N, \lambda > 0$ such that

$$\mid \Psi(t)(Z_n(t) - Z(t)) \mid \leq Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N,$$

then $Z(t)$ is said to be $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Remark 2.3. 1. It is easy to see that if $\Psi(t)$ and $\Psi^{-1}(t)$ are bounded on $\mathbb{R}_+$, then the $\Psi-$ stability and $\Psi-$ conditional (exponential) (asymptotic) stability are equivalent with the classical stability and conditional (exponential) (asymptotic) stability respectively.

2. In the same manner as in classical conditional asymptotic stability, we can speak about $\Psi-$ conditional exponential asymptotic stability of a linear matrix differential equation (3), (4) or (5).

Indeed, let $X(t)$, $Y(t)$ be two solutions of the equation (4). Suppose that the solution $X(t)$ is $\Psi-$ conditionally exponentialliy asymptotically stable on $\mathbb{R}_+$. From Definition 2.6, $X(t)$ is not $\Psi-$ stable on $\mathbb{R}_+$ and there exists a sequence $(X_n(t))$ of solutions of the equation defined on $\mathbb{R}_+$ such that $\lim_{n\to\infty}\Psi(t)X_n(t) = \Psi(t)X(t)$, uniformly on $\mathbb{R}_+$ and there exist the constants $N, \lambda > 0$ such that $\mid \Psi(t)(X_n(t) - X(t)) \mid \leq Ne^{-\lambda t}$, for all $t \in \mathbb{R}_+$ and $n \in N$.

Now, we consider the solutions of (4),

$$Y_n(t) = X_n(t) - X(t) + Y(t), t \in \mathbb{R}_+,\ n \in \mathbb{N}.$$

From Theorem 1, [6], we have that $Y(t)$ is not $\Psi-$ stable on $\mathbb{R}_+$ and $\lim_{n\to\infty}\Psi(t)Y_n(t) = \Psi(t)Y(t)$, uniformly on $\mathbb{R}_+$. In addition, $\mid \Psi(t)(Y_n(t) - Y(t)) \mid \leq Ne^{-\lambda t}$, for all $t \in \mathbb{R}_+$ and $n \in N$.

Thus, all solutions of (4) are $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The last cases are similar.

Definition 2.7. *([1]). Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

The important rules of calculation of the Kronecker product there are in next Lemma.

Lemma 2.1. *([1]). The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2). $(A \otimes B)^T = A^T \otimes B^T$;
3). $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;
4). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_p \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8). $(A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t)$; *($'$ denotes the derivative* $\frac{d}{dt}$*).*

Proof. See in [1].

Definition 2.8. *The application* $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, *defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, *is called the vectorization operator.*

Lemma 2.2. *([6]). The vectorization operator*

$$\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn},\ A \longrightarrow \mathcal{V}ec(A),$$

is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ *and* $\mathcal{V}ec^{-1}$ *are continuous operators.*

Proof. See Lemma 2, [6].

Lemma 2.3. *([11]). A function* $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ *is a continuous (differentiable) matrix function on* $\mathbb{R}_+$ *if and only if* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$, *defined by* $f(t) = \mathcal{V}ec(F(t))$, *is a continuous (differentiable) vector function on* $\mathbb{R}_+$.

Proof. It is a simple exercise.

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations.

Lemma 2.4. *If* $A, B, M \in \mathbb{M}_{n\times n}$, *then*

1). $\mathcal{V}ec(AMB) = (B^T \otimes A) \cdot \mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n) \cdot \mathcal{V}ec(M)$;

3). $\mathcal{V}ec(AM) = (I_n \otimes A) \cdot \mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A) \cdot \mathcal{V}ec(I_n)$.
Proof. It is a simple exercise.

The following lemmas play a vital role in the proofs of main results of present paper.

Lemma 2.5. *([6]). The matrix function $Z(t)$ is a solution on R_+ of (1) if and only if the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system*

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + f(t,z) \tag{6}$$

where $f(t,z) = \mathcal{V}ec\left(F(t,Z)\right)$, on the same interval $\mathbb{R}_+$.
Proof. See Lemma 5, [6].

Definition 2.9. *The above system (6) is called "corresponding Kronecker product system associated with (1)".*

Lemma 2.6. *([6]). For every matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{d\times d}$,*

$$\frac{1}{d} \mid \Psi(t)M(t) \mid \leq \parallel \left(I_d \otimes \Psi(t)\right) \mathcal{V}ec\left(M(t)\right) \parallel_{\mathbb{R}^{d^2}} \leq \mid \Psi(t)M(t) \mid, \forall t \geq 0. \tag{7}$$

Proof. See Lemma 6, [6].

Lemma 2.7. *The trivial solution of the equation (1) is $\Psi-$ conditionally exponentially asymptotically stable on R_+ if and only if the trivial solution of the corresponding Kronecker product system (6) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on R_+.*

Proof. First, suppose that the trivial solution of the equation (*1*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. According to Definition 2.6, this solution is not $\Psi-$ stable on $\mathbb{R}_+$ but there exists a sequence $(Z_n(t))$ of solutions of the equation defined on $\mathbb{R}_+$ such that

$$\lim_{n\to\infty} \Psi(t)Z_n(t) = O_d, \text{ uniformly on } R_+$$

and, in addition, there exist the constants $N, \lambda > 0$ such that

$$\mid \Psi(t)Z_n(t) \mid \leq Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

From Lemma 2.5 and Lemma 7, [7], the trivial solution of (6) is not $I_d \otimes \Psi-$ stable on $\mathbb{R}_+$. In addition, from the inequality (7), the solutions $z_n = \mathcal{V}ec\left(Z_n(t)\right)$, n = 1,2,...., of the system (6) satisfy

$$\parallel \left(I_d \otimes \Psi(t)\right) \mathcal{V}ec\left(Z_n(t)\right) \parallel_{\mathbb{R}^{d^2}} \quad \leq \quad \mid \Psi(t)Z_n(t) \mid, \quad \forall t \geq 0.$$

It follows that

$$\lim_{n\to\infty} (I_d \otimes \Psi(t)) z_n(t) = 0, \text{ uniformly on } \mathbb{R}_+$$

and, in addition,

$$\| (I_d \otimes \Psi(t)) z_n(t) \|_{\mathbb{R}^{d^2}} \leq N e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

From these results, Lemma 2.5 and Definition 2.3, it follows that the trivial solution of (6) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

Suppose, conversely, that the trivial solution of (6) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. According to Definition 2.3, this solution is not $I_d \otimes \Psi-$ stable on $\mathbb{R}_+$ but there exists a sequence $(z_n(t))$ of solutions of the system (6) defined on $\mathbb{R}_+$ such that

$$\lim_{n\to\infty} (I_d \otimes \Psi(t))\, z_n(t) = 0, \text{ uniformly on } \mathbb{R}_+$$

and, in addition, there exist the constants $N, \lambda > 0$ such that

$$\| (I_d \otimes \Psi(t)) z_n(t) \|_{\mathbb{R}^{d^2}} \leq N e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

From Lemma 2.5 and Lemma 7, [7], the trivial solution of (*1*) is not $\Psi-$ stable on $\mathbb{R}_+$. In addition, from the inequality (7), we have that

$$\frac{1}{d} \mid \Psi(t) Z_n(t) \mid \quad \leq \quad \| (I_d \otimes \Psi(t))\, z_n(t) \|_{\mathbb{R}^{d^2}}, t \in \mathbb{R}_+, n \in N,$$

where $Z_n(t) = \mathcal{V}ec^{-1}(z_n(t))$, $n = 1, 2, ...$ are solutions of the equation (1).

