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Regularity properties and integral inequalities related to (k, h_1, h_2)-convexity of functions

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Abstract. The class of (k, h_1, h_2)-convex functions is introduced, together with some particular classes of corresponding generalized convex dominated functions. Few regularity properties of (k, h_1, h_2)-convex functions are proved by means of Bernstein-Doetsch type results. Also we find conditions in which every local minimizer of a (k, h_1, h_2)-convex function is global. Classes of (k, h_1, h_2)-convex functions, which allow integral upper bounds of Hermite-Hadamard type, are identified. Hermite-Hadamard type inequalities are also obtained in a particular class of the (k, h_1, h_2)-convex dominated functions.

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1 Introduction

In what follows, $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ denote, respectively, the set of all real, rational, integer and natural numbers. If $k : (0, 1) \to \mathbb{R}$ is a given function then a subset D of a real linear space X is said to be k-convex (according to [14]) if $k(t)x + k(1 - t)y \in D$, whenever $x, y \in D$ and $t \in (0, 1)$. Let $k, h_1, h_2 : (0, 1) \to \mathbb{R}$ be three given functions and assume that $D \subseteq X$ is a k-convex set.

Definition 1.1. *A function $f : D \to \mathbb{R}$ is said to be (k, h_1, h_2)-convex if*

$$f(k(t)x + k(1-t)y) \leq h_1(t)f(x) + h_2(t)f(y), \tag{1.1}$$

for all $x, y \in D$ and $t \in (0, 1)$. If the inequality is strict then f is said to be strictly (k, h_1, h_2)-convex.

This concept extends the (h_1, h_2)-convexity defined in our paper [3]. If $k(t) = t$ then (1.1) becomes the definition of (h_1, h_2)-convex functions from [3]. Also, this definition extends the concept of (k, h)-convexity introduced in [14], which may be obtained from (1.1) by taking $h_1(t) = h(t)$ and $h_2(t) = h(1-t)$.

Many segmental convexity properties for functions are particular cases of (k, h_1, h_2)-convexity. If $k(t) = t$, $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in (0, 1)$, then Definition 1.1 gives the classically convex functions. If $k(t) = t$, $h_1(t) = h_2(t) = 1$ for all $t \in (0, 1)$, then Definition 1.1 identifies the $P(D)$ class introduced in [6]. Supposing that $X = \mathbb{R}$, $D = [0, +\infty)$, $s \in (0, 1]$, $k(t) = t^{\frac{1}{s}}$, $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in (0, 1)$ then (1.1) describes the s-convexity in the first sense (also known as Orlicz's convexity since it comes from [16]). Taking now $0 < s \leq 1$, $k(t) = t$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in (0, 1)$, Definition 1.1 gives the functions that are s-convex in the second sense (or Breckner-convex, originating in [2]). Suppose now that $k(t) = t$ and that $h : [0, 1] \to \mathbb{R}$ is a nonnegative function. If $h_1(t) = h(t)$ and $h_2(t) = h(1-t)$ then Definition 1.1 introduces the h-convexity defined by Varošanec in [21]. The Godunova-Levin $Q(D)$ class of functions (see [8]) is obtained from Definition 1.1 if $k(t) = t$, $h_1(t) = t^{-1}$ and $h_2(t) = (1-t)^{-1}$ for all $t \in (0, 1)$ and function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ interval. Combining the definition of the Godunova-Levin class and the Breckner-convexity the class of $s - Q(D)$ convexity was obtained in [15] by taking $0 < s \leq 1$, $k(t) = t$, $h_1(t) = t^{-s}$ and $h_2(t) = (1-t)^{-s}$ for $t \in (0, 1)$, which means that the inequality (1.1) becomes

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}, \tag{1.2}$$

for all $x, y \in I \subseteq \mathbb{R}$ and $t \in (0, 1)$.

In this paper we intend to study regularity and extremal properties within classes of functions having generalized convexity properties of the type introduced in Definition 1.1. In Section 2 of this paper we identify conditions in which boundedness and continuity of functions having (k, h_1, h_2)-convexity with respect to a set T occur. Conditions for Bernstein-Doetsch type result (see [1]) are identified in this case of generalized convexity. We identify conditions, in which every local minimizer of a (k, h_1, h_2)-convex function is a global one. In Section 3 we prove an integral inequality of Hermite-Hadamard type, which holds within the class of (k, h_1, h_2)-convex functions. Section 4 refers to functions that are (k, h_1, h_2)-convex dominated, deriving Hermite-Hadamard type inequalities in the framework provided by a particular function k.

