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Strong convergence of the Ishikawa iteration for Lipschitz α-hemicontractive mappings

Micah Okwuchukwu Osilike and Anthony Chibuike Onah

Abstract. A new class of *α-hemicontractive* maps T for which the strong convergence of the Ishikawa iteration algorithm to a fixed point of T is assured is introduced and studied. The study is a continuation of a recent study of a new class of α-demicontractive mappings T by L. Măruşter and Ş. Măruşter, *Mathematical and Computer Modeling* **54** (2011) 2486-2492 in which they proved strong convergence of the Mann iteration scheme to a fixed point of T. Our class of α-hemicontractive maps is more general than the class of α-demicontractive maps. No compactness assumption is imposed on the operator or it's domain, and no additional requirement is imposed on the set of fixed points.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||.||$. Let C be a nonempty closed convex subset of H.

Definition 1.1. *A mapping $T : C \to C$ is said to be* demicontractive *(see for example* [1]*) if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and there exists $k \in [0, 1)$*

such that

$$||Tx - p||^2 \leq ||x - p||^2 + k||x - Tx||^2, \ \forall \ x \in C \text{ and } \forall p \in F(T). \qquad (1.1)$$

The class of demicontractive maps coincides with the class of mappings satisfying *condition (A)* which was studied by Ş. Măruşter [2, 3].

Definition 1.2. *$T : C \to C$ is said to satisfy* condition (A) *if $F(T) \neq \emptyset$ and there exists $\lambda > 0$ such that*

$$\langle x - Tx, x - p \rangle \geq \lambda ||x - Tx||^2, \ \forall \ x \in C \text{ and } \forall p \in F(T). \qquad (1.2)$$

Definition 1.3. *T is said to be* hemicontractive *(see for example* [4]*) if $k = 1$ in (1.1).*

The class of demicontractive maps is a proper subclass of the class of hemicontractive maps (see for example [4]). The classes of demicontractive maps and hemicontractive maps have been studied by many authors (see for example [1–14]).

The Mann iteration scheme $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \ n \geq 1, \qquad (1.3)$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $(0, 1]$ satisfying some appropriate conditions has been used by several authors for the approximation of fixed points of demicontractive maps. It is now well known (see for example [15]) that Mann iteration scheme may not in general converge to a fixed point of a hemicontractive map in Hilbert spaces. For hemicontractive maps, the *Ishikawa iteration sequence* $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \ n \geq 1, \qquad (1.4)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in $[0, 1]$ is usually applicable.

Demicontractivity of T alone is not sufficient for the convergence of the Mann iteration to a fixed point of T even in finite dimensional spaces (see for example [17]). Some additional smoothness properties of T are necessary, like continuity or *demiclosedness principle*.

Definition 1.4. *$T : C \to H$ is said to be* demiclosed *at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to x^* in C and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$.*

In finite dimensional spaces, Măruşter proved the following:

Theorem 1.1. *([2]) Let $\Re^m$ be the Euclidean m-dimensional space and let $T : \Re^m \to \Re^m$ be a nonlinear mapping satisfying the conditions:*
(i) $(I - T)$ is demiclosed at 0;
(ii) T is demicontractive with constant k, or, equivalently T satisfies condition (A) with $\lambda = \frac{(1-k)}{2}$;
(iii) $0 < a < \alpha_n \leq b < 2\lambda = 1 - k$. Then the Mann iteration sequence $\{x_n\}$ converges to a point of $F(T)$ for any starting point $x_0 \in \Re^m$.

In infinite dimensional spaces, the two conditions (demicontractivity and demiclosedness principle) are not sufficient for strong convergence (see for example [3, 17]). The two conditions however ensure weak convergence of $\{x_n\}$ to a fixed point of T in real Hilbert spaces and some more general Banach spaces (see for example [1, 3, 8]). In order to obtain strong convergence, some additional conditions or some modifications of the standard Mann iteration are necessary. Such modifications have been considered by several authors (see for example [3, 5, 6, 8, 10–12, 18–20]).