It follows that

$$\lim_{n\to\infty} \Psi(t) Z_n(t) = O_d, \text{ uniformly on } R_+$$

and

$$\mid \Psi(t) Z_n(t) \mid \quad \leq \; d N e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

From these results, Lemma 2.5 and Definition 2.6, it follows that the trivial solution of (*1*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Lemma 2.8. *(*[6]*). Let X(t) and Y(t) be a fundamental matrices for the equations (4) and (5) respectively.*

Then, the matrix $Z(t) = Y^{\mathrm{T}}(t) \otimes X(t)$ *is a fundamental matrix for the corresponding Kronecker product system associated with (3), i.e. for the differential system*

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z \tag{8}$$

Proof. See Lemma 9, [6].

3 $\Psi-$ conditional exponential asymptotic stability of linear matrix differential equations (4) and (9)

The purpose of this section is to study the $\Psi-$ conditional exponential asymptotic stability of the linear matrix differential equations

$$Z' = A(t)Z$$

and

$$Z' = (A(t) + A_1(t))\,Z. \tag{9}$$

The conditions for $\Psi-$ conditional exponential asymptotic stability of the linear matrix differential equation (4) can be expressed in terms of solutions or in terms of a fundamental matrix for (4).

Theorem 3.1. *The linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+ if and only if it has a $\Psi-$ unbounded solution on R_+ and a nontrivial solution $Z_0(t)$ such that*

$$\mid \Psi(t)Z_0(t) \mid \ \leq \ Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+,$$

where N and λ are positive constants.

Proof. First, we shall prove the "only if" part.

Suppose that the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. Let $X(t)$ be a fundamental matrix for (4). From the above Definition 2.6, Remark 2.2 and Theorem 1, [7], it follows that $\mid \Psi(t)X(t) \mid$ is unbounded on $\mathbb{R}_+$. Thus, the linear equation (4) has at least one $\Psi-$ unbounded solution on $\mathbb{R}_+$. In addition, there exists a sequence $(Z_n(t))$ of nontrivial solutions of (4) such that

$$\lim_{n\to\infty} \Psi(t)Z_n(t) = O_d, \text{ uniformly on } R_+$$

and there exist positive constants N and λ such that

$$\mid \Psi(t)Z_n(t) \mid \ \leq \ Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

The proof of "only if" part is complete.

Now, we shall prove the "if" part.

Suppose that the linear matrix differential equation (4) has a $\Psi-$ unbounded solution on $\mathbb{R}_+$ and a nontrivial solution $Z_0(t)$ such that

$$\mid \Psi(t)Z_0(t) \mid \ \leq \ Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+$$

where N and λ are positive constants.

It follows that the fundamental matrix $X(t)$ for *(4)* is such that $\mid \Psi(t)X(t) \mid$ is unbounded on $\mathbb{R}_+$. Consequently, the linear matrix differential equation *(4)* is not $\Psi-$ stable on $\mathbb{R}_+$ (see Theorem 1, [7]). On the other hand, $(\frac{1}{n}Z_0(t))$ is a sequence of nontrivial solutions of *(4)* such that

$$\lim_{n\to\infty} \Psi(t)\left(\frac{1}{n}Z_0(t)\right) = O_d, \text{ uniformly on } R_+$$

and

$$\mid \Psi(t)\left(\frac{1}{n}Z_0(t)\right) \mid \;\leq\; Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

Thus, the linear matrix differential equation *(4)* is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Remark 3.1. There exists a similar results for the differential systems $z' = A(t)z$ (see Theorem 1, [13]).

Theorem 3.2. *Let $X(t)$ be a fundamental matrix for the linear matrix differential equation (4).*

Then, the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+ if and only if the following conditions are true:

a). there exists a projection $P_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\Psi(t)X(t)P_1$ is unbounded on R_+;

b). there exists a projection $P_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $P_2 \neq 0$, and two positive constants $\widetilde{N}$ and λ such that

$$\mid \Psi(t)X(t)P_2 \mid \;\leq\; \widetilde{N}e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+.$$

Proof. First, we shall prove the "only if" part.

From $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of *(4)* and Theorem 1, [7], it follows that the matrix $\Psi(t)X(t)$ is unbounded on $\mathbb{R}_+$. In addition, there exists a nontrivial solution $Z_0(t)$ of *(4)* such that

$$\mid \Psi(t)Z_0(t) \mid \;\leq\; Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+,$$

where N and λ are positive constants.

Thus, there exists a constant matrix $C \neq O_d$ such that $X(t)C$ is nontrivial solution of *(4)* on $\mathbb{R}_+$ and

$$\mid \Psi(t)X(t)C \mid \;\leq\; Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+.$$

Let the column $c_i = (c_{1i}, c_{2i}, ..., c_{di})^T \neq \theta$ of C. Let $c_{ji} = \| c_i \|$. Let P_2 be the nul matrix O_d in which the j column is replaced with the column $c_{ji}^{-1} c_i$. It is easy to see that $P_2 \neq 0$ is a projection and there exists a positive constant $\widetilde{N}$ such that

$$| \Psi(t)X(t)P_2 | \ \leq \ \widetilde{N}e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+.$$

Now, we shall prove the "if" part.

From the hypothesis a) and Theorem 1, [6], it follows that the linear matrix differential equation (*4*) is not $\Psi-$ stable on $\mathbb{R}_+$.

Let $Z_0(t)$ be a nontrivial solution on $\mathbb{R}_+$ of the linear matrix differential equation (*4*). Let (λ_n) be such that $\lambda_n \in \mathbb{R} \setminus \{1\}$, $\lim_{n\to\infty} \lambda_n = 1$ and let $(Z_n(t))$ be defined by

$$Z_n(t) = X(t)P_2X^{-1}(0)(\lambda_n Z_0(0)) + X(t)(I - P_2)X^{-1}(0)Z_0(0), \ t \geq 0.$$

It is easy to see that $Z_n(t)$ are solutions of the linear matrix differential equation (*4*). For $n \in \mathbb{N}$ and $t \geq 0$, we have

$$\begin{aligned}
&| \Psi(t)Z_n(t) - \Psi(t)Z_0(t) | = \\
&= | \Psi(t)X(t)P_2X^{-1}(0)(\lambda_n Z_0(0)) + \\
&+ \Psi(t)X(t)(I - P_2)X^{-1}(0)Z_0(0) - \Psi(t)X(t)X^{-1}(0)Z_0(0) | = \\
&= | \Psi(t)X(t)P_2X^{-1}(0)\left((\lambda_n - 1)Z_0(0)\right) | \leq \\
&\leq | \lambda_n - 1 || \Psi(t)X(t)P_2 || X^{-1}(0)Z_0(0) | \leq \\
&\leq \widetilde{N} | \lambda_n - 1 | e^{-\lambda t} | X^{-1}(0)Z_0(0) |.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \Psi(t)Z_n(t) = \Psi(t)Z_0(t), \text{ uniformly on } \mathbb{R}_+$$

and

$$| \Psi(t)\left(Z_n(t) - Z_0(t)\right) | \leq \overline{N}e^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+ \text{ and } n \in N.$$

Thus, the linear matrix differential equation (*4*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Theorem 3.3. *Let $X(t)$ be a fundamental matrix for the linear matrix differential equation (4).*

Then, the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+ if and only if there exist a projection $P : \mathbb{R}^d \longrightarrow \mathbb{R}^d, P \neq O_d$ and two positive constants $\widetilde{N}$ and λ such that

a). $\Psi(t)X(t)(I - P)$ is unbounded on R_+;

b). $| \Psi(t)X(t)P | \leq \widetilde{N}e^{-\lambda t}$, for all $t \in \mathbb{R}_+$.

Proof. It results from the above Theorem.

Remark 3.2. 1. There exists a similar results for the differential systems $z' = A(t)z$ (see Theorem 2 and Theorem 3, [13]).

2. The above Theorems generalize a similar results in connection with the classical conditional exponential asymptotic stability and $\Psi-$ conditional exponential asymptotic stability of vectorial differential equations $z' = A(t)z$ to matrix differential equations (4).