2 Regularity properties of the (k, h_1, h_2)-convex functions

In [10] is introduced the more general concept of (k, h)-convexity with respect to a subset T of a real linear space X. The set T is supposed to verify the property that it contains the element $1 - t$ whenever $t \in T$. Functions $k, h : T \to \mathbb{R}$ and Definition 1.1 is supposed to hold for $h_1 = h(t)$ and $h_2(t) = h(1 - t)$, for every $t \in T$. Regularity properties of the (k, h)-convex functions are studied in [10].
In the sequel we suppose that $(X, \|\cdot\|)$ is a real or complex normed space and $T \subseteq \mathbb{R}$ such that $1 - t \in T$ if and only if $t \in T$. Let $k, h_1, h_2 : T \to \mathbb{R}$ be three given functions. Consider a set $D \subseteq X$, which is k-convex. In this section we study few smoothness properties of the (k, h_1, h_2)-convex functions with respect to T, i.e. functions $f : D \to \mathbb{R}$ that verify (1.1) for all $x, y \in D$ and $t \in T$. If $T = \{t\}$ is a singleton set then a function that verifies (1.1) is called (k, h_1, h_2)-convex functions with respect to t. For example, if $T = \{\frac{1}{2}\}$, $h_1(t) = t$, $h_2(t) = 1 - t$, then the (k, h_1, h_2)-convex functions with respect to $\frac{1}{2}$ become the Jensen-convex functions [12].

Theorem 2.1. *Let $k, h_1, h_2 : T \to \mathbb{R}$ such that $k(t) + k(1 - t) = 1$ for all $t \in T$. Let $f : D \to \mathbb{R}$ be a (k, h_1, h_2)-convex function with respect to T. Then*

1. *if $h_1(t) + h_2(t) \geq 1$ for all $t \in T$ and there is a point $t_0 \in T$ such as $h_1(t_0) + h_2(t_0) > 1$ then f is nonnegative;*

2. *if $h_1(t) + h_2(t) \leq 1$ for all $t \in T$ and there is a point $t_0 \in T$ such as $h_1(t_0) + h_2(t_0) < 1$ then f is non-positive;*

3. *if there are $t_1, t_2 \in T$ such that $h_1(t_1) + h_2(t_1) > 1$ and $h_1(t_2) + h_2(t_2) < 1$ then f is constant 0.*

Proof. 1. Let $x \in D$ be an arbitrary element. From the (k, h_1, h_2)-convexity of function f one gets

$$f(x) = f(k(t_0)x + k(1-t_0)x)$$
$$\leq h_1(t_0)f(x) + h_2(t_0)f(x) = f(x)(h_1(t_0) + h_2(t_0)),$$

which means that

$$f(x)(h_1(t_0) + h_2(t_0) - 1) \geq 0.$$

Since $h_1(t_0) + h_2(t_0) - 1 > 0$ it follows that $f(x) \geq 0$.

2. In a similar manner as above, since $h_1(t_0) + h_2(t_0) - 1 < 0$ it follows that $f(x) \leq 0$.

3. The result is an immediate consequence of the two previous cases.

□

Let us remind that a function $f : D \to \mathbb{R}$, with $D \subseteq X$, ia locally upper bounded (or locally bounded from above) if for each point $p \in D$, there exist $r > 0$ and a neighborhood $B(p, r) = \{x \in X | \|x - p\| < r\}$ such that f is bounded from above on $B(p, r)$.

Theorem 2.2. *Let $t \in T$ be fixed, $k, h_1, h_2 : T \to \mathbb{R}$ be non-negative functions such as:*

1. *$k(t)k(1-t) \neq 0$ and $k(t) + k(1-t) = 1$;*
2. *$h_1(t)h_2(t) \neq 0$.*

Let $D \subseteq X$ be a non-empty, open and k-convex set, and $f : D \to \mathbb{R}$ be a function that is (k, h_1, h_2)-convex with respect to t. Then if f is locally bounded from above at a point $p \in D$ and if $h_1(t)+h_2(t) < 1$ or $h_1(t)+h_2(t) \geq 1$ then f is locally bounded at every point of D.

