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Perturbed Companions of Ostrowski's Inequality for Absolutely Continuous Functions (I)

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Abstract. Perturbed companions of Ostrowski's inequality for absolutely continuous functions whose derivatives are either bounded or of bounded variation and applications are given.

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1 Introduction

In [16] we established the following companion of Ostrowski inequality [26] for Lebesgue *sup-norm*:

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous function on*

$[a,b]$. *If* $f' \in L_\infty [a,b]$, *then we have the inequalities*

$$\left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{b-a} \left[\frac{(x-a)^2}{2} \|f'\|_{[a,x],\infty} + \left(\frac{a+b}{2} - x \right)^2 \|f'\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2} \|f'\|_{[a+b-x,b],\infty} \right] \quad (1.1)$$

$$\le \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_{[a,b],\infty} \\ \\ \left[\frac{1}{2^{\alpha-1}} \left(\frac{x-a}{b-a} \right)^{2\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right]^{\frac{1}{\alpha}} \\ \times \left[\|f'\|_{[a,x],\infty}^{\beta} + \|f'\|_{[x,a+b-x],\infty}^{\beta} + \|f'\|_{[a+b-x,b],\infty}^{\beta} \right]^{\frac{1}{\beta}} (b-a) \\ \qquad \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \max \left\{ \frac{1}{2} \left(\frac{x-a}{b-a} \right)^2, \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right\} \\ \times \left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,a+b-x],\infty} + \|f'\|_{[a+b-x,b],\infty} \right] (b-a) \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$, *where*

$$\|g\|_{[c,d],\infty} := ess \sup_{t \in [c,d]} |g(s)|.$$

The inequality (1.1), the first inequality in (1.1) and the constant $\frac{1}{8}$ *are sharp.*

If in Theorem 1 we choose $x = a$, then we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{4} (b-a) \|f'\|_{[a,b],\infty} \quad (1.2)$$

with $\frac{1}{4}$ as a sharp constant (see for example [20, p. 25]).

If in the same theorem we now choose $x = \frac{a+b}{2}$, then we get

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} \tag{1.3}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [15] by a different argument.

It is natural to consider the following corollary.

Corollary 2. *With the assumptions in Theorem 1, one has the inequality:*

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty}. \tag{1.4}$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In the same paper [16] we established the corresponding inequalities for Lebesgue p-norms with $p \ge 1$ as well as have provided some applications for cumulative distribution functions and some quadrature rules.

For a monograph devoted to Ostrowski type inequalities, see [20].

For research papers on Ostrowski's inequality see [1]-[19], [21]-[23] and [25].

Motivated by the above results, we investigate in this paper some perturbed versions of the inequality (1.1). Applications for cumulative distribution function are provided as well.

2 Some Identities

The following identity holds.

Lemma 3. *Assume that $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function* *on* $[a,b]$. *Then we have the equality*

$$\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.1}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_3(x)\right]dt,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_j(x)$, $j=1,2,3$ *complex numbers.*

Proof. Using the integration by parts formula for Lebesgue integral, we have

$$\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt$$
$$=\int_a^x (t-a)f'(t)\,dt-\lambda_1(x)\int_a^x (t-a)\,dt$$
$$=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}\lambda_1(x)(x-a)^2,$$

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt$$
$$=\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt$$
$$=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x}f(t)\,dt$$
$$-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt$$
$$=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x}f(t)\,dt,$$

since, by symmetry

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt=0$$

and

$$
\begin{aligned}
&\int_{a+b-x}^{b} (t-b)\left[f'(t)-\lambda_3(x)\right] dt \\
&= \int_{a+b-x}^{b} (t-b) f'(t)\, dt - \lambda_3(x) \int_{a+b-x}^{b} (t-b)\, dt \\
&= (x-a) f(a+b-x) - \int_{a+b-x}^{b} f(t)\, dt + \frac{1}{2}\lambda_3(x)(x-a)^2 .
\end{aligned}
$$

Summing the above equalities, we deduce

$$
\begin{aligned}
&\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right] dt + \int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right] dt \\
&+ \int_{a+b-x}^{b} (t-b)\left[f'(t)-\lambda_3(x)\right] dt \\
&= (b-a)\frac{f(x)+f(a+b-x)}{2} - \int_a^b f(t)\, dt + \frac{1}{2}\left[\lambda_3(x)-\lambda_1(x)\right](x-a)^2 ,
\end{aligned}
$$

which is equivalent with the desired identity (2.1). ∎

The following particular cases are of interest:

