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The hyperstability of AQ-Jensen functional equation on 2-divisible abelian group and inner product spaces

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Abstract. In this paper, we prove the hyperstability of the following mixed additive-quadratic-Jensen functional equation

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y)$$

in the class of functions from an 2-divisible abelian group G into a Banach space.

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1 Introduction and Preliminaries

The study of stability problems for functional equations is related to a question of Ulam [29] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [16]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings. In 1978, Th.

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M. Rassias [27] generalized the result of Hyers by considering the stability problem for unbounded Cauchy differences. This phenomenon of stability introduced by Th. M. Rassias [27] is called the Hyers-Ulam-Rassias stability.

Theorem 1.1 ([27, Th. M. Rassias])**.** *Let $f : E_1 \to E_2$ be a mapping from a real normed vector space E_1 into a Banach space E_2 satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{1.1}$$

for all $x, y \in E_1$, where θ and p are constants with $\theta > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p, \tag{1.2}$$

for all $x \in E_1$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \to f(tx)$ from $\mathbb{R}$ into E_2 is continuous for each fixed $x \in E_1$, then T is linear.

In 1994, a generalization of Rassias' theorem was obtained by Găvruta [13], who replaced $\theta(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 11, 14, 17–22, 24, 25, 28]).

Recently, interesting results concerning additive-quadratic-Jensen type functional equation (briefly, AQ-Jensen functional equation)

$$2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y), \tag{1.3}$$

have been obtained in [1] and [30].

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. The term hyperstability was used for first time probably in [23]. However, it seems that the first hyperstability result was published in [5] and concerned the ring homomorphisms. The hyperstability results for Cauchy equation were investigated by Brzdek in [7, 8, 10]. Gselmann in [15] studied the hyperstability of the parametric fundamental equation of information. In [4] Bahyrycz and Piszczek provided the hyperstability of the Jensen functional equation. In [12] EL-Fassi and Kabbaj studied the hyperstability of a Cauchy-Jensen type functional equation in Banach spaces.

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}^*$ be the set of positive integers. We note that $\mathbb{N}_{m_0}$ (with $m_0 \in \mathbb{N}^*$) the set of all integers greater than or equal to m_0. Let $\mathbb{R}_+ := [0, \infty)$ be the set of nonnegative real numbers and $\mathbb{R}_{*+} := (0, \infty)$ the set of positive real numbers and Y^X denotes the family of all functions mapping from a nonempty set X into a nonempty set Y.

J. Brzdek and K. Ciepliński [9] introduced the following definition, which describes the main ideas of such a hyperstability notion for equations in several variables.

Definition 1.1. *Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon : X^n \to \mathbb{R}_+$ (with $n \in \mathbb{N}^*$) be an arbitrary function, and let $\mathcal{F}_1$, $\mathcal{F}_2$ be two operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation*

$$\mathcal{F}_1\varphi(x_1, ..., x_n) = \mathcal{F}_2\varphi(x_1, ..., x_n), \quad (x_1, ..., x_n \in X) \tag{1.4}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d(\mathcal{F}_1\varphi_0(x_1, ..., x_n), \mathcal{F}_2\varphi_0(x_1, ..., x_n)) \leq \varepsilon(x_1, ..., x_n), \quad (x_1, ..., x_n \in X)$$

fulfills equation (1.4) on X.

In this article, we introduce the following definition, which describes the main ideas of the concept of hyperstability for equations in several variables.

Definition 1.2. *Let X be a nonempty set, (Y, d) be a metric space, $\Sigma \subset \mathbb{R}_{*+}^{X^n}$ be a nonempty subset and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n} We say that the operator equation*

$$\mathcal{F}_1\varphi(x_1, ..., x_n) = \mathcal{F}_2\varphi(x_1, ..., x_n), \quad (x_1, ..., x_n \in X) \tag{1.5}$$

is Σ-hyperstable for the pair (X, Y) provided for any $\varepsilon \in \Sigma$ and $\varphi_0 \in \mathcal{D}$ satisfies the inequality

$$d(\mathcal{F}_1\varphi_0(x_1, ..., x_n), \mathcal{F}_2\varphi_0(x_1, ..., x_n)) \leq \varepsilon(x_1, ..., x_n), \quad (x_1, ..., x_n \in X)$$

fulfills equation (1.5) on X.