Proof. The conclusion of locally upper boundedness is a consequence of Theorem 2.1 in the case $h_1(t) + h_2(t) < 1$. As consequence, we take into account the case $h_1(t) + h_2(t) \geq 1$.
In order to prove the locally boundedness from above on D we construct the sequence of subsets $\{D_n\}_{n\in\mathbb{N}} \subseteq D$ as follows:

$$D_0 := \{p\}, \qquad D_{n+1} := k(t)D_n + k(1-t)D. \tag{2.1}$$

We prove that

$$D = \bigcup_{n=1}^{\infty} D_n. \tag{2.2}$$

Since the relation $\bigcup_{n=1}^{\infty} D_n \subseteq D$ is obvious, we check the converse inclusion. From (2.1) one gets

$$D_n = (k(t))^n p + (1 - (k(t))^n)D$$

by induction. For a fixed point $x \in D$ one defines the sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_n := \frac{x - (k(t))^n p}{1 - (k(t))^n}.$$

Obviously, $\lim_{n\to\infty} x_n = x$, by the hypothesis on function k. Since D is open, one gets that $x_n \in D$ for some n. Therefore,

$$x = (k(t))^n p + (1 - (k(t))^n)x_n \in (k(t))^n p + (1 - (k(t))^n)D = D_n.$$

So, the reverse inclusion occurs and (2.2) as well.
Let us come back to the properties of function f. By hypothesis we have that f is locally upper bounded at $p \in D_0$. We proceed by induction on n to prove that f is upper bounded at each point of D. Assume that f is locally upper bounded at each point of D_n for some n. From (2.1) one gets that for $x \in D_{n+1}$ there are $x_0 \in D_n$ and $y_0 \in D$ such that $x = k(t)x_0 + k(1-t)y_0$. From the inductive hypothesis it follows that there are $r > 0$ and $M_0 \geq 0$ such that $f(x_1) \leq M_0$ for $\|x_0 - x_1\| < r$. Then if $x_1 \in B_0 := B(x_0, r)$, by the (k, h_1, h_2)-convexity of f with respect to t one has

$$\begin{aligned} f(k(t)x_1 + k(1-t)y_0) &\leq h_1(t)f(x_1) + h_2(t)f(y_0) \\ &\leq h_1(t)M_0 + h_2(t)f(y_0) =: M. \end{aligned}$$

As consequence, for

$$y \in B := k(t)B_0 + k(1-t)y_0 = B(k(t)x_0 + k(1-t)y_0, k(t)r) = B(x, k(t)r),$$

one obtains $f(y) \leq M$, which means that f is locally bounded from above on D_{n+1}. So, by (2.2) f is locally bounded from above on D.
Let us investigate now the locally boundedness from below of f. Let $z \in D$. Since f is locally upper bounded at z, there are $r > 0$ and $M > 0$ such that

$$\sup_{x\in B(z,r)} f(x) \leq M.$$

Suppose that $x \in B(z, k(1-t)r)$ and let

$$y := \frac{z - k(t)x}{k(1-t)} \in B(z, r).$$

The (k, h_1, h_2)-convexity of f with respect to t implies $f(z) \leq h_1(t)f(x) + h_2(t)f(y)$, which means that

$$f(x) \geq \frac{1}{h_1(t)}f(z) - \frac{h_2(t)}{h_1(t)}f(y) \geq \frac{1}{h_1(t)}f(z) - \frac{h_2(t)}{h_1(t)}M =: M_1,$$

which means that the function is locally bounded from below at z. Since z was arbitrarily chosen it follows that f is locally bounded from below at any point of D. □

The next result contains a sufficient condition for the local boundedness to imply the continuity within the class of the (k, h_1, h_2)-convex functions with respect to a set T.

Theorem 2.3. *Let $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ be a sequence such that $\lim_{n\to\infty} t_n = 0$ and let $T = \{t_n\}_{n\in\mathbb{N}}$. Let $k, h_1, h_2 : [0,1] \to \mathbb{R}$ be three non-negative, continuous functions such as:*

1. *$h_1(t_n)h_2(t_n) \neq 0$ for every $n \in \mathbb{N}$;*
2. *$k(t_n) + k(1 - t_n) = 1$ for every $n \in \mathbb{N}$;*
3. *$\lim_{t\to 0} h_1(t) = 0$, $\lim_{t\to 1} h_1(t) = 1$;*
4. *$\lim_{t\to 0} h_2(t) = 1$, $\lim_{t\to 1} h_2(t) = 0$.*

Let $D \subseteq X$ a non-empty, open and k-convex set. If $f : D \to \mathbb{R}$ is (k, h_1, h_2)-convex with respect to T and locally bounded from above at a point of D, then f is continuous on D.