Corollary 4. *With the assumption of Lemma 3 we have the equalities*

$$
\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\, dt = \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right] dt, \tag{2.2}
$$

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_3-\lambda_1) - \frac{1}{b-a}\int_a^b f(t)\, dt \qquad (2.3)\\
&= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\lambda_1\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} (t-b)\left[f'(t)-\lambda_3\right] dt,
\end{aligned}
$$

and

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.4}$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_3\right]dt$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

Corollary 5. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$. *Then we have the equality*

$$\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.5}$$
$$=\frac{1}{b-a}\int_a^{x}(t-a)\left[f'(t)-\lambda_1(x)\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^{b}(t-b)\left[f'(t)-\lambda_1(x)\right]dt,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2$ *complex numbers.*

Remark 1. *We get from (2.3) the following particular case:*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.6}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}(t-b)\left[f'(t)-\lambda_1\right]dt,$$

for any $\lambda_1 \in \mathbb{C}$*, while from (2.4) we get*

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.7}$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_1\right]dt$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$*.*

3 Inequalities for Bounded Derivatives

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma)$$
$$:=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 6. *For any* $\gamma, \Gamma \in \mathbb{C}$*,* $\gamma \neq \Gamma$*, we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Gamma-\gamma\right|^{2}-\left|z-\frac{\gamma+\Gamma}{2}\right|^{2}=\operatorname{Re}\left[\left(\Gamma-z\right)\left(\bar{z}-\bar{\gamma}\right)\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. ∎

On making use of the complex numbers field properties we can also state that:

Corollary 7. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}\left(\gamma,\Gamma\right)=\{f:[a,b]\to\mathbb{C}\mid\ &\left(\operatorname{Re}\Gamma-\operatorname{Re}f\left(t\right)\right)\left(\operatorname{Re}f\left(t\right)-\operatorname{Re}\gamma\right)\\&+\left(\operatorname{Im}\Gamma-\operatorname{Im}f\left(t\right)\right)\left(\operatorname{Im}f\left(t\right)-\operatorname{Im}\gamma\right)\geq 0 \text{ for a.e. } t\in[a,b]\}.\end{aligned}\tag{3.2}$$

Now, if we assume that $\operatorname{Re}\left(\Gamma\right)\geq\operatorname{Re}\left(\gamma\right)$ and $\operatorname{Im}\left(\Gamma\right)\geq\operatorname{Im}\left(\gamma\right)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}\left(\gamma,\Gamma\right):=\{f:[a,b]\to\mathbb{C}\mid\ &\operatorname{Re}\left(\Gamma\right)\geq\operatorname{Re}f\left(t\right)\geq\operatorname{Re}\left(\gamma\right)\\&\text{and }\operatorname{Im}\left(\Gamma\right)\geq\operatorname{Im}f\left(t\right)\geq\operatorname{Im}\left(\gamma\right)\text{ for a.e. }t\in[a,b]\}.\end{aligned}\tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}\left(\gamma,\Gamma\right)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}\left(\gamma,\Gamma\right)\subseteq\bar{U}_{[a,b]}\left(\gamma,\Gamma\right).\tag{3.4}$$

Theorem 8. *Assume that $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$ and $x\in\left[a,\frac{a+b}{2}\right]$. If there exist the complex numbers $\gamma_j\left(x\right)\neq\Gamma_j\left(x\right)$, $j=1,2,3$ such that*

$$\begin{aligned}f'\in\ &\bar{\Delta}_{[a,x]}\left(\gamma_1\left(x\right),\Gamma_1\left(x\right)\right)\cap\bar{\Delta}_{[x,a+b-x]}\left(\gamma_2\left(x\right),\Gamma_2\left(x\right)\right)\\&\cap\bar{\Delta}_{[a+b-x,b]}\left(\gamma_3\left(x\right),\Gamma_3\left(x\right)\right),\end{aligned}\tag{3.5}$$

then we have the inequality

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{4}\left(x-a\right)^{2}\frac{\gamma_3\left(x\right)+\Gamma_3\left(x\right)-\gamma_1\left(x\right)-\Gamma_1\left(x\right)}{b-a}\right.\\&\left.-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\\&\leq\frac{1}{4\left(b-a\right)}\left[\left|\Gamma_1\left(x\right)-\gamma_1\left(x\right)\right|\left(x-a\right)^{2}+2\left|\Gamma_2\left(x\right)-\gamma_2\left(x\right)\right|\left(\frac{a+b}{2}-x\right)^{2}\right.\\&\left.+\left|\Gamma_3\left(x\right)-\gamma_3\left(x\right)\right|\left(x-a\right)^{2}\right].\end{aligned}\tag{3.6}$$