Indeed, consider in *(4)*

$$Z = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 \\ z_2 & z_2 & \cdots & z_2 \\ \cdots & \cdots & \cdots & \cdots \\ z_d & z_d & \cdots & z_d \end{pmatrix}$$

Now, the definitions and conditions for $\Psi-$ boundedness or $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of z are the same for $\Psi-$ boundedness or $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of Z.

Sufficient conditions for $\Psi-$ conditional exponential asymptotic stability of equation *(4)* are given in the following theorems.

Theorem 3.4. *Suppose that there exist two supplementary projections $P_i : R^d \longrightarrow R^d$, $P_1 \neq 0, P_2 \neq 0$ and a constant $K > 0$ such that the fundamental matrix $X(t)$ for the linear matrix differential equation (4) satisfies the condition*

$$\int_0^t \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid ds + \int_t^\infty \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid ds \leq K$$

for all $t \in \mathbb{R}_+$.

Then, the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. From Lemmas 10 and 11, [6], results that there exists a positive constant N such that $\mid \Psi(t)X(t)P_1 \mid \leq Ne^{-K^{-1}t}$, for all $t \in R_+$ and the matrix $\Psi(t)X(t)P_2$ is unbounded on $\mathbb{R}_+$.

Now, the Theorem results from the above Theorem 3.2.

Example 3.1. Consider the linear matrix differential equation (4) in which $A(t) = A$ is a $d \times d$ real constant matrix which has characteristic roots with diferent real parts. In this case there exists, e.g., an interval $(\alpha, \beta) \subset \mathbb{R}$ such that for $\lambda \in (\alpha, \beta)$, $\Psi(t) = e^{-\lambda t}I_d$ and $X(t) = e^{tA}$, there exist supplementary projections $P_1 \neq 0$, $P_2 \neq 0$ and a positive constant K such that

$$\begin{cases} \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ Ke^{(\alpha-\lambda)(t-s)}, \ \ for\ 0 \leq s \leq t \\ \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ Ke^{(\lambda-\beta)(s-t)}, \ \ for\ 0 \leq t \leq s \end{cases}.$$

Then, for $t \in \mathbb{R}_+$,

$$\int_0^t \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid ds + \int_t^\infty \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid ds \leq$$
$$\leq \int_0^t Ke^{(\alpha-\lambda)(t-s)}ds + \int_t^\infty Ke^{(\lambda-\beta)(s-t)}ds < K\left(\tfrac{1}{\lambda-\alpha} + \tfrac{1}{\beta-\lambda}\right).$$

Thus, from the above Theorem, the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

A similar situation is if $A(t)$ in (4) is a $d \times d$ real continuous periodic matrix (see [2], Chapter III – Stability).

In general case, we have the following:

Theorem 3.5. *If there exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_1 \neq 0, P_2 \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrix X(t) of (4) satisfies the conditions*

$$\begin{cases} \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ K_1e^{-\lambda_1(t-s)}, \ \ for\ 0 \leq s \leq t \\ \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ K_2e^{-\lambda_2(s-t)}, \ \ for\ 0 \leq t \leq s \end{cases},$$

then, the linear matrix differential equation (4) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. It follows from the above Theorem 3.4.

Sufficient conditions for $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of the linear matrix differential equation (9) are given in the following theorem.

Theorem 3.6. *Suppose that:*

(1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_1 \neq 0, P_2 \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrix X(t) of (4) satisfies the conditions

$$\begin{cases} \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ K_1e^{-\lambda_1(t-s)}, \ \ for\ 0 \leq s \leq t \\ \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid & \leq \ K_2e^{-\lambda_2(s-t)}, \ \ for\ 0 \leq t \leq s \end{cases};$$

(2). $A_1(t)$ is a $d \times d$ continuous matrix function on R_+ and satisfies one of following conditions

(i). $\sup\limits_{t \geq 0} \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}$,

(ii). $\lim\limits_{t\to\infty} \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid = 0$,

(iii). $\int_0^\infty \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid dt$ *is convergent.*

Then, the linear matrix differential equation (9) is $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.

Proof. It is similar with the proofs of Theorems 6, 7, [13].

Remark 3.3. These Theorems are variants of Theorems 4, 5, 6, 7, [13], for linear matrix differential equations.

Thus, the above results can be considered as a generalization of a well-known results in connection with the classical conditional exponential asymptotic stability for differential systems (see e.g. [2])

Remark 3.4. If the linear equation (4) is only $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$, then the perturbed equation (9) can't be $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. This is shown by the Example from [13], in variant for a linear matrix differential equation (4) and (9).

4 $\Psi-$ conditional exponential asymptotic stability of linear Lyapunov matrix differential equations

The purpose of this section is to study the $\Psi-$ conditional exponential asymptotic stability of the linear Lyapunov matrix differential equations (2) and (3).

Theorem 4.1. *The linear Lyapunov matrix differential equation (3) is* $\Psi-$ *conditionally exponentially asymptotically stable on* R_+ *if and only if the corresponding Kronecker product system (8) is* $I_d \otimes \Psi-$ *conditionally exponentially asymptotically stable on* R_+ .

Proof. It results from Lemma 2.7 and Remark 2.3.

The conditions for $\Psi-$ conditional exponential asymptotic stability of the linear Lyapunov matrix differential equation (3) can be expressed in terms of solutions or in terms of a fundamental matrices for (4) and (5).

Theorem 4.2. *The linear Lyapunov matrix differential equation (3) is* $\Psi-$ *conditionally exponentially asymptotically stable on* R_+ *if and only if it has a* $\Psi-$ *unbounded solution on* R_+ *and there exist two positive constants* N, λ *and a nontrivial solution* $Z_0(t)$ *on* R_+ *such that*

$$\mid \Psi(t)Z_0(t) \mid \leq Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+.$$

Proof. It results from Theorem 4.1, Theorem 3.1 (variant for systems), Lemmas 2.5 and 2.6.

Theorem 4.3. *Suppose that the fundamental matrices X(t) and Y(t) for the equations (4) and (5) respectively satisfy the following conditions:*
a). there exists a projection $Q_1 : R^{d^2} \longrightarrow R^{d^2}$ *such that*

$$\left(Y^T(t) \otimes \Psi(t)X(t)\right) Q_1$$

is unbounded on R_+;
b). there exist two positive constants N, λ *and a projection* $Q_2 : R^{d^2} \longrightarrow R^{d^2}$, $Q_2 \neq 0$, *such that*

$$\mid \left[Y^T(t) \otimes (\Psi(t)X(t))\right] Q_2 \mid \leq Ne^{-\lambda t}, \text{ for all } t \in \mathbb{R}_+.$$

Then, the equation (3) is $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.
Proof. It results from Theorem 4.1, Theorem 3.2 (variant for systems) and Lemmas 2.5, 2.6, 2.8.

Remark 4.1. It is easy to prove that the projection Q_1 have the form $Q_1 = I_d \otimes P_1$, where $P_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a projection (see Theorem 1, [6]).

Sufficient conditions for $\Psi-$ conditional exponential asymptotic stability of equation (3) are given in the following theorems.

Theorem 4.4. *Let* $X(t)$ *and* $Y(t)$ *be fundamental matrices for the equations (4) and (5) respectively. Suppose that there exist two supplementary projections* $P_i : R^d \longrightarrow R^d$, $P_1 \neq 0, P_2 \neq 0$ *and a constant* $K > 0$ *such that,* for $t \geq 0$,

$$\int_0^t \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \mid ds+$$

$$+\int_t^\infty \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \mid ds \leq K.$$

Then, the equation (3) is $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.
Proof. From Theorem 4.1, we know that the equation (*3*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$ if and only if the corresponding Kronecker product system (8) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Lemma 2.8, we know that $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the system (8). The hypotheses ensure, via Theorem 3.4 (variant for systems) that the system (8) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Theorem 4.1, the equation (*3*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Theorem 4.5. *Let $X(t)$ and $Y(t)$ be fundamental matrices for the equations (4) and (5) respectively. Suppose that there exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that*

$$\begin{cases} \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)) \mid\leq K_1e^{-\lambda_1(t-s)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \\ \\ \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)) \mid\leq K_2e^{-\lambda_2(s-t)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq t \leq s \end{cases}.$$

Then, the linear Lyapunov matrix differential equation (3) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. It results from the above Theorem.