Proof. Without loss of generality one may assume that $h_2(t) > 0$. Let $x_0 \in D$ such as f is locally upper bounded at x_0. Then there is a neighborhood U of x_0 and a constant $M > 0$ such as $f(x) \leq M$ for every $x \in U$. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$h_1(t_n)M + [h_2(t_n) - 1]f(x_0) < \varepsilon,$$

for $n \geq n_0$, which means that

$$\frac{h_1(t_n)}{h_2(t_n)}M + \left[1 - \frac{1}{h_2(t_n)}\right]f(x_0) < \varepsilon.$$

Let V be a neighborhood of the origin of space X such that $x_0 + V \subseteq U$ and denote by $U' = x_0 + k(t_n)V$. We intend to prove that $|f(x) - f(x_0| < \varepsilon$ for

every $x \in U'$.
Suppose that $x \in U'$. Since

$$y - x_0 = \frac{1}{k(t_n)}(x - x_0) \in \frac{1}{k(t_n)}k(t_n)V = V,$$

$$z - x_0 = \frac{1 - k(t_n)}{k(t_n)}(x_0 - x) \in \frac{1 - k(t_n)}{k(t_n)}k(t_n)V = (1 - k(t_n))V \subseteq V,$$

there are $y, z \in x_0 + V$ such as

$$x = k(t_n)y + k(1 - t_n)x_0 = k(t_n)y + (1 - k(t_n))x_0,$$

$$x_0 = k(t_n)z + k(1 - t_n)x = k(t_n)z + (1 - k(t_n))x.$$

From the (k, h_1, h_2)-convexity of f with respect to T one gets

$$f(x) \leq h_1(t_n)f(y) + h_2(t_n)f(x_0) \leq h_1(t_n)M + h_2(t_n)f(x_0),$$

$$f(x_0) \leq h_1(t_n)f(z) + h_2(t_n)f(x) \leq h_1(t_n)M + h_2(t_n)f(x).$$

These inequalities together with the limit hypothesis imply that

$$f(x) - f(x_0) \leq h_1(t_n)M + \Big[h_2(t_n) - 1\Big]f(x_0) < \varepsilon \tag{2.3}$$

and

$$f(x) \geq \frac{f(x_0) - h_1(t_n)M}{h_2(t_n)}.$$

From these two inequalities one gets

$$f(x) - f(x_0) \geq \Big[\frac{1}{1 - h_1(t_n)} - 1\Big]f(x_0) - \frac{h_1(t_n)}{h_2(t_n)}M > -\varepsilon. \tag{2.4}$$

From (2.3), (2.4) and the limit hypothesis one concludes that $|f(x) - f(x_0| < \varepsilon$, which means that f is continuous at x_0, as required. □

Remark 2.1. Almost all the particular cases of (k, h_1, h_2)-convexity mentioned in Section 1 of this paper, in which $k(t) = t$, satisfy the hypotheses of Theorem 2.2 and Theorem 2.3. So, the classic convexity, the s-convexity of second kind have all the regularity properties discussed in the above proved theorems. The limit hypotheses from Theorem 2.3 do not occur in case of the Godunova-Levin class and also in $P(D)$ class. There are counterexamples of non-negative functions belonging to the Godunova-Levin class that are monotone but are not continuous. The two theorems identify conditions for function h such as the h-convexity defined in [21] have these regularity properties. They also provide conditions for h_1 and h_2 such as the same regularity properties occur in case of the (h_1, h_2)-convexity defined in [3].

Remark 2.2. The limit conditions are not necessary, since there are cases of known convexities, in which they do not fulfill. For example, this happens in case of Jensen-convex functions, but the property of Bernstein-Doetsch type is valid in this case (see [1]).

The next result states conditions, in which every local minimizer of a (k, h_1, h_2)-convex function is a global one, as in the case of the convex functions in the classical sense.

Theorem 2.4. *Let $k, h_1, h_2 : [0, 1] \to \mathbb{R}$ be three non-negative and continuous functions such as*

$$\lim_{t \to 0+} k(t) = 0, \qquad \lim_{t \to 1-} k(t) = 1$$

and $h_1(t) + h_2(t) \leq 1$, for all $t \in [0, 1]$. Let $D \subseteq X$ be a non-empty, open and k-convex set. Then every local minimizer of a (k, h_1, h_2)-convex function $f : D \to \mathbb{R}$ is a global one. More, if f is strictly (k, h_1, h_2)-convex then there is at most one global minimum.