In [3] the existence of a nonzero solution $h \in H,\ h \neq 0$, of the variational inequality

$$\langle x - Tx, h\rangle \leq 0,\ \forall x \in H \tag{1.5}$$

is required as an additional condition for strong convergence. The results of [3] has been extended by some authors to either more general Banach spaces or to the Ishikawa iteration scheme (see for example [8, 11–13]). We note however that the existence of a nonzero solution of the variational inequality (1.5) exists only in very particular cases.

In exploring more conditions that may be less restrictive than the condition of the existence of a nonzero solution of (1.5), L. Măruşter and Ş. Măruşter [17] introduced a new concept of demicontractivity called *α-demicontractivity.*

Definition 1.5. *A mapping $T : C \to C$ is said to be* α-demicontractive [17] *if $F(T) \neq \emptyset$ and there exist $\lambda > 0$, $\alpha \geq 1$ such that*

$$\langle x - Tx, x - \alpha p\rangle \geq \lambda||x - Tx||^2,\ \forall x \in C \text{ and } \forall p \in F(T) \tag{1.6}$$

Clearly (1.6) is equivalent to

$$||Tx - \alpha p||^2 \leq ||x - \alpha p||^2 + k||x - Tx||^2,\ \forall x \in C \text{ and } \forall p \in F(T), \tag{1.7}$$

where $k = 1-2\lambda \in [0, 1)$. It is easy to observe that if T is α-demicontractive, then $\alpha p \in F(T)\ \forall p \in F(T)$ such that αp remains in the domain $D(T)$ of T. Since if T is demicontractive, then $F(T)$ is closed and convex, it follows that if T is both demicontractive (1-demicontractive) and α-demicontractive, $\alpha > 1$

then the line segment $(1-t)p+t\alpha p$, $t \in [0,1]$, is contained in $F(T)$, $\forall p \in F(T)$ such that αp remains in the domain $D(T)$ of T. In [17] an example of an α-demicontractive mapping with $\alpha > 1$ which is not demicontractive is given and it is easy to observe that there are demicontractive (1-demicontractive) maps which are not α-demicontractive for $\alpha > 1$ (see for example ([4], Example 2.2)). For other properties of this new class of demicontractive mappings, the reader may consult [17].

In [17] the authors proved the following strong convergence theorem:

Theorem 1.2 ([17] Theorem 5)**.** *Let C be a closed convex subset of a real Hilbert space H and let let $T : C \to C$ be a demicontractive mapping with constant k, or, equivalently T satisfies condition (A) with $\lambda = \frac{(1-k)}{2}$. Let T be α-demicontractive for some $\alpha > 1$ and let $(I-T)$ be demiclosed at 0. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ which satisfy the condition $0 < a \leq \alpha_n \leq b < 2\lambda = 1-k$. Then for suitable $x_0 \in C$, the sequence $\{x_n\}_{n=1}^{\infty}$ of the Mann iteration sequence given by (1.3) converges strongly to a fixed point of T.*

It is our purpose in this paper to study the more general class of α-demicontractive mappings for which $k = 1$ and which we call α-hemicontractive mappings following the usual terminology. For this more general class of mapppings, we prove strong convergence theorem similar to Theorem 1.2 using the Ishikawa iteration scheme.

2 Main Results

Definition 2.1. *We say that a mapping $T : C \to C$ is* α-hemicontractive *if $F(T) \neq \emptyset$ and there exists $\alpha \geq 1$ such that*

$$||Tx - \alpha p||^2 \leq ||x - \alpha p||^2 + ||x - Tx||^2, \ \forall x \in C \text{ and } \forall p \in F(T). \tag{2.1}$$

Observe that (2.1) is equivalent to

$$\langle x - Tx, x - \alpha p\rangle \geq 0, \ \forall x \in C \text{ and } \forall p \in F(T). \tag{2.2}$$

We discuss the following examples.