Proof. Taking the modulus in the equality (2.1) written for

$$\lambda_1(x) = \frac{\gamma_1(x)+\Gamma_1(x)}{2}, \lambda_2(x) = \frac{\gamma_2(x)+\Gamma_2(x)}{2},$$
$$\lambda_3(x) = \frac{\gamma_3(x)+\Gamma_3(x)}{2}$$

and utilizing the condition (3.5) we have

$$\begin{aligned}
&\left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{4}(x-a)^2 \frac{\gamma_3(x)+\Gamma_3(x)-\gamma_1(x)-\Gamma_1(x)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{\gamma_1(x)+\Gamma_1(x)}{2}\right| dt\\
&+\frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-\frac{\gamma_2(x)+\Gamma_2(x)}{2}\right| dt\\
&+\frac{1}{b-a}\int_{a+b-x}^b (b-t)\left|f'(t)-\frac{\gamma_3(x)+\Gamma_3(x)}{2}\right| dt\\
&\le \frac{1}{4(b-a)}\left|\Gamma_1(x)-\gamma_1(x)\right|(x-a)^2\\
&+\frac{2}{4(b-a)}\left|\Gamma_2(x)-\gamma_2(x)\right|\left(\frac{a+b}{2}-x\right)^2\\
&+\frac{1}{4(b-a)}\left|\Gamma_3(x)-\gamma_3(x)\right|(x-a)^2
\end{aligned}$$

and the inequality (3.6) is proved. ∎

Corollary 9. *Assume that $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$ and $x\in\left[a,\frac{a+b}{2}\right]$. If there exist the complex numbers $\gamma_j(x)\ne\Gamma_j(x)$, $j=1,2$ such that*

$$\begin{aligned}
f' \in\ & \bar{\Delta}_{[a,x]}(\gamma_1(x),\Gamma_1(x)) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2(x),\Gamma_2(x))\\
&\cap \bar{\Delta}_{[a+b-x,b]}(\gamma_1(x),\Gamma_1(x)),
\end{aligned} \tag{3.7}$$

then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{2}[f(x)+f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2(b-a)}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|(x-a)^2 + \left|\Gamma_2(x)-\gamma_2(x)\right|\left(\frac{a+b}{2}-x\right)^2\right].
\end{aligned} \tag{3.8}$$

Remark 2. *Assume that $f : [a, b] \to \mathbb{C}$ is an absolutely continuous function on $[a, b]$.*

If there exist the complex numbers $\gamma_2 \neq \Gamma_2$ such that $f' \in \bar{\Delta}_{[a,b]}(\gamma_2, \Gamma_2)$, then

$$\left| \frac{1}{2} [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{8} (b-a) |\Gamma_2 - \gamma_2| . \tag{3.9}$$

If there exist the complex numbers $\gamma_j \neq \Gamma_j$, $j = 1, 3$ such that

$$f' \in \bar{\Delta}_{\left[a, \frac{a+b}{2}\right]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{\left[\frac{a+b}{2}, b\right]}(\gamma_3, \Gamma_3),$$

then we have the inequality

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)(\Gamma_3 + \gamma_3 - \Gamma_1 - \gamma_1) - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \tag{3.10}$$
$$\leq \frac{1}{16}(b-a)\left[|\Gamma_1 - \gamma_1| + |\Gamma_3 - \gamma_3|\right].$$

In particular, if $f' \in \bar{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$ then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \leq \frac{1}{8}(b-a)|\Gamma_1 - \gamma_1| . \tag{3.11}$$

If there exist the complex numbers $\gamma_j \neq \Gamma_j$, $j = 1, 2, 3$ such that

$$f' \in \bar{\Delta}_{\left[a, \frac{3a+b}{4}\right]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{\left[\frac{a+3b}{4}, b\right]}(\gamma_3, \Gamma_3),$$

then

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{64}(b-a)(\Gamma_3 + \gamma_3 - \Gamma_1 - \gamma_1) \right. \tag{3.12}$$
$$\left. - \frac{1}{b-a}\int_a^b f(t)\, dt \right|$$
$$\leq \frac{1}{64}(b-a)\left[|\Gamma_1 - \gamma_1| + 2|\Gamma_2 - \gamma_2| + |\Gamma_3 - \gamma_3|\right].$$