Theorem 4.6. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrix $X(t)$ for the equation (4) satisfies the conditions:

$$\begin{cases} \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid & \leq & K_1e^{-\lambda_1(t-s)} & \text{for } 0 \leq s \leq t \\ \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid & \leq & K_2e^{-\lambda_2(s-t)} & \text{for } 0 \leq t \leq s \end{cases};$$

2). The matrix $B(t)$ satisfies one of the following conditions:

a). $\sup\limits_{t\geq 0} \mid B^T(t) \mid < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}$

b). $\lim\limits_{t\to\infty} \mid B(t) \mid = 0.$

c). $\int_0^\infty \mid B(t) \mid dt$ *is convergent.*

Then, the linear Lyapunov matrix differential equation (3) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. We will use Theorem 4.1. We write the corresponding Kronecker product system (8) for the equation (*3*) in the form

$$z' = (I_d \otimes A(t))\, z + \left(B^T(t) \otimes I_d\right) z.$$

Now, using Lemma 2.8 and Theorem 3.6, it results that this system is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Theorem 4.1, it follows that the equation (*3*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Theorem 4.7. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrix $Y(t)$ for the equation (5) satisfies the conditions:

$$\begin{cases} \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)P_1\Psi^{-1}(s)) \mid & \leq \; K_1 e^{-\lambda_1(t-s)} \quad \text{for } 0 \leq s \leq t \\ \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)P_2\Psi^{-1}(s)) \mid & \leq \; K_2 e^{-\lambda_2(s-t)} \quad \text{for } 0 \leq t \leq s \end{cases};$$

2). The matrix $A(t)$ satisfies one of the following conditions:

a). $\sup\limits_{t\geq 0} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}$;

b). $\lim\limits_{t\to\infty} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid = 0$;

c). $\int_0^\infty \mid \Psi(t)A(t)\Psi^{-1}(t) \mid dt$ *is convergent.*

Then, the linear Lyapunov matrix differential equation (3) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. We will use Theorem 4.1. We write the corresponding Kronecker product system (8) for the equation (*3*) in the form

$$z' = \left(B^T(t) \otimes I_d\right) z + (I_d \otimes A(t))\, z$$

Now, using Lemma 2.8 and Theorem 3.6, it results that this system is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Theorem 4.1, it follows that the equation (*3*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Theorem 4.8. *Suppose that are satisfied the following hypotheses:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and the positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrices $X(t)$ and $Y(t)$ for the equations (4) and (5) respectively satisfy the following conditions:

$$\begin{cases} \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)) \mid \leq K_1 e^{-\lambda_1(t-s)}, \\ \qquad\qquad\qquad\qquad 0 \leq s \leq t \\ \\ \mid (Y^T(t)\left(Y^T\right)^{-1}(s)) \otimes (\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)) \mid \leq K_2 e^{-\lambda_2(s-t)}, \\ \qquad\qquad\qquad\qquad 0 \leq t \leq s \end{cases};$$

2). $A_1(t)$ and $B_1(t)$ are continuous $d \times d$ matrices functions on R_+ and satisfy one of the following conditions:

i). $M = \sup_{t \geq 0} \mid I_d \otimes (\Psi(t) A_1(t) \Psi^{-1}(t)) + B_1^T(t) \otimes I_d \mid < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2} \right)^{-1}$;

ii). $\lim_{t \to \infty} \mid I_d \otimes (\Psi(t) A_1(t) \Psi^{-1}(t)) + B_1^T(t) \otimes I_d \mid = 0$;

iii). $\int_0^\infty \mid I_d \otimes (\Psi(t) A_1(t) \Psi^{-1}(t)) + B_1^T(t) \otimes I_d \mid dt$ is convergent.

Then, the linear Lyapunov matrix differential equation (2) is $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. From Theorem 4.1, we know that the equation (*2*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$ if and only if the corresponding Kronecker product system associated with (*2*), i.e.

$$z' = \left[I_d \otimes (A(t) + A_1(t)) + (B(t) + B_1(t))^T \otimes I_d \right] z$$

or

$$z' = \left[I_d \otimes A(t) + B^T(t) \otimes I_d \right] z + \left[I_d \otimes A_1(t) + B_1^T(t) \otimes I_d \right] z \tag{10}$$

is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Lemma 2.8, we know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d \right) z.$$

The hypotheses ensure, via Theorem 3.6, that the system (10) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Theorem 4.1, the equation (*2*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Remark 4.2. These Theorems generalize similar results in connection with the classical conditional exponential asymptotic stability and $\Psi-$ conditional exponential asymptotic stability for differential systems in [2] and [13].

Remark 4.3. If the linear Lyapunov matrix differential equation (3) is only $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$, then the perturbed equation (*2*) can't be $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

This is shown by the next Example, transformed after an equation due to O. Perron [18].

Example 4.1. Consider the equation (3) with

$$A(t) = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \text{ and } B(t) = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, a fundamental matrices for the homogeneous equations (4) and (5) are

$$X(t) = \begin{pmatrix} e^{(t+1)(\sin\ln(t+1)-\frac{1}{2})} & 0 \\ 0 & e^{-\frac{1}{4}(t+1)} \end{pmatrix} \text{ and } Y(t) = \begin{pmatrix} e^{-\frac{1}{4}(t+1)} & 0 \\ 0 & e^{t+1} \end{pmatrix}$$

respectively.

Let Ψ be the matrix

$$\Psi(t) = \begin{pmatrix} e^{\frac{1}{2}(t+1)} & 0 \\ 0 & e^{-\frac{1}{2}(t+1)} \end{pmatrix}.$$

From Lemma 2.8, the matrix

$$Y^T(t) \otimes X(t) = \begin{pmatrix} u(t) & 0 & 0 & 0 \\ 0 & e^{-\frac{1}{2}(t+1)} & 0 & 0 \\ 0 & 0 & v(t) & 0 \\ 0 & 0 & 0 & e^{\frac{3}{4}(t+1)} \end{pmatrix},$$

where $u(t) = e^{(t+1)(\sin\ln(t+1)-\frac{3}{4})}$, $v(t) = e^{(t+1)(\sin\ln(t+1)+\frac{1}{2})}$, is a fundamental matrix for the system (8), i.e. the corresponding Kronecker product system associated with equation (3).

We have

$$(I_2 \otimes \Psi(t))\left(Y^T(t) \otimes X(t)\right) = \begin{pmatrix} \overline{u}(t) & 0 & 0 & 0 \\ 0 & e^{-(t+1)} & 0 & 0 \\ 0 & 0 & \overline{v}(t) & 0 \\ 0 & 0 & 0 & e^{\frac{1}{4}(t+1)} \end{pmatrix},$$

where $\overline{u}(t) = e^{(t+1)(\sin\ln(t+1)-\frac{1}{4})}$, $\overline{v}(t) = e^{(t+1)(\sin\ln(t+1)+1)}$.

If we take $Q_1(t) = \operatorname{diag}\,[1,0,0,0]$, it is easy to see that the matrix

$$\left[Y^T(t) \otimes (\Psi(t)X(t))\right] Q_1(t)$$

is unbounded on $\mathbb{R}_+$ (because $\lim_{n\to\infty} \overline{u}(-1 + e^{\frac{\pi}{2}+2n\pi}) = +\infty$).