Example 2.1. ([4], Example 2.4) Let $\Re$ denote the reals with the usual norm and let $C = [0,1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T is hemicontractive (1-hemicontractive) but not α-hemicontractive for some $\alpha > 1$. T is neither demicontractive (1-demicontractive) nor α-demicontractive for some $\alpha > 1$.

Example 2.2. Let $\Re$ denote the reals with the usual norm and let $C = [1,4] \subset \Re$. Define $T : C \to C$ by;

$$Tx = \begin{cases} x^2, & 1 \leq x \leq 2 \\ \\ 1, & 2 < x \leq 4 \end{cases}$$

Then T is $2-$hemicontractive (i.e., T is α-hemicontractive with $\alpha = 2$). T is not hemicontractive (1-hemicontractive).

Example 2.3. ([21]) Let $\Re$ denote the reals with the usual norm and let $C = (-\infty, 1)$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{x}{1-x}, & x \in (-\infty, 0], \\ \frac{x}{x-1}, & x \in [0, 1). \end{cases}$$

Then T is hemicontractive (1-hemicontractive) and is also α-hemicontractive for all $\alpha > 1$. T is neither demicontractive (1-demicontractive) nor α-demicontractive for some $\alpha > 1$.

Remark 2.1. It is easy to verify that if T is *hemicontractive* and

$$\langle x - Tx, p \rangle \leq 0, \ \forall \ (x,p) \in C \times F(T),$$

then T is *α-hemicontractive* for all $\alpha > 1$.

In [13] the authors proved the following:

Theorem 2.1. *(*[13]*, Theorem 1.) Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a Lipschitz hemicontractive mapping. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$* and *$\{c'_n\}$ be real sequences in $[0,1]$ satisfying the conditions:*
(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \ n \geq 1,$
(ii) $0 < \epsilon \leq b'_n \leq b_n \leq b < 1, \ \forall n \geq 1,$ for some $\epsilon > 0$ and for some $b \in (0, \frac{1}{[(\sqrt{1+L^2})+1]}),$
(iii) $\sum_{n=1}^{\infty} c_n < \infty, \ \sum_{n=1}^{\infty} c'_n < \infty.$
Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in C and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = a'_n x_n + b'_n T[a_n x_n + b_n T x_n + c_n u_n] + c'_n v_n, \ n \geq 1. \tag{2.3}$$

Then $\lim_{n \to \infty} ||x_n - Tx_n|| = 0.$

Remark 2.2. If in Theorem 2.1 $(I-T)$ is demiclosed at 0, then $\{x_n\}$ converges weakly to a fixed point of T (see for example Theorem 2 of [13]).

As in the case of demicontractive maps, hemicontractiveness and the demiclosedness principle are not sufficient to obtain strong convergence of the Ishikawa scheme to a fixed point of T. Additional conditions are required on the map and or the subset C. In [22] the authors assumed that the interior of $F(T)$ is nonempty ($\mathrm{int}(F(T)) \neq \emptyset$) to achieve strong convergence. This appears very restrictive since even in $\Re$ with the usual norm, Lipschitz hemicontractive maps with finite number of fixed points do not enjoy this condition that $\mathrm{int}(F(T)) \neq \emptyset$.

Remark 2.3. The assumption that T is demicontractive (1 - demicontractive) in Theorem 1.2 is to ensure the weak convergence of $\{x_n\}$ to a point $p \in F(T)$. This assumption appears unnecessary since the following argument shows that weak convergence of $\{x_n\}$ to a point $p \in F(T)$ is guaranteed if T is α-demicontractive for some $\alpha > 1$. If $T : C \to C$ is α-demicontractive with some $\alpha > 1$, let $p \in F(T)$ be arbitrary. Then using the well known identity

$$||(1-t)x + ty||^2 = (1-t)||x||^2 + t||y||^2 - t(1-t)||x-y||^2 \tag{2.4}$$

which holds for all x, y in H and for all t in $[0,1]$ we obtain

$$\begin{aligned} ||x_{n+1} - \alpha p||^2 &= ||(1-\alpha_n)(x_n - \alpha p) + \alpha_n(Tx_n - \alpha p)||^2 \\ &= (1-\alpha_n)||x_n - \alpha p||^2 + \alpha_n||Tx_n - \alpha p||^2 \\ &\quad -\alpha_n(1-\alpha_n)||x_n - Tx_n||^2 \\ &\leq ||x_n - \alpha p||^2 - \alpha_n[1-\alpha_n - k]||x_n - Tx_n||^2 \\ &\leq ||x_n - \alpha p||^2 - a[1-k-b]||x_n - Tx_n||^2. \end{aligned} \tag{2.5}$$