A function $H : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called homogeneous of degree a real number p if it satisfies $H(tu, tv) = t^p H(u, v)$ for all $t \in \mathbb{R}_{*+}$ and $u, v \in \mathbb{R}_+$. In the sequel, we assume that $G = (G, +)$ is an 2-divisible abelian group, E is an arbitrary real Banach space, $H : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a symmetric homogeneous function of degree $p < 0$ for which there exists a positive integer m_0 such that

$$\inf\{\varepsilon(x, mx) : m \in \mathbb{N}_{m_0}\} = 0 \tag{1.6}$$

for all $x \in G$ and $\gamma : G \to \mathbb{R}_{*+}$ is a function satisfying

(C1) $\gamma\left(\frac{kx}{2}\right) = \frac{|k|}{2}\gamma(x)$ for all $x \in G$ and $k \in \mathbb{Z}\backslash\{0\}$,

(C2) $\gamma(x+y) \leq \gamma(x) + \gamma(y)$ for all $x, y \in G$.

We will denote by Σ the set of all functions $\varepsilon : G^2 \to \mathbb{R}_+$ for which there exists a constant $c \in \mathbb{R}_+$ such that

$$\varepsilon(x, y) = cH(\gamma(x), \gamma(y)) \;\; x, y \in G. \tag{1.7}$$

By the conditions **(C1)** and **(C2)**, we notice that:

$$\varepsilon\left(\frac{kx}{2}, \frac{ky}{2}\right) = \left|\frac{k}{2}\right|^p \varepsilon(x, y)$$

for all $x, y \in G$ and $k \in \mathbb{Z}\backslash\{0\}$.

In this paper, we present the hyperstability results for the mixed additive-quadratic-Jensen type functional equation (1.3) in the class of functions from an 2-divisible commutative group $(G, +)$ into a Banach space E.

The method of the proof of the main results is motivated by an idea used in [7–10, 26]. It is based on a fixed point theorem for functional spaces obtained by Brzdek et al. (see [6, Theorem 1]).

First, we take the following three hypotheses (all notations come from [6]).

(H1) U is a nonempty set, V is a Banach space, $f_1, ...f_k : U \to U$ and $L_1, ...L_k : U \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : V^U \to V^U$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in V^U$, $x \in U$.

(H3) $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x))$$

for all $\delta \in \mathbb{R}_+^U$, $x \in U$.

The mentioned fixed point theorem is stated as follows.

Theorem 1.2. *Let hypotheses* ***(H1)-(H3)*** *be valid and functions* $\varepsilon : U \to \mathbb{R}_+$ *and let* $\varphi : U \to V$ *fulfil the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in U$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in U.$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in U.$$

Moreover

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \quad x \in U.$$

2 Hyperstability Results of eq (1.3)

The following theorems are the main results in this paper and concern the $\Sigma-$hyperstability of equation (1.3).

Theorem 2.1. *Let* G *be an 2-divisible abelian group and* E *be a Banach space. If* $f : G \to E$ *satisfies*

$$\left\|2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(y) - f(x)\right\| \leq \varepsilon(x,y) \quad (2.1)$$

for all $x, y \in G$ *and* $\varepsilon \in \Sigma$*, then* f *is a solution of (1.3) on* G.

Proof. Let $\varepsilon \in \Sigma$, then there exists a constant $c \in \mathbb{R}_+$ such that $\varepsilon(x,y) = cH(\gamma(x),\gamma(y))$. Replacing (x,y) by (x,mx), with $m \in \mathbb{N}_1$, in (2.1), we get

$$\left\|2f\left(\frac{1+m}{2}x\right) + f\left(\frac{1-m}{2}x\right) + f\left(\frac{m-1}{2}x\right) - f(mx) - f(x)\right\|$$
$$\leq \varepsilon(x,mx) = cH(\gamma(x),\gamma(mx)) := \varepsilon_m(x) \quad (2.2)$$

for all $x \in G$. Further put

$$\mathcal{T}\xi(x) := 2\xi\left(\frac{1+m}{2}x\right) + \xi\left(\frac{1-m}{2}x\right) + \xi\left(\frac{m-1}{2}x\right) - \xi(mx), \quad x \in G,\ \xi \in E^G.$$

Then the inequality (2.2) takes the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in G.$$

Now, we define an operator $\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G$ by

$$\begin{aligned}\Lambda\delta(x) :=& 2\delta\left(\frac{1+m}{2}x\right) + \delta\left(\frac{1-m}{2}x\right) + \delta\left(\frac{m-1}{2}x\right)\\ &+ \delta(mx), \ x \in G, \quad \delta \in \mathbb{R}_+^G. \qquad (2.3)\end{aligned}$$