Proof. Let $x_0 \in D$ be a local minimizer of f. Then there is $r > 0$ such that $f(x_0) \leq f(x)$ for every $x \in B(x_0, r)$. Let us suppose that x_0 is not a global minimizer. Then there is $x_1 \in D$ such that $f(x_0) > f(x_1)$. From the (k, h_1, h_2)-convexity condition on function f, taking into account that $f(x_1) - f(x_0) < 0$, it follows that

$$f(k(t)x_0 + k(1-t)x_1) \leq h_1(t)f(x_0) + h_2(t)f(x_1)$$

$$\leq (1 - h_2(t))f(x_0) + h_2(t)f(x_1) = f(x_0) + h_2(t)\left[f(x_1) - f(x_0)\right] < f(x_0).$$

The limit conditions on function k imply that one can chose t in a sufficiently small neighborhood of 1 such that $k(t)x_0 + k(1-t)x_1 \in B(x_0, r)$. This is a contradiction with the fact that x_0 is a local minimizer.
If the convexity property of f is strict, supposing that there are two global minimizers $x_1 \neq x_2$, one gets

$$f(k(t)x_1 + k(1-t)x_2) \leq h_1(t)f(x_1) + h_2(t)f(x_2)$$

$$= \left[h_1(t) + h_2(t)\right] f(x_1) \leq f(x_1).$$

This is a contradiction with the extremal property of x_1. □

Corollary 2.5. *The local minimizer is a global one in case of any convex function in the classical sense. If the convexity is strict then the function has at most one global minimum.*

Corollary 2.6. *The local minimizer is a global one in case of any Orlicz-convex function. If the Orlicz-convexity is strict then the function has at most one global minimum.*

Corollary 2.7. *The local minimizer is a global one in case of any* (k, h)*-convex function in the sense of* [10] *and* [14] *if* k *satisfies the hypothesis of Theorem 2.4. Similar remark is valid in case of the* h*-convexity form* [21] *and also in case of the* (h_1, h_2)*-convexity from* [3] *and* [20]*. If these generalized convexities are strict then the function has at most one global minimum.*

3 Hermite-Hadamard type upper bounds for (k, h_1, h_2)-convex functions

Let us consider the space $X = \mathbb{R}$ and the function $k : [0, 1] \to [0, 1]$. Let $I \subseteq \mathbb{R}$ an open interval such that I is k-convex. In the sequel, $L_1(I)$ denotes the set of those functions $f : I \to \mathbb{R}$, which are Lebesque integrable over I. In this section we derive the following Hermite-Hadamard type integral upper bound inequality:

Theorem 3.1. *Let* $k, h_1, h_2 : [0, 1] \to [0, 1]$ *be three non-negative functions,* $h_1, h_2 \in L_1([0, 1])$*. Let* $I \subseteq \mathbb{R}$ *an open* k*-convex interval and a function* $f : I \to \mathbb{R}$*, which is* (k, h_1, h_2)*-convex on* I *and* $f \in L_1(I)$*. Then the following inequality holds:*

$$\int_0^1 f(k(t)x + k(1-t)y)dt \le \frac{f(x) + f(y)}{2} \int_0^1 [h_1(t) + h_2(t)]dt, \tag{3.1}$$

whenever $x, y \in I$*,* $x < y$*.*

Proof. Let us consider $x, y \in I$, $x < y$. Since f is (k, h_1, h_2)-convex on I one has

$$f(k(t)x + k(1-t)y) \le h_1 f(x) + h_2 f(y),$$
$$f(k(1-t)x + k(t)y) \le h_2 f(x) + h_1 f(y).$$

Computing the sum of these two inequalities and integrating the resulted inequality side by side over $[0, 1]$ with respect to t, one gets:

$$\int_0^1 f(k(t)x + k(1-t)y)dt + \int_0^1 f(k(1-t)x + k(t)y)dt$$
$$\le [f(x) + f(y)] \int_0^1 [h_1(t) + h_2(t)]\, dt.$$

In the second integral we perform the change of variable $u = 1 - t$ and the result is

$$2\int_0^1 f(k(t)x + k(1-t)y)dt \leq [f(x) + f(y)] \int_0^1 [h_1(t) + h_2(t)]\, dt,$$

which is the expected result. $\square$

Corollary 3.2. *If $k(t) = t$, $h_1(t) = t$ and $h_2(t) = 1 - t$, we obtain the case of the classic convex functions. In this case, (3.1) becomes*

$$\frac{1}{y-x}\int_x^y f(u)du \leq \frac{f(x) + f(y)}{2}, \tag{3.2}$$

$x, y \in I$, $x < y$, which is the upper bound of the classical Hermite-Hadamard inequality, according to [9] *and* [11].