It follows from (2.5) that $\lim\limits_{n\to\infty} ||x_n - \alpha p||$ exists for all $p \in F(T)$, and $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = 0$. Since $\{x_n\}$ is bounded, it has a subsequence say $\{u_n\}_{n=1}^{\infty}$ which converges weakly to a point $u \in C$. Since $\lim\limits_{n\to\infty} ||u_n - Tu_n|| = 0$ and $(I-T)$ is demiclosed at 0, then $u \in F(T)$. To conclude that $\{x_n\}$ converges weakly to u, it suffices to show that if $\{x_n\}$ has any other subsequence $\{v_n\}_{n=1}^{\infty}$ which converges weakly to v, then $u = v$. Observe that we also have that $v \in F(T)$ and thus $\lim\limits_{n\to\infty} ||x_n - \alpha u||$ and $\lim\limits_{n\to\infty} ||x_n - \alpha v||$ exist. Let $\lim\limits_{n\to\infty} ||x_n - \alpha u|| = d_1$ and $\lim\limits_{n\to\infty} ||x_n - \alpha v|| = d_2$ and consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by $a_n = ||u_n - \alpha u||^2 - ||v_n - \alpha u||^2 - ||u_n - \alpha v||^2 + ||v_n - \alpha v||^2,\ n \geq 1$. Observe that $\lim\limits_{n\to\infty} a_n = 0$. Observe also that $a_n = -2\alpha\langle u_n - v_n, u - v\rangle$ and the weak convergence of $\{u_n\}$ and $\{v_n\}$ to u and v respectively imply that $\lim_{n\to\infty} a_n = -\alpha||u-v||^2$. Hence $-\alpha||u-v||^2 = 0$ and $u = v$.

We now prove the following.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be an L-Lipschitzian and α-hemicontractive mapping with $\alpha > 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ which satisfy the condition $0 < \epsilon \leq \alpha_n \leq \beta_n \leq b < 1$ for some $\epsilon > 0$ and for some $b \in (0, \frac{1}{\sqrt{1+L^2}+1})$. Let $(I-T)$ be demiclosed at 0. Then for suitable $x_1 \in K$, the sequence $\{x_n\}$ given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n],\ n \geq 1 \tag{2.6}$$

converges strongly to a point in $F(T)$.

Proof. Let $G_n x_n := T[(1-\beta_n)x_n + \beta_n Tx_n],\ n \geq 1$. Then for all $p \in F(T)$ we have

$$\begin{aligned}
||G_n x_n - \alpha p||^2 &= ||T[(1-\beta_n)x_n + \beta_n Tx_n] - \alpha p||^2 \\
&\leq ||(1-\beta_n)x_n + \beta_n Tx_n - \alpha p||^2 \\
&\quad + ||(1-\beta_n)x_n + \beta_n Tx_n - G_n x_n||^2
\end{aligned}$$