This operator has the form described in (**H3**) with $k = 4$ and $f_1(x) = \frac{1+m}{2}x$, $f_2(x) = \frac{1-m}{2}x$, $f_3(x) = \frac{m-1}{2}x$, $f_4(x) = mx$, $L_1(x) = 2$ and $L_2(x) = L_3(x) = L_4(x) = 1$ for all $x \in G$.
Moreover, for every $\xi, \mu \in E^G$ and $x \in G$, we obtain

$$\begin{aligned}\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| =& \|2(\xi-\mu)(f_1(x)) + (\xi-\mu)(f_2(x))(\xi-\mu)(f_3(x)) - (\xi-\mu)(f_4(x))\|\\ \leq& 2\,\|(\xi-\mu)(f_1(x))\| + \|(\xi-\mu)(f_2(x))\| + \|(\xi-\mu)(f_3(x))\|\\ &+ \|(\xi-\mu)(f_4(x))\|\\ =& \sum_{i=1}^{4} L_i(x)\,\|(\xi-\mu)(f_i(x))\|\,.\end{aligned}$$

So, **(H2)** is valid. Not that for some $p < 0$, we have

$$\lim_{n\to\infty}\left(2\left(\frac{1+m}{2}\right)^p + 2\left(\frac{m-1}{2}\right)^p + m^p\right) = 0,$$

then, there exists $m_0 \in \mathbb{N}^*$ such that

$$A_m := 2\left(\frac{1+m}{2}\right)^p + 2\left(\frac{m-1}{2}\right)^p + m^p < 1, \quad \text{for all } m \geq m_0.$$

Therefore, in view of (1.7) and (2.3), it is easily to check that

$$\begin{aligned}\Lambda\varepsilon_m(x) =& 2\varepsilon_m\left(\frac{1+m}{2}x\right) + \varepsilon_m\left(\frac{1-m}{2}x\right) + \varepsilon_m\left(\frac{m-1}{2}x\right) + \varepsilon_m(mx)\\ =& 2H\left(\gamma\left(\frac{1+m}{2}x\right), \gamma\left(m\frac{1+m}{2}x\right)\right) + H\left(\gamma\left(\frac{1-m}{2}x\right), \gamma\left(m\frac{1-m}{2}x\right)\right)\\ &+ H\left(\gamma\left(\frac{m-1}{2}x\right), \gamma\left(m\frac{m-1}{2}x\right)\right) + H(\gamma(mx), \gamma(m.mx))\\ =& 2H\left(\frac{1+m}{2}\gamma(x), \frac{1+m}{2}\gamma(mx)\right) + 2H\left(\frac{m-1}{2}\gamma(x), \frac{m-1}{2}\gamma(mx)\right)\\ &+ H(m\gamma(x), m\gamma(mx))\\ =& \left(2\left(\frac{1+m}{2}\right)^p + 2\left(\frac{m-1}{2}\right)^p + m^p\right) H(\gamma(x), \gamma(mx))\\ =& A_m\varepsilon_m(x) \qquad (2.4)\end{aligned}$$

for all $x \in G$ and $m \geq m_0$. Therefore, we obtain that

$$\begin{aligned} \varepsilon_m^*(x) :&= \sum_{n=0}^{\infty} \Lambda^n \varepsilon_m(x) \\ &= \varepsilon_m(x) \sum_{n=0}^{\infty} (A_m)^n \\ &= \frac{\varepsilon_m(x)}{1 - A_m} < \infty. \end{aligned}$$

for all $x \in G$ and $m \geq m_0$. Thus, according to Theorem 1.2, for each $m \geq m_0$ there exists a unique solution $F_m : G \to E$ of the equation

$$F_m(x) = 2F_m\left(\frac{1+m}{2}x\right) + F_m\left(\frac{1-m}{2}x\right) + F_m\left(\frac{m-1}{2}x\right) - F_m(mx)$$

for all $x \in G$, such that

$$\|f(x) - F_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - A_m}, \quad x \in G.$$

Moreover $F_m(x) = \lim_{n\to\infty} \mathcal{T}^n f(x)$ for all $x \in G$.