Remark 3.1. From the case of the classic Hermite-Hadamard inequality (3.2), which is sharp for linear functions and, which is a particular case of (3.1), one concludes that (3.1) is sharp.

Corollary 3.3. *If $k(t) = t$, $h_1(t) = h_2(t) = 1$ for all $t \in [0, 1]$ then the (k, h_1, h_2)-convexity identifies the $P(I)$-class. In this case (3.1) becomes*

$$\frac{1}{y-x}\int_x^y f(u)du \leq f(x) + f(y), \tag{3.3}$$

for all $x, y \in I$, $x < y$. The integral inequality (3.3) was proved in [6].

Corollary 3.4. *Let $f : [0, +\infty) \to \mathbb{R}$ and $s \in (0, 1]$. The Orlicz-convexity is a (k, h_1, h_2)-convexity, with $k(t) = t^{\frac{1}{s}}$, $h_1(t) = t$, $h_2(t) = 1 - t$ for all $t \in [0, 1]$. In this case, the inequality (3.1) becomes*

$$\int_0^1 f(t^{\frac{1}{s}}x + (1-t)^{\frac{1}{s}}y)dt \leq \frac{f(x) + f(y)}{2}, \tag{3.4}$$

for all $x, y \in I$, $x < y$. This inequality seems to be a new one.

Corollary 3.5. *As in* [2], *suppose that $0 < s \leq 1$. A function $f : I \to \mathbb{R}$ is Breckner-convex, or s-convex of second kind if $k(t) = t$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in (0, 1)$, and the inequality (3.1) becomes*

$$\frac{1}{y-x}\int_x^y f(u)du \leq \frac{f(x) + f(y)}{s+1}. \tag{3.5}$$

for all $x, y \in I$, $x < y$. This Hermite-Hadamard type inequality for Breckner-convex functions was proved in [4].

Corollary 3.6. *Let us suppose, as in* [21]*, that* $h : [0, 1] \to \mathbb{R}$ *is a nonnegative function. A function* $f : I \to \mathbb{R}$ *is* h*-convex on* I *if* $k(t) = t$, $h_1(t) = h(t)$ *and* $h_2(t) = h(1 - t)$ *for all* $t \in [0, 1]$*. In this case, the inequality (3.1) has the form*

$$\frac{1}{y-x}\int_x^y f(u)du \le [f(x) + f(y)]\int_0^1 h(t)dt. \tag{3.6}$$

for all $x, y \in I$, $x < y$*. This inequality was derived for the first time in* [19]*.*

Corollary 3.7. *Let us suppose, as in* [10] *and* [14]*, that* $k, h : [0, 1] \to \mathbb{R}$ *are nonnegative functions,* $h_1(t) = h(t)$*, and* $h_2(t) = h(1 - t)$ *for all* $t \in [0, 1]$*. So, we are in case of the* (k, h)*-convexity. In this case, the inequality (3.1) has the form*

$$\int_0^1 f(k(t)x + k(1-t)y)dt \le [f(x) + f(y)]\int_0^1 h(t)dt, \tag{3.7}$$

for all $x, y \in I$, $x < y$*. This inequality was derived for the first time in* [14]*.*

Corollary 3.8. *Let us consider* $k(t) = t$ *and two non-negative functions* $h_1, h_2 : [0, 1] \to \mathbb{R}$*. The* (k, h_1, h_2)*-convexity becomes in this case the* (h_1, h_2)*-convexity, introduced in* [3] *independently and simultaneously with* [20]*, in which it is a particular case. The inequality (3.1) yields to*

$$\frac{2}{y-x}\int_x^y f(u)du \le [f(x) + f(y)]\int_0^1 [h_1(t) + h_2(t)]dt, \tag{3.8}$$

for all $x, y \in I$, $x < y$*. Inequality (3.8) seems to be new.*

4 (h_1, h_2)-convex dominated functions and Hermite-Hadamard like inequalities

In this section we suppose that $k : [0, 1] \to [0, 1]$ is the particular function $k(t) = t$. Let $h_1, h_2 : [0, 1] \to \mathbb{R}$ be two non-negative functions. In this case the (k, h_1, h_2)-convexity will be called, as in [3], (h_1, h_2)-convexity. Let $I \subseteq \mathbb{R}$ be an interval and $g : I \subset \mathbb{R} \to [0, \infty)$ be a (h_1, h_2)-convex function.