$$\begin{aligned}
&= ||(1-\beta_n)(x_n - \alpha p) + \beta_n(Tx_n - \alpha p)||^2 \\
&\quad + ||(1-\beta_n)(x_n - G_n x_n) \\
&\quad + \beta_n(Tx_n - T[(1-\beta_n)x_n + \beta_n Tx_n])||^2 \\
&= (1-\beta_n)||x_n - \alpha p||^2 + \beta_n||Tx_n - \alpha p||^2 \\
&\quad - \beta_n(1-\beta_n)||x_n - Tx_n||^2 + (1-\beta_n)||x_n - G_n x_n||^2 \\
&\quad + \beta_n||Tx_n - T[(1-\beta_n)x_n + \beta_n Tx_n]||^2 \\
&\quad - \beta_n(1-\beta_n)||x_n - Tx_n||^2 \\
&\leq ||x_n - \alpha p||^2 + \beta_n||x_n - Tx_n||^2 - \beta_n(1-\beta_n)||x_n - Tx_n||^2 \\
&\quad + (1-\beta_n)||x_n - G_n x_n||^2 + L^2\beta_n^3||x_n - Tx_n||^2 \\
&\quad - \beta_n(1-\beta_n)||x_n - Tx_n||^2 \\
&= ||x_n - \alpha p||^2 + (1-\beta_n)||x_n - G_n x_n||^2 \\
&\quad - \beta_n[1 - 2\beta_n - \beta_n^2 L^2]||x_n - Tx_n||^2. \qquad (2.7)
\end{aligned}$$

Using the condition on $\{\beta_n\}$ in (2.7) we obtain

$$||G_n x_n - \alpha p||^2 \leq ||x_n - \alpha p||^2 + (1-\beta_n)||x_n - G_n x_n||^2. \tag{2.8}$$

It follows easily from (2.8) that

$$\langle x_n - G_n x_n, x_n - \alpha p\rangle \geq \frac{\beta_n}{2}||x_n - G_n x_n||^2. \tag{2.9}$$

Using (2.7) we obtain for arbitrary $p \in F(T)$ that

$$\begin{aligned} ||x_{n+1} - \alpha p||^2 &= ||(1-\alpha_n)(x_n - \alpha p) + \alpha_n(G_n x_n - \alpha p)||^2 \\ &= (1-\alpha_n)||x_n - \alpha p||^2 + \alpha_n||G_n x_n - \alpha p||^2 \\ &\quad -\alpha_n(1-\alpha_n)||x_n - G_n x_n||^2 \\ &\leq (1-\alpha_n)||x_n - \alpha p||^2 + \alpha_n\Big[||x_n - \alpha p||^2 \\ &\quad +(1-\beta_n)||x_n - G_n x_n||^2 \\ &\quad -\beta_n(1 - 2\beta_n - \beta_n^2 L^2)||x_n - Tx_n||^2\Big] \\ &\quad -\alpha_n(1-\alpha_n)||x_n - G_n x_n||^2 \end{aligned}$$

$$\begin{aligned} &= ||x_n - \alpha p||^2 - \alpha_n(\beta_n - \alpha_n)||x_n - G_n x_n||^2 \\ &\quad -\alpha_n\beta_n[1 - 2\beta_n - \beta_n^2 L^2]||x_n - Tx_n||^2 \\ &\leq ||x_n - \alpha p||^2 - \alpha_n\beta_n[1 - 2\beta_n - \beta_n^2 L^2]||x_n - Tx_n||^2 \\ &\leq ||x_n - \alpha p||^2 - \epsilon^2[1 - 2b - b^2 L^2]||x_n - Tx_n||^2. \qquad (2.10) \end{aligned}$$

It follows from (2.10) that $\lim_{n\to\infty} ||x_n - \alpha p||$ exists for all $p \in F(T)$, and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

Thus as in Remark 2.3 we obtain that $\{x_n\}$ converges weakly to some $p \in F(T)$. We prove that

$$\langle x_n - p, p\rangle \geq \frac{1}{2(\alpha - 1)}||x_n - p||^2, \ \forall n \geq 1. \tag{2.11}$$

We choose $x_1 \in C$ (see for example [17]) such that

$$\langle x_1 - p, p\rangle \geq \frac{1}{2(\alpha - 1)}||x_1 - p||^2. \tag{2.12}$$

Suitable $x_1 \in C$ exists since if $P_C : H \to C$ is the proximity map (projection map from H onto C), then for $\lambda \in \Re$ such that $1 < \lambda \leq 2\alpha - 1$, we can choose $x_1 = P_C(\lambda p)$.