To prove that the function F_m satisfies the functional equation (1.3) on G, it suffices to prove the following inequality

$$\left\|2\mathcal{T}^n f\left(\frac{x+y}{2}\right) + \mathcal{T}^n f\left(\frac{x-y}{2}\right) + \mathcal{T}^n f\left(\frac{y-x}{2}\right) - \mathcal{T}^n f(y) - \mathcal{T}^n f(x)\right\| \leq (A_m)^n \varepsilon(x,y) \tag{2.5}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Indeed, if $n = 0$, then (2.5) is simply (2.1). So, fix $n \in \mathbb{N}$ and suppose that (2.5) holds for n. Then

$$\begin{aligned} &\left\|2\mathcal{T}^{n+1} f\left(\frac{x+y}{2}\right) + \mathcal{T}^{n+1} f\left(\frac{x-y}{2}\right) + \mathcal{T}^{n+1} f\left(\frac{y-x}{2}\right) - \mathcal{T}^{n+1} f(y) - \mathcal{T}^{n+1} f(x)\right\| \\ =&\Big\|2\Big[\, 2\mathcal{T}^n f\left(\frac{m+1}{2}\frac{x+y}{2}\right) + \mathcal{T}^n f\left(\frac{1-m}{2}\frac{x+y}{2}\right) + \mathcal{T}^n f\left(\frac{m-1}{2}\frac{x+y}{2}\right) \\ &- \mathcal{T}^n f\left(m\frac{x+y}{2}\right)\Big] + \Big[2\mathcal{T}^n f\left(\frac{m+1}{2}\frac{x-y}{2}\right) + \mathcal{T}^n f\left(\frac{1-m}{2}\frac{x-y}{2}\right) \\ &+ \mathcal{T}^n f\left(\frac{m-1}{2}\frac{x-y}{2}\right) - \mathcal{T}^n f\left(m\frac{x-y}{2}\right)\Big] + \Big[2\mathcal{T}^n f\left(\frac{m+1}{2}\frac{y-x}{2}\right) \\ &+ \mathcal{T}^n f\left(\frac{1-m}{2}\frac{y-x}{2}\right) + \mathcal{T}^n f\left(\frac{m-1}{2}\frac{y-x}{2}\right) - \mathcal{T}^n f\left(m\frac{y-x}{2}\right)\Big] \\ &- \Big[2\mathcal{T}^n f\left(\frac{m+1}{2}y\right) + \mathcal{T}^n f\left(\frac{1-m}{2}y\right) + \mathcal{T}^n f\left(\frac{m-1}{2}y\right) - \mathcal{T}^n f(my)\Big] \end{aligned}$$

$$
\begin{aligned}
&-\left[2\mathcal{T}^n f\left(\frac{m+1}{2}x\right)+\mathcal{T}^n f\left(\frac{1-m}{2}x\right)+\mathcal{T}^n f\left(\frac{m-1}{2}x\right)-\mathcal{T}^n f(mx)\right]\Bigg\|\\
\leq&2\Bigg\|2\mathcal{T}^n f\left(\frac{m+1}{2}\frac{x+y}{2}\right)+\mathcal{T}^n f\left(\frac{m+1}{2}\frac{x-y}{2}\right)+\mathcal{T}^n f\left(\frac{m+1}{2}\frac{y-x}{2}\right)\\
&-\mathcal{T}^n f\left(\frac{m+1}{2}y\right)-\mathcal{T}^n f\left(\frac{m+1}{2}x\right)\Bigg\|+\Bigg\|2\mathcal{T}^n f\left(\frac{1-m}{2}\frac{x+y}{2}\right)\\
&+\mathcal{T}^n f\left(\frac{1-m}{2}\frac{x-y}{2}\right)+\mathcal{T}^n f\left(\frac{1-m}{2}\frac{y-x}{2}\right)-\mathcal{T}^n f\left(\frac{1-m}{2}y\right)\\
&-\mathcal{T}^n f\left(\frac{1-m}{2}x\right)\Bigg\|+\Bigg\|2\mathcal{T}^n f\left(\frac{m-1}{2}\frac{x+y}{2}\right)+\mathcal{T}^n f\left(\frac{m-1}{2}\frac{x-y}{2}\right)\\
&+\mathcal{T}^n f\left(\frac{m-1}{2}\frac{y-x}{2}\right)-\mathcal{T}^n f\left(\frac{m-1}{2}y\right)-\mathcal{T}^n f\left(\frac{m-1}{2}x\right)\Bigg\|\\
&+\Bigg\|2\mathcal{T}^n f\left(m\frac{x+y}{2}\right)+\mathcal{T}^n f\left(m\frac{x-y}{2}\right)+\mathcal{T}^n f\left(m\frac{y-x}{2}\right)-\mathcal{T}^n f(my)-\mathcal{T}^n f(mx)\Bigg\|\\
\leq&(A_m)^n\left[2\varepsilon\left(\frac{m+1}{2}x,\frac{m+1}{2}y\right)+\varepsilon\left(\frac{1-m}{2}x,\frac{1-m}{2}y\right)+\varepsilon\left(\frac{m-1}{2}x,\frac{m-1}{2}y\right)\right.\\
&\qquad\left.+\varepsilon(mx,my)\right]\\
=&(A_m)^{n+1}\varepsilon(x,y)
\end{aligned}
$$