If we take $Q_2(t) = \operatorname{diag}\,[0,1,0,0]$, it is easy to see that

$$\mid \left[Y^T(t) \otimes (\Psi(t)X(t))\right] Q_2(t) \mid = e^{-(t+1)}, \text{ for all } t \geq 0.$$

From Theorem 4.3 it results that the equation (3) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

Now, if we take

$$A_1(t) = \begin{pmatrix} 0 & be^{-\frac{1}{4}(t+1)} \\ 0 & 0 \end{pmatrix},$$

where $b \in \mathbb{R}$, $b \neq 0$, then, a fundamental matrix for the perturbed corresponding Kronecker product system associated with equation (2) is

$$Z_0(t) = \begin{pmatrix} bu(t)\int_1^{t+1} e^{-s\sin\ln s}ds & u(t) & 0 & 0 \\ e^{-\frac{1}{2}(t+1)} & 0 & 0 & 0 \\ 0 & 0 & bv(t)\int_1^{t+1} e^{-s\sin\ln s}ds & v(t) \\ 0 & 0 & e^{\frac{3}{4}(t+1)} & 0 \end{pmatrix}.$$

As in Example 4.1, [12], we have that the all columns of $(I_2 \otimes \Psi)\, Z_0(t)$ are unbounded on $\mathbb{R}_+$. From this and Theorem 3.1 (variant for systems), the corresponding Kronecker product system associated with equation (2) is not $I_2 \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

From Theorem 4.1, it follows that the perturbed equation (2) is not $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

Finally, we have

$$\Psi(t)A_1(t)\Psi^{-1}(t) = \begin{pmatrix} 0 & be^{-\frac{1}{2}(t+1)} \\ 0 & 0 \end{pmatrix}.$$

Thus, $A_1(t)$ and $B_1(t) = O_2$ satisfy the condition 2) of Theorem 4.8.

5 $\Psi-$ conditional exponential asymptotic stability of non–linear Lyapunov matrix differential equations

The purpose of this section is to study the $\Psi-$ conditional exponential asymptotic stability of the nonlinear matrix differential equations (1) and (11).

Theorem 5.1. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$, $i = 1, 2$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrix $U(t)$ for the equation (4), $U(0) = I_d$, satisfies the conditions:

$$\begin{cases} |\ \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s)\ | & \leq & K_1e^{-\lambda_1(t-s)} & \text{for } 0 \leq s \leq t \\ |\ \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s)\ | & \leq & K_2e^{-\lambda_2(s-t)} & \text{for } 0 \leq t \leq s \end{cases};$$

2). *The continuous function* $F : R_+ \times M_{d\times d} \longrightarrow M_{d\times d}$ *is such that* $F(t, O_d) = O_d$ *and satisfies the Lypschitz condition*

$$\mid \Psi(t)\,(F(t,X_1) - F(t,X_2)) \mid\leq \gamma(t) \mid \Psi(t)(X_1 - X_2) \mid,$$

for all $t \geq 0$ *and* $X_1, X_2 \in M_{d\times d}$, *where* $\gamma : R_+ \longrightarrow R_+$ *is a continuous function such that*

$$\gamma_0 \;=\; \sup_{t\geq 0} \gamma(t) < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}.$$

Then, all $\Psi-$ *bounded solutions of the equation*

$$X' = A(t)X + F(t,X) \tag{11}$$

are $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.

If, in addition, the continuous matrix function $B(t)$ *is such that*

$$\sup_{t\geq 0}\ (\gamma(t) + \mid B(t) \mid) < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1},$$

then, all $\Psi-$ *bounded solutions of the the nonlinear Lyapunov matrix differential equation (1) are* $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.

Proof. We put

$$S = \{X : \mathbb{R}_+ \longrightarrow \mathbb{M}_{d\times d} \mid X \text{ is continuous and } \Psi - \text{ bounded on } \mathbb{R}_+\}.$$

Define on the set S a norm by

$$\mid\mid X \mid\mid \;=\; \sup_{t\geq 0}\ \mid \Psi(t)X(t) \mid .$$

It is well-known that $(S, \mid\mid \cdot \mid\mid)$ is a Banach real space.

For $X \in S$, we define

$$(TX)\,(t) = \int_0^t U(t)P_1U^{-1}(s)F(s,X(s))ds - \int_t^\infty U(t)P_2U^{-1}(s)F(s,X(s))ds,$$

for all $t \geq 0$.

For $v \geq t \geq 0$,

$\mid \int_t^v U(t)P_2U^{-1}(s)F(s,X(s))ds \mid =$

$=\mid \Psi^{-1}(t) \int_t^v \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s,X(s))ds \mid \leq$

$\leq \mid \Psi^{-1}(t) \mid \int_t^v \mid \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)F(s,X(s)) \mid ds \leq$

$\leq \mid \Psi^{-1}(t) \mid \int_t^v K_2 e^{-\lambda_2(s-t)}\gamma(s) \mid \Psi(s)X(s) \mid ds \leq$
$\leq \mid \Psi^{-1}(t) \mid\mid\mid X \mid\mid K_2\gamma_0 \int_t^v e^{-\lambda_2(s-t)}ds.$

From the first assumption, it follows that the integral

$$\int_t^\infty U(t)P_2U^{-1}(s)F(s,X(s))ds$$

is convergent for all $X \in S$ and $t \geq 0$.

From hypotheses, $(TX)(t)$ exists and is continuous differentiable on $\mathbb{R}_+$.

For $X \in S$ and $t \geq 0$,

$\mid \Psi(t)(TX)(t) \mid =$
$= \mid \int_0^t \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s,X(s))ds -$
$- \int_t^\infty \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s,X(s))ds \mid \leq$
$\leq \int_0^t \mid \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)F(s,X(s)) \mid ds +$
$+ \int_t^\infty \mid \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)F(s,X(s))ds \mid ds \leq$
$\leq \int_0^t K_1e^{-\lambda_1(t-s)}\gamma(s) \mid \Psi(s)X(s) \mid ds + \int_t^\infty K_2e^{-\lambda_2(s-t)}\gamma(s) \mid \Psi(s)X(s) \mid ds \leq$
$\leq \gamma_0 \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right) \mid\mid X \mid\mid$

This shows that $TS \subset S$.

Moreover, for any two $\Psi-$ bounded continuous functions $X_1(t), X_2(t)$,

$\mid \Psi(t)\left((TX_1)(t) - (TX_2)(t)\right) \mid =$
$= \mid \int_0^t \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s,X_1(s)) - F(s,X_2(s))\right)ds -$
$- \int_t^\infty \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s,X_1(s)) - F(s,X_2(s))\right)ds \mid \leq$
$\leq \int_0^t \mid \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s,X_1(s)) - F(s,X_2(s))\right) \mid ds +$
$+ \int_t^\infty \mid \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s,X_1(s)) - F(s,X_2(s))\right) \mid ds \leq$
$\leq \int_0^t K_1e^{-\lambda_1(t-s)}\gamma(s) \mid \Psi(s)\left(X_1(s) - X_2(s)\right) \mid ds +$
$+ \int_t^\infty K_2e^{-\lambda_2(s-t)}\gamma(s) \mid \Psi(s)\left(X_1(s) - X_2(s)\right) \mid ds \leq$
$\leq \gamma_0 \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right) \mid\mid X_1 - X_2 \mid\mid .$

It follows that

$$\mid\mid TX_1 - TX_2 \mid\mid \ \leq \ \gamma_0 \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right) \mid\mid X_1 - X_2 \mid\mid, \text{ for all } X_1,\ X_2 \in S.$$

Since $M = \gamma_0 \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right) < 1$, it follows that the operator T is a contraction of the Banach space $(S, \ \mid\mid \cdot \mid\mid)$.

It follows by the contraction principle that for any $\Psi-$ bounded continuous function $Y(t)$ on $\mathbb{R}_+$, the integral equation

$$X = Y + TX \tag{12}$$

has a unique solution $X \in S$.