Definition 4.1. *The real function $f : I \subset \mathbb{R} \to [0,\infty)$ is said to be a (h_1,h_2)-convex dominated function by g on I, if*

$$|h_1(t)f(x) + h_2(t)f(y) - f(tx + (1-t)y)| \tag{4.1}$$
$$\leq h_1(t)g(x) + h_2(t)g(y) - g(tx + (1-t)y), \quad \forall x, y \in I, t \in (0,1).$$

Many particular cases are in the literature. For $h_1(t) = t$, $h_2(t) = 1-t$ in Definition 4.1, we have the definition of convex dominated functions [7]. For $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ in Definition 4.1, we have the definition of s-convex dominated functions by g, discussed in [13]. For $h_1(t) = 1 = h_2(t)$ in Definition 4.1, we have the definition of $P(D)$-dominated by g functions [18]. For $h_1(t) = t^{-1}$ and $h_2(t) = (1-t)^{-1}$, $t \in (0,1)$, in Definition 4.1, we have the definition of $Q(I)$-dominated functions [18]. For $h_1(t) = t^{-s}$ and $h_2(t) = (1-t)^{-s}$, $t \in (0,1)$, in Definition 4.1, we have the definition of $s-Q(I)$-dominated functions by g, which appears to be new in the literature.

Definition 4.2. *Let $g : I \subset \mathbb{R} \to [0,\infty)$ be a $s-Q(I)$-function. The real function $f : I \subset \mathbb{R} \to [0,\infty)$ is said to be $s-Q(I)$-dominated function by g on I, if*

$$\left|\frac{1}{t^s}f(x) + \frac{1}{(1-t)^s}f(y) - f(tx + (1-t)y)\right| \tag{4.2}$$
$$\leq \frac{1}{t^s}g(x) + \frac{1}{(1-t)^s}g(y) - g(tx + (1-t)y), \quad \forall x, y \in I, s \in [0,1], t \in (0,1).$$

Hermite-Hadamard type inequalities are derived for more classes of generalized convex dominated functions in [5], [17], [18], [19].

Theorem 4.1. *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non-negative functions, $g : I \subset \mathbb{R} \to [0,\infty)$ be (h_1,h_2)-convex functions. Let $f : I \subset \mathbb{R} \to [0,\infty)$ be (g,h_1,h_2)-convex dominated function on I where $f \in L_1[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{h_1\left(\frac{1}{2}\right) + h_2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\right| \tag{4.3}$$
$$\leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - \frac{1}{h_1\left(\frac{1}{2}\right) + h_2\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right).$$

Proof. Using $t = \frac{1}{2}$, $x = \mu a + (1-\mu)b$ and $y = (1-\mu)a + \mu b$ where $\mu \in [0,1]$ in the definition of (g,h_1,h_2)-convex dominated function, we have

$$\left|h_1\left(\frac{1}{2}\right)f(\mu a + (1-\mu)b) + h_2\left(\frac{1}{2}\right)f((1-\mu)a + \mu b) - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq h_1\Big(\frac{1}{2}\Big)g(\mu a+(1-\mu)b)+h_2\Big(\frac{1}{2}\Big)g((1-\mu)a+\mu b)-g\Big(\frac{a+b}{2}\Big).$$

Integrating above inequality with respect to μ on $[0,1]$, we have

$$\left|\left[h_1\Big(\frac{1}{2}\Big)+h_2\Big(\frac{1}{2}\Big)\right]\frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x-f\Big(\frac{a+b}{2}\Big)\right|$$

$$\leq\left[h_1\Big(\frac{1}{2}\Big)+h_2\Big(\frac{1}{2}\Big)\right]\frac{1}{b-a}\int\limits_a^b g(x)\mathrm{d}x-g\Big(\frac{a+b}{2}\Big).$$

This implies that

$$\left|\frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x-\frac{1}{h_1\big(\frac{1}{2}\big)+h_2\big(\frac{1}{2}\big)}f\Big(\frac{a+b}{2}\Big)\right|$$

$$\leq\frac{1}{b-a}\int\limits_a^b g(x)\mathrm{d}x-\frac{1}{h_1\big(\frac{1}{2}\big)+h_2\big(\frac{1}{2}\big)}g\Big(\frac{a+b}{2}\Big).$$