for all $x,y\in G$. Thus, by induction, we have shown that (2.5) holds for all $x,y\in G$ and for all $n\in\mathbb{N}$. Letting $n\to\infty$ in (2.5), we obtain

$$2F_m\left(\frac{x+y}{2}\right)+F_m\left(\frac{x-y}{2}\right)+F_m\left(\frac{y-x}{2}\right)=F_m(y)+F_m(x)\tag{2.6}$$

for all $x,y\in G$. So, we find a sequence $(F_m)_{m\geq m_0}$ satisfies (1.3) on G such that

$$\|f(x)-F_m(x)\|\leq\frac{\varepsilon_m(x)}{1-A_m},\ \ x\in G,\ \ m\geq m_0.$$

Next, we prove that $F_m=F_k$ for all $m,k\in\mathbb{N}_{m_0}$. Let us fix $m,k\in\mathbb{N}_{m_0}$ and note that F_m and F_k satisfy (2.6). Hence, by replacing (x,y) by (x,mx) in (2.6), we get $\mathcal{T}F_m(x)=F_m(x)$, $\mathcal{T}F_k(x)=F_k(x)$ for all $x\in G$ and

$$\|F_m(x)-F_k(x)\|\leq\frac{\varepsilon_m(x)}{1-A_m}+\frac{\varepsilon_k(x)}{1-A_k}$$

for all $x \in G$. It follows, by linearity of Λ and (2.4) that

$$\begin{aligned}\|F_m(x) - F_k(x)\| &= \|\mathcal{T}^n F_m(x) - \mathcal{T}^n F_k(x)\| \\ &\leq \frac{\Lambda^n \varepsilon_m(x)}{1 - A_m} + \frac{\Lambda^n \varepsilon_k(x)}{1 - A_k} \\ &\leq (A_m)^n \left[\frac{\varepsilon_m(x)}{1 - A_m} + \frac{\varepsilon_k(x)}{1 - A_k}\right]\end{aligned}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we get $F_m = F_k =: F$. Thus, we have

$$\|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - A_m}, \quad x \in G, \quad m \geq m_0$$

and the function F is a solution of (1.3).

To prove the uniqueness of the function F, let us assume that there exists a function $F' : G \to E$ which satisfies (1.3) and the inequality

$$\|f(x) - F'(x)\| \leq \frac{\varepsilon_m(x)}{1 - A_m}, \quad x \in G, \quad m \geq m_0.$$

Then

$$\|F(x) - F'(x)\| \leq \frac{2\varepsilon_m(x)}{1 - A_m}, \quad x \in G, \quad m \geq m_0.$$

Further $\mathcal{T}F'(x) = F'(x)$ for all $x \in G$. Therefore, with a fixed $m \in \mathbb{N}_{m_0}$

$$\begin{aligned}\|F(x) - F'(x)\| &= \|\mathcal{T}^n F(x) - \mathcal{T}^n F'(x)\| \\ &\leq \frac{2\Lambda^n \varepsilon_m(x)}{1 - A_m} \\ &\leq 2(A_m)^n \times \frac{\varepsilon_m(x)}{1 - A_m}\end{aligned}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we get $F = F'$, which yields

$$\|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - A_m}, \quad x \in G, \quad m \geq m_0.$$

Next, in view of (1.6), we have

$$\inf \left\{\frac{\varepsilon_m(x)}{1 - A_m} : m \geq m_0\right\} = 0$$

for all $x \in G$, this means that $f(x) = F(x)$ for $x \in G$, which implies that f satisfies the functional equation (1.3) on G and the proof of the theorem is complete. $\square$

In a similar way we can prove that Theorem (2.1) holds if the inequality (2.1) is defined on $G\backslash\{0\} := G_0$.