Furthermore, by the definition of T, $X(t) - Y(t)$ is continuous differentiable and

$$(X(t) - Y(t))' = A(t)\,(X(t) - Y(t)) + F(t, X(t)),\ t \geq 0.$$

Hence, if $Y(t)$ is a $\Psi-$ bounded solution of (4), then the corresponding solution $X(t)$ of (12) is a $\Psi-$ bounded solution of (11). Conversely, if $X(t)$ is a $\Psi-$ bounded solution of (11), the function $Y(t)$ defined by (12) is a $\Psi-$ bounded solution of (4).

Thus, the equation (12) establishes a 1–1 correspondence C between the $\Psi-$ bounded solutions of (4) and (11): $X = CY$.

If we subtract from (12) the analogous equation $X_0 = Y_0 + TX_0$, we get

$$|| X - X_0 || \ \leq \ || Y - Y_0 || + || TX - TX_0 || \ \leq$$

$$\leq \ || Y - Y_0 || + M \, || X - X_0 ||,$$

i.e.

$$(1 - M) \, || X - X_0 || \ \leq \ || Y - Y_0 ||$$

and

$$|| Y - Y_0 || \ \leq \ || X - X_0 || + || TX - TX_0 || \ \leq \ (1 + M) \, || X - X_0 || \, .$$

Thus,

$$(1 + M)^{-1} \, || Y - Y_0 || \ \leq \ || X - X_0 || \ \leq \ (1 - M)^{-1} \, || Y - Y_0 || \, . \quad (13)$$

This shows that the correspondence C is bicontinuous on the interval $\mathbb{R}_+$.

Now, we prove that all $\Psi-$ bounded solutions of (11) tend to zero $\Psi-$ exponentially as $t \to \infty$.

Let $X(t)$ be a $\Psi-$ bounded solution of (11). Let $Y(t)$ be defined by (12); this function is a $\Psi-$ bounded solution of (4). Let $Z(t) = Y(t) - U(t)P_1X(0)$, for $t \geq 0$. It is easy to see that Z(t) is a $\Psi-$ bounded solution of (4). Since

$P_1Z(0) = P_1\,(Y(0) - P_1X(0)) = P_1\,(X(0) - (TX)(0)) - P_1^2X(0) =$

$= P_1X(0) - P_1\left(-\int_0^\infty P_2U^{-1}(s)F(s, X(s))ds\right) - P_1X(0) = O,$

it follows that

$Z(t) = U(t)Z(0) = U(t)(P_1 + P_2)Z(0) = U(t)P_2Z(0).$

If $P_2Z(0) \neq 0$, from Lemma 11, [6], it follows that $\limsup_{t\to\infty} \mid \Psi(t)Z(t) \mid = +\infty$, which is a contradiction.

Thus, $P_2Z(0)=0$ and then, $Z(t)=0$. Hence,
$X(t)=U(t)P_1X(0)+$
$+\int_0^t U(t)P_1U^{-1}(s)F(s,X(s))ds-\int_t^\infty U(t)P_2U^{-1}(s)F(s,X(s))ds$, $t\geq 0$.
Then, for $t\geq 0$,

$$\begin{aligned}&\mid \Psi(t)X(t)\mid\leq\mid \Psi(t)U(t)P_1\Psi^{-1}(0)\mid\mid \Psi(0)X(0)\mid+\\&+\int_0^t\mid \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s)\mid\mid \Psi(s)F(s,X(s))\mid ds+\\&+\int_t^\infty\mid \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s)\mid\mid \Psi(s)F(s,X(s))ds\mid ds\leq\\&\leq K_1e^{-\lambda_1 t}\mid \Psi(0)X(0)\mid+\\&+K_1\int_0^t e^{-\lambda_1(t-s)}\gamma(s)\mid \Psi(s)X(s)\mid ds+\\&K_2\int_t^\infty e^{-\lambda_2(s-t)}\mid \Psi(s)F(s,X(s))ds\mid ds.\end{aligned}$$

Now, we choose $b=\frac{K_1}{\frac{1}{\gamma_0}-\frac{K_2}{\lambda_2}}\in(0,\lambda_1)$ and $c=\lambda_1-b\in(0,\lambda_1)$.
Thus, by putting $m(t)=\sup\limits_{0\leq s\leq t} e^{cs}\mid \Psi(s)X(s)\mid$, we obtain

$$\begin{aligned}&e^{ct}\mid \Psi(t)X(t)\mid\leq K_1e^{ct}e^{-\lambda_1 t}\mid \Psi(0)X(0)\mid+\\&+K_1\int_0^t e^{ct}e^{-\lambda_1(t-s)}e^{-cs}\gamma(s)e^{cs}\mid \Psi(s)X(s)\mid ds+\\&+K_2\int_t^\infty e^{ct}e^{-\lambda_2(s-t)}e^{-cs}\gamma(s)e^{cs}\mid \Psi(s)X(s)\mid ds\end{aligned}$$

or

$$\begin{aligned}&e^{ct}\mid \Psi(t)X(t)\mid\leq K_1e^{-bt}\mid \Psi(0)X(0)\mid+\\&+K_1\gamma_0m(t)b^{-1}\left(1-e^{-bt}\right)+K_2\gamma_0\int_t^\infty e^{(c+\lambda_2)(t-s)}m(s)ds, \text{ for } t\geq 0.\end{aligned}$$

For a fixed $t\geq 0$, there are two cases to consider:
Case 1: for any $u\in[0,t]$, $e^{cu}\mid \Psi(u)X(u)\mid\leq e^{ct}\mid \Psi(t)X(t)\mid$.
In this case, $m(t)=e^{ct}\mid \Psi(t)X(t)\mid$. It follows that

$$\begin{aligned}&m(t)\leq K_1e^{-bt}\mid \Psi(0)X(0)\mid+m(t)\left(1-e^{-bt}\right)+\\&K_2\gamma_0\int_t^\infty e^{(c+\lambda_2)(t-s)}m(s)ds.\end{aligned}$$

Thus,

$$m(t)\leq K_1\mid \Psi(0)X(0)\mid+K_2\gamma_0e^{(\lambda_1+\lambda_2)t}\int_t^\infty e^{-(c+\lambda_2)s}m(s)ds.$$

Case 2: there exists $u_0\in[0,t)$, such that

$$e^{cu_0}\mid \Psi(u_0)X(u_0)\mid>e^{ct}\mid \Psi(t)X(t)\mid.$$

In this case, there exists $u_1\in[0,t)$, such that

$$m(t)=e^{cu_1}\mid \Psi(u_1)X(u_1)\mid=m(u_1)$$

and, in addition, $m(s)=m(t)$ for all $s\in[u_1,t]$.
It follows that

$$\begin{aligned}&m(t)=e^{cu_1}\mid \Psi(u_1)X(u_1)\mid\leq K_1e^{-bu_1}\mid \Psi(0)X(0)\mid+\\&+K_1\gamma_0m(u_1)b^{-1}\left(1-e^{-bu_1}\right)+K_2\gamma_0\int_{u_1}^\infty e^{(c+\lambda_2)(u_1-s)}m(s)ds.\end{aligned}$$

Because
$\int_{u_1}^\infty e^{(c+\lambda_2)(u_1-s)}m(s)ds=\int_{u_1}^t e^{(c+\lambda_2)(u_1-s)}m(t)ds+\int_t^\infty e^{(c+\lambda_2)(u_1-s)}m(s)ds=$

$= m(t)\left(\frac{e^{(c+\lambda_2)(u_1-t)}}{-(c+\lambda_2)} + \frac{1}{(c+\lambda_2)}\right) + \int_t^\infty e^{(c+\lambda_2)(u_1-s)}m(s)ds,$

we have

$m(t) \leq \frac{b}{\gamma_0} \mid \Psi(0)X(0) \mid + \frac{bK_2}{K_1}e^{(\lambda_1+\lambda_2)t}\int_t^\infty e^{-(c+\lambda_2)s}m(s)ds.$

In any case, the function $m(t)$ satisfies the inequality

$$m(t) \leq \frac{b}{\gamma_0} \mid \Psi(0)X(0) \mid + \frac{bK_2}{K_1}e^{\alpha t}\int_t^\infty e^{-\beta s}m(s)ds,\ t \geq 0,$$

where $\alpha = \lambda_1 + \lambda_2 > 0$ and $\beta = c + \lambda_2 > 0$.