In particular, if $\gamma_3 = \gamma_1$ and $\Gamma_3 = \Gamma_1$, then

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \tag{3.13}$$
$$\leq \frac{1}{32}(b-a)\left[|\Gamma_1 - \gamma_1| + |\Gamma_2 - \gamma_2|\right],$$

provided

$$f' \in \bar{\Delta}_{\left[a,\frac{3a+b}{4}\right]}\left(\gamma_1,\Gamma_1\right) \cap \bar{\Delta}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}\left(\gamma_2,\Gamma_2\right) \cap \bar{\Delta}_{\left[\frac{a+3b}{4},b\right]}\left(\gamma_1,\Gamma_1\right).$$

Moreover, if $f' \in \bar{\Delta}_{[a,b]}\left(\gamma_1,\Gamma_1\right)$ *then*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)\left|\Gamma_1-\gamma_1\right|. \tag{3.14}$$

The case of real-valued functions is of interest.

Remark 3. *If the function* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous and if there exist the constants* $l<L$ *such that* $l \le f'(t) \le L$ *for almost every* $t\in[a,b]$, *then we have the inequalities*

$$\left|\frac{1}{2}\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)(L-l), \tag{3.15}$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)(L-l) \tag{3.16}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)(L-l). \tag{3.17}$$

These results improve the corresponding inequalities from Introduction.

4 Inequalities for Derivatives of Bounded Variation

Assume that $f:I\to\mathbb{C}$ is an absolutely continuous function on $[a,b]\subset \mathring{I}$, the interior of I. Then from (2.1) we have for $\lambda_1(x)=f'(a)$, $\lambda_2(x)=$

$\frac{f'(x)+f'(a+b-x)}{2}$ and $\lambda_3(x) = f'(b)$ the equality

$$\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{4.1}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-f'(b)\right]dt,$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

We can state the following result.

Theorem 10. *Assume that $f : I \to \mathbb{C}$ is an absolutely continuous function on $[a,b] \subset \mathring{I}$. If the derivative f' is of bounded variation on $[a,b]$, then*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4.2}$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t (f')\,dt+\frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f')$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b (f')\,dt$$

$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x (f')+\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f')+(x-a)^2\bigvee_{a+b-x}^b (f')\right]$$
$$\leq \begin{cases} \frac{1}{2(b-a)}\max\left\{(x-a)^2,\left(x-\frac{a+b}{2}\right)^2\right\}\bigvee_a^b (f') \\[2ex] \frac{1}{2(b-a)}\max\left\{\bigvee_a^x (f'),\bigvee_x^{a+b-x}(f'),\bigvee_{a+b-x}^b (f')\right\}\left[2(x-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. If we take the modulus in (4.1) we get

$$\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{2}\left(x-a\right)^{2}\frac{f'\left(b\right)-f'\left(a\right)}{b-a}-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right| \tag{4.3}$$
$$\leq\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\left|f'\left(t\right)-f'\left(a\right)\right|dt$$
$$+\frac{1}{b-a}\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'\left(t\right)-\frac{f'\left(x\right)+f'\left(a+b-x\right)}{2}\right|dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^{b}\left(b-t\right)\left|f'\left(t\right)-f'\left(b\right)\right|dt:=K,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Let $x\in\left(a,\frac{a+b}{2}\right)$. Since f' is of bounded variation on $[a,b]$, then

$$\left|f'\left(t\right)-f'\left(a\right)\right|\leq\bigvee_{a}^{t}\left(f'\right),$$

for any $t\in[a,x]$ and

$$\left|f'\left(t\right)-\frac{f'\left(x\right)+f'\left(a+b-x\right)}{2}\right|$$
$$=\left|\frac{f'\left(t\right)-f'\left(x\right)+f'\left(t\right)-f'\left(a+b-x\right)}{2}\right|$$
$$\leq\frac{1}{2}\left[\left|f'\left(t\right)-f'\left(x\right)\right|+\left|f'\left(a+b-x\right)-f'\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_{x}^{a+b-x}\left(f'\right)$$

for any $t\in[x,a+b-x]$.