Theorem 2.2. *Let G be an 2-divisible abelian group and E be a Banach space. Let Σ be the set of all functions $\varepsilon : G_0 \to \mathbb{R}_+$ which satisfy the conditions as stated in the Section (1). If $f : G \to E$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(y) - f(x) \right\| \leq \varepsilon(x,y) \quad (2.7)$$

for all $x, y \in G_0$ and $\varepsilon \in \Sigma$, then f is a solution of (1.3) on G_0.

3 Applications

In this section we give some applications of the Theorem 2.2, with the case:

$$\varepsilon(x,y) = \theta\|x\|^p.\|y\|^q$$

where $\theta \in \mathbb{R}_+$, $p, q \in \mathbb{R}$ and $x, y \neq 0$.

Corollary 3.1. *Let E_1 and E_2 ba a normed space and a Banach space, respectively. Assume $S := (S, +)$ is an 2-divisible subgroup of the group $(E_1, +)$, $p, q \in \mathbb{R}$, $p + q < 0$ and $\theta \geq 0$. If $f : S \to E_2$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(y) - f(x) \right\| \leq \theta\|x\|^p\|y\|^q \quad (3.1)$$

for all $x, y \in S\backslash\{0\}$, then f is a solution of (1.3) on $S\backslash\{0\}$.

Proof. Let Σ the set of all functions $\varepsilon : S\backslash\{0\} \times S\backslash\{0\} \to \mathbb{R}_+$ such that

$$\varepsilon(x,y) = \theta\|x\|^p\|y\|^q$$

for some $\theta \in \mathbb{R}_+$ and for all $x, y \in S\backslash\{0\}$. Define $H : \mathbb{R}^2_{+*} \to \mathbb{R}_+$ by $H(u,v) = cu^pv^q$ for some $c \geq 0$, $p + q < 0$ and for all $u, v \in \mathbb{R}_{+*}$ and $\gamma : S \to \mathbb{R}_+$ by $\gamma(x) = \|x\|$ for all $x \in S$. It is easily seen that H is monotonically symmetric homogeneous function of degree $p + q < 0$ and conditions indicated in the start of the second section are fulfilled. Therefore every function $f : S\backslash\{0\} \to E_2$ satisfying (3.1) is a solution of the functional equation (1.3) on $S\backslash\{0\}$. $\square$

We know that any norm that satisfies the parallelogram law is bound to have been originated from a scalar product. The following corollary gives a characterization of the inner product space, which is one of the applications of Corollary 3.1.

Corollary 3.2. *Let X be a normed space and $X_0 = X\backslash\{0\}$. Write*

$$D(x,y) = \left| 2\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 + \left\|\frac{y-x}{2}\right\|^2 - \|y\|^2 - \|x\|^2 \right|$$

for all $x, y \in X$. Assume that

$$\sup_{x,y\in X_0} \frac{D(x,y)}{\|x\|^p\|y\|^q} < \infty$$

for somme $p, q \in \mathbb{R}$ and $p + q < 0$. Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$. Then from Corollary 3.1, we easily derive f is a solution of the functional equation (1.3). That implies $D(x,y) = 0$. Thus, the norm $\|.\|$ on X satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.$$

Therefore, X is an inner product space. □

Corollary 3.3. *Let G be an 2-divisible commutative group and E be a Banach space. Let Σ be the set of all functions $\varepsilon : G \to \mathbb{R}_+$ which satisfy the conditions as stated in the Section (1) and $F : G^2 \to E$ be a mapping such that $F(x_0, y_0) \neq 0$ for some $x_0, y_0 \in G$ and*

$$\|F(x,y)\| \le \varepsilon(x,y), \tag{3.2}$$

for all $x, y \in G$. Then the functional equation

$$2h\left(\frac{x+y}{2}\right) + h\left(\frac{x-y}{2}\right) + h\left(\frac{y-x}{2}\right) = F(x,y) + h(y) + h(x), \quad x, y, z \in G \tag{3.3}$$

has no solution in the class of functions $h : G \to E$.

Proof. Suppose that $h : G \to E$ is a solution to (3.3). Then (2.1) holds, and consequently, according to the above theorems, h is Jensen on G, which means that $F(x_0, y_0) = 0$. This is a contradiction. □

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