From Theorem I. 19,[14], (page 114), it follows that $m(t) \leq u(t)$, for $t \geq 0$, where $u(t)$ is the solution of Volterra integral equation

$$u(t) = \frac{b}{\gamma_0} \mid \Psi(0)X(0) \mid + \frac{bK_2}{K_1}e^{\alpha t}\int_t^\infty e^{-\beta s}u(s)ds$$

or the linear differential equation

$$u' = \left(\alpha - \frac{bK_2}{K_1}e^{(\alpha-\beta)t}\right)u - \frac{\alpha b}{\gamma_0} \mid \Psi(0)X(0) \mid$$

with the condition $\lim\limits_{t\to\infty} u(t) = 0$.

From a result of O. Perron [18], it follows that

$$u(t) = \frac{\alpha b}{\gamma_0} \mid \Psi(0)X(0) \mid \int_t^\infty e^{\alpha(t-s)-bK_2K_1^{-1}(\alpha-\beta)^{-1}\left(e^{(\alpha-\beta)t}-e^{(\alpha-\beta)s}\right)}ds,\ t \geq 0.$$

Since $\lim\limits_{t\to\infty} u(t) = 0$, it follows that there exists a constant $N > 0$ such that

$$\mid \Psi(t)X(t) \mid \leq N \mid \Psi(0)X(0) \mid e^{-ct},\ t \geq 0. \tag{14}$$

We specify that the constants N and c do not depend on the solution $X(t)$.

Now, we finish the proof.

Let $X(t)$ be a $\Psi-$ bounded solution of (11). This solution is $\Psi-$ unstable on $\mathbb{R}_+$.

Indeed, if not, it is $\Psi-$ stable on $\mathbb{R}_+$. Thus, for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that, any solution $\widetilde{X}(t)$ of the equation (11) which satisfies the inequality $\mid \Psi(t_0)(\widetilde{X}(t_0) - X(t_0)) \mid < \delta$, exists and satisfies the inequality $\mid \Psi(t)(\widetilde{X}(t) - X(t)) \mid < \varepsilon$ for all $t \geq t_0$.

Let $Z_0 \in \mathbb{M}_{d\times d}$ be such that $P_1 Z_0 = O_d$ and $0 < \mid \Psi(0)Z_0 \mid < \delta(\varepsilon, 0)$ and let $\widetilde{X}(t)$ be the solution of (11) with the initial condition $\widetilde{X}(0) = X(0) + Z_0$. Then, $\mid \Psi(t)(\widetilde{X}(t) - X(t)) \mid < \varepsilon$ for all $t \geq 0$.

Let $Y(t)$ be the function $Y(t) = \widetilde{X}(t) - X(t) - T(\widetilde{X}(t) - X(t))$, $t \geq 0$.

Clearly, $Y(t)$ is a $\Psi-$ bounded solution on $\mathbb{R}_+$ of (4). It is easy to see that $P_1Y(0) = O_d$. If $P_2Y(0) \neq O_d$, from Lemma 11, [6], it follows that $\limsup_{t\to\infty} \mid \Psi(t)Y(t) \mid = \infty$, which is a contradiction. Thus, $P_2Y(0) = O_d$ and then

$$\widetilde{X}(t) - X(t) = T(\widetilde{X}(t) - X(t)),\ t \geq 0.$$

It follows that $\widetilde{X}(t) = X(t)$, $t \geq 0$, which is a contradiction.

This shows that the solution $X(t)$ is $\Psi-$ unstable on $\mathbb{R}_+$.

Let $Y = X - TX$ be. From Theorem 3.5, it follows that there exists a sequence (Y_n) of solutions of (4) defined on $\mathbb{R}_+$ such that

$$\lim_{n\to\infty} \Psi(t)Y_n(t) = \Psi(t)Y(t),\ \text{uniformly on } \mathbb{R}_+$$

and there exist the positive constants $\overline{N}$ and λ such that

$$\mid \Psi(t)\,(Y_n(t) - Y(t)) \mid \ \leq\ \overline{N}e^{-\lambda t},\ \text{for all } t \geq 0 \text{ and } n \in N.$$

Let $X_n = CY_n$ be. From (13), it follows that the sequence (X_n) of solutions of (11) defined on $\mathbb{R}_+$ is such that

$$\lim_{n\to\infty} \Psi(t)X_n(t) = \Psi(t)X(t),\ \text{uniformly on } \mathbb{R}_+.$$

Let $R > 0$ such that

$$\mid \Psi(0)X_n(0) \mid \ <\ R,\ \text{for all } n \in N.$$

From (14),

$$\mid \Psi(t)\,(X_n(t) - X(t)) \mid \ \leq\ \mid \Psi(t)X_n(t) \mid + \mid \Psi(t)X(t) \mid \leq$$
$$\leq Ne^{-ct}\,(\mid \Psi(0)X_n(0) \mid + \mid \Psi(0)X(0) \mid) \leq 2RNe^{-ct},$$

for all $t \geq 0$ and $n \in N$.

Thus, the solution $X(t)$ is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof of the first part is complete.

The last part results from the above, if we put $XB(t) + F(t, X)$ instead of $F(t, X)$.

The proof is complete.

Remark 5.1. The Theorem contains as a particular case a result concerning $\Psi-$ conditional exponential asymptotic stability of all $\Psi-$ bounded solutions of the differential system

$$x' = A(t)x + f(t, x).$$

Indeed, consider in (11)

$$X = \begin{pmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \cdots & \cdots & \cdots & \cdots \\ x_d & x_d & \cdots & x_d \end{pmatrix}$$

and

$$F(t,X) = \begin{pmatrix} f_1(t,x) & f_1(t,x) & \cdots & f_1(t,x) \\ f_2(t,x) & f_2(t,x) & \cdots & f_2(t,x) \\ \cdots & \cdots & \cdots & \cdots \\ f_d(t,x) & f_d(t,x) & \cdots & f_d(t,x) \end{pmatrix},$$

where $x = (x_1, x_2, \cdots, x_d)^T$ and $f(t,x) = (f_1(t,x), f_2(t,x), \cdots, f_d(t,x))^T$.

Now, the definitions and conditions for $\Psi-$ boundedness or $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of x are the same for $\Psi-$ boundedness or $\Psi-$ conditional exponential asymptotic stability on $\mathbb{R}_+$ of X.

We mention that in Theorem 8, [13], there exist a result concerning $\Psi-$ conditional exponential asymptotic stability of all $\Psi-$ bounded solutions of the nonlinear Volterra integro-differential system

$$x' = A(t)x + \int_0^t F(t,s,x(s))ds.$$

Corollary 5.1. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0, i = 1,2$ *and positive constants* K_1, K_2, λ_1, λ_2 *such that the fundamental matrix* $U(t)$ *for the equation (4),* $U(0) = I_d$, *satisfies the conditions:*

$$\begin{cases} \mid \Psi(t)U(t)P_1U^{-1}(s)\Psi^{-1}(s) \mid & \leq K_1 e^{-\lambda_1(t-s)} \quad \text{for } 0 \leq s \leq t \\ \mid \Psi(t)U(t)P_2U^{-1}(s)\Psi^{-1}(s) \mid & \leq K_2 e^{-\lambda_2(s-t)} \quad \text{for } 0 \leq t \leq s \end{cases};$$

2). The continuous matrix $B(t)$ *satisfies the condition*

$$\sup_{t \geq 0} \mid B(t) \mid < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2} \right)^{-1}.$$

Then, the linear Lyapunov matrix differential equation (3) is $\Psi-$ *conditionally exponentially asymptotically stable on* R_+.