Then since the proximity map, P_C is firmly nonexpansive (i.e., $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$), it is easy to verify that x_1 satisfies (2.12).

The proof of (2.11) now follows by induction since if we assume (2.11), then from (2.9) and $\alpha > 1$ we obtain

$$
\begin{aligned}
& \langle x_n - G_n x_n, x_n - \alpha p\rangle \geq \frac{\beta_n}{2}||x_n - G_n x_n||^2 \geq \frac{\alpha_n}{2}||x_n - G_n x_n||^2 \\
\Rightarrow\ & \langle x_n - G_n x_n, x_n - p - (\alpha - 1)p\rangle \geq \frac{\alpha_n}{2}||x_n - G_n x_n||^2 \\
\Rightarrow\ & -(\alpha - 1)\langle x_n - G_n x_n, p\rangle \geq -\langle x_n - G_n x_n, x_n - p\rangle \\
& + \frac{\alpha_n}{2}||x_n - G_n x_n||^2 \\
\Rightarrow\ & -\langle x_n - G_n x_n, p\rangle \geq -\frac{1}{(\alpha - 1)}\langle x_n - G_n x_n, x_n - p\rangle \\
& + \frac{\alpha_n}{2(\alpha - 1)}||x_n - G_n x_n||^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ & -\alpha_n\langle x_n - G_n x_n, p\rangle \geq -\frac{\alpha_n}{(\alpha - 1)}\langle x_n - G_n x_n, x_n - p\rangle \\
& + \frac{\alpha_n^2}{2(\alpha - 1)}||x_n - G_n x_n||^2,
\end{aligned}
$$

and with the inductive hypothesis (2.11) we obtain

$$
\begin{aligned}
& \langle x_n - p - \alpha_n(x_n - G_n x_n), p\rangle \geq \frac{1}{2(\alpha - 1)}||x_n - p||^2 \\
& \qquad - \frac{\alpha_n}{(\alpha - 1)}\langle x_n - G_n x_n, x_n - p\rangle \\
& \qquad + \frac{\alpha_n^2}{2(\alpha - 1)}||x_n - G_n x_n||^2, \\
\Rightarrow\ & \langle (1 - \alpha_n)x_n + \alpha_n G_n x_n - p, p\rangle \geq \frac{1}{2(\alpha - 1)}\Big[||x_n - p||^2 \\
& \qquad - 2\alpha_n\langle x_n - G_n x_n, x_n - p\rangle \\
& \qquad + \alpha_n^2||x_n - G_n x_n||^2\Big] \\
\Rightarrow\ & \langle (1 - \alpha_n)x_n + \alpha_n G_n x_n - p, p\rangle \geq \frac{1}{2(\alpha - 1)}||x_n - p - \alpha_n(x_n - G_n x_n)||^2 \\
\Rightarrow\ & \langle x_{n+1} - p, p\rangle \geq \frac{1}{2(\alpha - 1)}||x_{n+1} - p||^2.
\end{aligned}
$$

Since $\{x_n\}$ converges weakly to p, we have that $\lim_{n\to\infty} ||x_n - p|| = 0$. □

Remark 2.4. If C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is an L-Lipschizian α-hemicontractive map, $\alpha \geq 1$, then

$\alpha p \in F(T)$ for all $p \in F(T)$ such that αp remains in the domain $D(T)$ of T. Furthermore, since if T is L-Lipschtzian hemicontractive, then $F(T)$ is closed and convex, it follows that if T is both L-Lipschitzian hemictractive and α-hemicontractive, $\alpha > 1$, then the line segment $(1-t)p + t(\alpha p)$, $t \in [0,1]$, is contained in $F(T)$ for all $p \in F(T)$ such that αp remains in the domain $D(T)$ of T.

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Micah Okwuchukwu Osilike

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

E-mail: `osilike@yahoo.com, micah.osilike@unn.edu.ng`

Anthony Chibuike Onah

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

E-mail: `onah87@gmail.com`

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