This completes the proof. □

Theorem 4.2. *Let $h_1,h_2:[0,1]\to\mathbb{R}$ be two non-negative functions and $g:I\subset\mathbb{R}\to[0,\infty)$ be a (h_1,h_2)-convex function. Let $f:I\subset\mathbb{R}\to[0,\infty)$ be (g,h_1,h_2)-convex dominated function on I where $f\in L_1[a,b]$, then*

$$\left|f(a)\int\limits_0^1 h_1(t)\mathrm{d}t+f(b)\int\limits_0^1 h_2(t)\mathrm{d}t-\frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x\right| \tag{4.4}$$

$$\leq g(a)\int\limits_0^1 h_1(t)\mathrm{d}t+g(b)\int\limits_0^1 h_2(t)\mathrm{d}t-\frac{1}{b-a}\int\limits_a^b g(x)\mathrm{d}x.$$

Proof. Let $x=a$ and $y=b$ in the definition of (g,h_1,h_2)-convex dominated function, we have

$$|h_1(t)f(a)+h_2(t)f(b)-f(ta+(1-t)b)|$$

$$\leq h_1(t)g(a)+h_2(t)g(b)-g(ta+(1-t)b).$$

Integrating above inequalities with respect to t on $[0,1]$, we have

$$\left|f(a)\int_0^1 h_1(t)\mathrm{d}t + f(b)\int_0^1 h_2(t)\mathrm{d}t - \int_0^1 f(ta+(1-t)b)\mathrm{d}t\right|$$

$$\leq g(a)\int_0^1 h_1(t)\mathrm{d}t + g(b)\int_0^1 h_2(t)\mathrm{d}t - \int_0^1 g(ta+(1-t)b)\mathrm{d}t.$$

This implies that

$$\left|f(a)\int_0^1 h_1(t)\mathrm{d}t + f(b)\int_0^1 h_2(t)\mathrm{d}t - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\leq g(a)\int_0^1 h_1(t)\mathrm{d}t + g(b)\int_0^1 h_2(t)\mathrm{d}t - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x.$$

This completes the proof. □

Corollary 4.3. *Under the conditions of Theorem 4.1 and of Theorem 4.2, if* $h_1(t)=t$,$h_2(t)=1-t$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - g\Big(\frac{a+b}{2}\Big), \tag{4.5}$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x. \tag{4.6}$$

These inequalities were proved for the first time in [7].

Corollary 4.4. *Under the conditions of Theorem 4.1 and of Theorem 4.2, if* $h_1(t)=t^s$ *and* $h_2(t)=(1-t)^s$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - 2^{s-1}f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - 2^{s-1}g\Big(\frac{a+b}{2}\Big), \tag{4.7}$$

$$\left|\frac{f(a)+f(b)}{s+1} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{s+1} - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x. \tag{4.8}$$

These inequalities were derived for the first time in [13].

Corollary 4.5. *Under the conditions of Theorem 4.1 and of Theorem 4.2, if $h_1(t) = 1 = h_2(t)$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - \frac{1}{2}g\Big(\frac{a+b}{2}\Big), \tag{4.9}$$

$$\left|f(a)+f(b) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq g(a)+g(b) - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x. \tag{4.10}$$

These inequalities were proved for the first time in [18].

Corollary 4.6. *Under the conditions of Theorem 4.1 and of Theorem 4.2, if $h_1(t) = t^{-1}$ and $h_2(t) = (1-t)^{-1}$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{4}f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - \frac{1}{4}g\Big(\frac{a+b}{2}\Big), \tag{4.11}$$

This inequality was derived for the first time in [18].

Corollary 4.7. *Under the conditions of Theorem 4.1 and of Theorem 4.2, if $h_1(t) = t^{-s}$ and $h_2(t) = (1-t)^{-s}$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2^{s+1}}f\Big(\frac{a+b}{2}\Big)\right| \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x - \frac{1}{2^{s+1}}g\Big(\frac{a+b}{2}\Big), \tag{4.12}$$

$$\left|\frac{f(a)+f(b)}{1-s} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{g(a)+g(b)}{1-s} - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x. \tag{4.13}$$

These inequalities are new.

We suggest that it may be possible to derive inequalities of Hermite-Hadamard type in case of (k, h_1, h_2)-convex dominated functions, if there is a point t in which $k(t) \neq t$, under suitable hypotheses on function k.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

Gabriela Cristescu participated by coordinating the research and elaborating Section 2, Mihail Găianu elaborated Section 3 and Muhammad Uzair Awan elaborated Section 4. All authors read and approved the final manuscript.

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