Proof. It results from the above Theorem, if we take $F(t,X) = O_d$.

Corollary 5.2. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0, i = 1,2$ *and positive constants* K_1, K_2, λ_1, λ_2 *such that:*

$$\begin{cases} \mid \Psi(t)P_1\Psi^{-1}(s) \mid & \leq \ K_1 e^{-\lambda_1(t-s)} \quad for\ 0 \leq s \leq t \\ \mid \Psi(t)P_2\Psi^{-1}(s) \mid & \leq \ K_2 e^{-\lambda_2(s-t)} \quad for\ 0 \leq t \leq s \end{cases};$$

2). The continuous function $F : R_+ \times M_{d\times d} \longrightarrow M_{d\times d}$ is such that $F(t, O_d) = O_d$ and satisfies the Lipschitz condition

$$\mid \Psi(t)\left(F(t, X_1) - F(t, X_2)\right) \mid \leq \gamma(t) \mid \Psi(t)(X_1 - X_2) \mid,$$

for all $t \geq 0$ and $X_1, X_2 \in M_{d\times d}$, where $\gamma : R_+ \longrightarrow R_+$ is a continuous function.

3). The continuous matrix function $A(t)$ satisfies the condition

$$\sup_{t\geq 0} \left(\mid \Psi(t)A(t)\Psi^{-1}(t) \mid \ + \ \gamma(t)\right) < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}.$$

Then, all $\Psi-$ bounded solutions of the equation (11) are $\Psi-$ conditionally exponentially exponentially asymptotically stable on R_+.

Proof. It results from the above Theorem, if we take O_d instead of $A(t)$ and $A(t)X + F(t, X)$ instead of $F(t, X)$.

Corollary 5.3. *If in the above Corollary, the hypothesis 3) is replaced with*

3'). the continuous matrices $A(t)$ and $B(t)$ satisfy the condition

$$\sup_{t\geq 0} \left(\mid \Psi(t)A(t)\Psi^{-1}(t) \mid \ + \ \mid B(t) \mid \ + \ \gamma(t)\right) < \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1},$$

then, all $\Psi-$ bounded solutions of the the nonlinear Lyapunov matrix differential equation (1) are $\Psi-$ conditionally exponentially asymptotically stable on R_+.

Proof. It results from the above Corollary, if we take O_d instead of $A(t)$ and $A(t)X + XB(t) + F(t, X)$ instead of $F(t, X)$.

Theorem 5.2. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrices $X(t)$ and $Y(t)$ (with $X(0) = Y(0) = I_d$) for the equations (4) and (5) respectively satisfy the condition

$$\begin{cases} \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \mid & \leq \ K_1 e^{-\lambda_1(t-s)}, \\ & for\ 0 \leq s \leq t \\ \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \mid & \leq \ K_2 e^{-\lambda_2(s-t)}, \\ & for\ 0 \leq t \leq s \end{cases};$$

2). The continuous function $F : \mathbb{R}_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ is such that $F(t, O_d) = O_d$ and satisfies the Lipschitz condition

$$| \Psi(t) \left(F(t, X_1) - F(t, X_2)\right) | \;\; \leq \; \gamma(t) \mid \Psi(t)(X_1 - X_2) \mid,$$

for all $t \geq 0$ and $X_1, X_2 \in \mathbb{M}_{d\times d}$, where $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function such that

$$\sup_{t\geq 0} \gamma(t) < d^{-1} \left(\frac{K_1}{\lambda_1} + \frac{K_2}{\lambda_2}\right)^{-1}.$$

Then, the trivial solution of the equation (1) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

Proof. From Lemma 2.7, we know that the trivial solution of the equation (*1*) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$ if and only if the trivial solution of the corresponding Kronecker product system (6) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

In system (6), we will apply the Theorem 5.1, variant for a differential system (see Remark 5.1).

From Lemma 2.8, we know that the matrix $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear homogeneous system associated with (6), i.e. for the differential system (8).

The hypothesis 1 ensures the hypothesis 1 of Theorem 5.1 (with $I_d \otimes \Psi$ instead of Ψ).

Now, let

$$f(t, x) = \mathcal{V}ec\left(F(t, X)\right), \; x = \mathcal{V}ec(X),$$

for $t \in \mathbb{R}_+$ and $X \in \mathbb{M}_{d\times d}$.

From hypothesis 2 and Lemma 2.6, it follows that

$$\begin{aligned}
&\| \left(I_d \otimes \Psi(t)\right)\left(f(t, x_1) - f(t, x_2)\right) \|_{\mathbb{R}^{d^2}} = \\
&= \| \left(I_d \otimes \Psi(t)\right)\left(\mathcal{V}ec\left(F(t, X_1)\right) - \mathcal{V}ec\left(F(t, X_2)\right)\right) \|_{\mathbb{R}^{d^2}} = \\
&= \| \left(I_d \otimes \Psi(t)\right)\mathcal{V}ec\left(F(t, X_1) - F(t, X_2)\right) \|_{\mathbb{R}^{d^2}} \leq \\
&\leq | \Psi(t)\left(F(t, X_1) - F(t, X_2)\right) | \leq \gamma(t) \mid \Psi(t)\left(X_1 - X_2\right) \mid \leq \\
&\leq d\gamma(t) \| \left(I_d \otimes \Psi(t)\right)\mathcal{V}ec(X_1 - X_2) \|_{\mathbb{R}^{d^2}} = \\
&\leq d\gamma(t) \| \left(I_d \otimes \Psi(t)\right)\left(\mathcal{V}ec\left(X_1\right) - \mathcal{V}ec\left(X_2\right)\right) \|_{\mathbb{R}^{d^2}} = \\
&= d\gamma(t) \| \left(I_d \otimes \Psi(t)\right)\left(x_1 - x_2\right) \|_{\mathbb{R}^{d^2}},
\end{aligned}$$

for all $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^{d^2}$.

Thus, is ensured the hypothesis 2 of Theorem 5.1.

From Theorem 5.1, variant for differential systems, the trivial solution of the system (6) is $I_d \otimes \Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$. From Lemma 2.7 again, results that the trivial solution of (1) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

The proof is now complete.

Remark 5.2. For $F(t,X) = O_d$, one obtain Theorem 4.5.

Corollary 5.4. *Suppose that:*

1). There exist supplementary projections $P_i : R^d \longrightarrow R^d$, $P_i \neq 0$ and positive constants K_1, K_2, λ_1, λ_2 such that the fundamental matrices $X(t)$ and $Y(t)$ (with $X(0) = Y(0) = I_d$) for the equations (4) and (5) respectively satisfy the condition

$$\begin{cases} \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \mid & \leq K_1e^{-\lambda_1(t-s)}, \\ & \textit{for } 0 \leq s \leq t \\ \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \mid & \leq K_2e^{-\lambda_2(s-t)}, \\ & \textit{for } 0 \leq t \leq s \end{cases};$$

2). $A_1(t)$ and $B_1(t)$ are continuous $d \times d$ matrices functions on $\mathbb{R}_+$ and satisfy the condition

$$\sup_{t\geq 0}\left[\mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid + \mid B_1(t) \mid\right] < d^{-1}\left(\frac{K_1}{\lambda_1}+\frac{K_2}{\lambda_2}\right)^{-1}.$$

Then, the equation (2) is $\Psi-$ conditionally exponentially asymptotically stable on $\mathbb{R}_+$.

Proof. It results from the above Theorem.

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