We also have

$$\left|f'\left(t\right)-f'\left(b\right)\right|\leq\bigvee_{t}^{b}\left(f'\right),\ t\in[a+b-x,b].$$

Then we get

$$
\begin{aligned}
K &\leq \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t (f')\,dt + \frac{1}{2(b-a)}\bigvee_x^{a+b-x}(f')\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\
&+ \frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b (f')\,dt \\
&\leq \frac{1}{b-a}\bigvee_a^x (f')\int_a^x (t-a)\,dt + \frac{1}{2(b-a)}\bigvee_x^{a+b-x}(f')\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\
&+ \frac{1}{b-a}\bigvee_{a+b-x}^b (f')\int_{a+b-x}^b (b-t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2(b-a)}(x-a)^2\bigvee_a^x (f') + \frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') \\
&+ \frac{1}{2(b-a)}(x-a)^2\bigvee_{a+b-x}^b (f'),
\end{aligned}
$$

which proves the first two inequalities in (4.2).

The last part is obvious by the maximum properties. ∎

Corollary 11. *With the assumptions of Theorem 10 we have*

$$
\left|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)\bigvee_a^b (f'), \tag{4.4}
$$

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)[f'(b)-f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\bigvee_a^t (f')\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\bigvee_t^b (f')\,dt \\
&\leq \frac{1}{8}(b-a)\bigvee_a^b (f')
\end{aligned} \tag{4.5}
$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)\left[f'(b)-f'(a)\right]\right. \tag{4.6}$$
$$\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\bigvee_a^t(f')\,dt+\frac{1}{32}(b-a)\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f')$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b(b-t)\bigvee_t^b(f')\,dt$$
$$\leq \frac{1}{32}(b-a)\bigvee_a^b(f').$$

5 Applications for PDF

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the *probability density function* (PDF) $f:[a,b]\to[0,\infty)$ and with the *cumulative distribution function* (CDF) $F(x)=\Pr(X\leq x)=\int_a^x f(t)\,dt$. We know that F is monotonic nondecreasing and absolutely continuous on $[a,b]$, $F'=f$ almost everywhere on $[a,b]$ and $F(a)=0$, $F(b)=\int_a^b f(t)\,dt=1$.

Assume that $g:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ and there exist the constants $m<M$ such that

$$m\leq g'(t)\leq M \text{ for almost every } t\in[a,b]$$

then, by Corollary 9, we have the inequality

$$\left|\frac{1}{2}\left[g(x)+g(a+b-x)\right]-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{5.1}$$
$$\leq\frac{1}{2}(M-m)\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proposition 12. *Let X be a random variable taking values in the finite interval $[a,b]$, with PDF $f:[a,b]\to[0,\infty)$ and with CDF $F(x)=\Pr(X\le x)=\int_a^x f(t)\,dt$. If there exist the constants $m<M$ such that*

$$m \le f(t) \le M \text{ for almost every } t\in[a,b]$$

then,

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \le \frac{1}{2}(M-m)\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a) \tag{5.2}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $E(X)=\int_a^b t\,dF(t)$ is the expectation of X.

Proof. Follows from (5.1) for $g=F$ and by taking into account that

$$\int_a^b F(t)\,dt = b-E(X).$$

∎

Corollary 13. *With the assumptions in Proposition 12, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right| \le \frac{1}{16}(M-m)(b-a) \tag{5.3}$$

Utilising Theorem 10 we can also state:

Proposition 14. *If PDF $f:[a,b]\to[0,\infty)$ is of bounded variation on $[a,b]$,* *then*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f(b)-f(a)}{b-a}-\frac{b-E(X)}{b-a}\right| \qquad (5.4)\\
&\leq \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t (f)\,dt+\frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f)\\
&+\frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b (f)\,dt\\
&\leq \frac{1}{2(b-a)}\\
&\times\left[(x-a)^2\bigvee_a^x (f)+\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f)+(x-a)^2\bigvee_{a+b-x}^b (f)\right]\\
&\leq \frac{1}{2(b-a)}\\
&\times\begin{cases}\max\left\{(x-a)^2,\left(x-\frac{a+b}{2}\right)^2\right\}\bigvee_a^b (f),\\ \\ \max\left\{\bigvee_a^x (f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b (f)\right\}\left[2(x-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\end{cases}
\end{aligned}
$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Finally we have:

Corollary 15. *With the assumptions in Proposition 14, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)[f(b)-f(a)]\right. \tag{5.5}$$
$$\left.-\frac{b-E(X)}{b-a}\right|$$
$$\leq \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\bigvee_a^t(f)\,dt+\frac{1}{32}(b-a)\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f)$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(b-t)\bigvee_t^b(f)\,dt$$
$$\leq \frac{1}{32}(b-a)\bigvee_a^b(f).$$

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