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Existence of $\Psi-$ Bounded Solutions for a Lyapunov Matrix Differential Equation

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Abstract. It is proved a necessary and sufficient condition for the existence of at least one $\Psi-$ bounded solution of a linear nonhomogeneous Lyapunov matrix differential equation. In addition, it is given a result in connection with the asymptotic behavior of the $\Psi-$ bounded solutions of this equation.

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1 Introduction

The purpose of our paper is to give a necessary and sufficient condition so that the nonhomogeneous Lyapunov matrix differential equation

$$X' = A(t)X + XB(t) + F(t) \tag{1.1}$$

has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi-$ bounded matrix function F on $\mathbb{R}_+ = [0, \infty)$.

In present paper, Ψ will be a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

Recently, the existence of at least one $\Psi-$ bounded solution of equation (1.1) on $\mathbb{R}_+$ or $\mathbb{R}$ for various types of functions F has been studied in [4]-[6],

[8]. In [7], the authors have been studied the problem of $\Psi-$ boundedness of solutions for the corresponding Kronecker product system (2.1) associated with (1.1) (see a comment in [4]).

The approach used in our paper is essentially based on the technique of Kronecker product of matrices (which has been successfully applied in similar problems - see, e.g. [4]-[8]) and on a decomposition of the underlying space at the initial moment (see, e.g. [4]-[8] for finite-dimensional spaces and [9]-[10] in general case of Banach spaces).

Thus, we obtain results which contain and extend the recent results regarding the boundedness of solutions of the equation (1.1) (see [2]-[4], [7]).

2 Preliminaries

In this section we present some basic definitions, notations, hypotheses and results which are useful later on.

Let $\mathbb{R}^n$ be the Euclidean n - dimensional space. For $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, let $\| x \| = \max\{| x_1 |, | x_2 |, ..., | x_n |\}$ be the norm of x (^{T} denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all $m \times n$ matrices with real entries.

For a $n \times n$ real matrix $A = (a_{ij})$, we define $| A |$ by $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$.

It is well-known that $| A | = \max\limits_{1\leq i\leq n}\{\sum\limits_{j=1}^{n} | a_{ij}|\}$.

Definition 2.1. ([1]) *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B written $A \otimes B$ is defined to be the block partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 2.1. *The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;

2). $(A \otimes B)^T = A^T \otimes B^T$;

3). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
4). $(A \otimes B) \cdot (C \otimes D) = AC \otimes BD$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_p \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8). $(A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t)$; *(here,* $'$ *denotes derivative* $\frac{d}{dt}$ *).*

Proof. See in [1]. $\square$

Definition 2.2. *The application* $Vec : \mathbb{M}_{m \times n} \longrightarrow \mathbb{R}^{mn}$, *defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij})$, *is called the vectorization operator.*

Lemma 2.2. *The vectorization operator* $Vec : \mathbb{M}_{n \times n} \longrightarrow \mathbb{R}^{n^2}$, *is a linear and one-to-one operator. In addition,* $\mathcal{V}ec$ *and* $\mathcal{V}ec^{-1}$ *are continuous operators.*

Proof. See Lemma 2, [4]. $\square$

Remark 2.1. Using Definition 2.2, we can see that if F is a continuous matrix function on $\mathbb{R}_+$, then $f = Vec(F)$ is a continuous vector function on $\mathbb{R}_+$ and reciprocally.

Lemma 2.3. *If* $A, B, M \in \mathbb{M}_{n \times n}$, *then*
1). $\mathcal{V}ec(AMB) = (B^T \otimes A) \cdot \mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n) \cdot \mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A) \cdot \mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A) \cdot \mathcal{V}ec(I_n)$.

Proof. It is a simple exercise. $\square$

Let $\Psi_{\mathrm{i}} : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., n$, be continuous functions and

$$\Psi = \mathrm{diag}\ [\Psi_1, \Psi_2, \cdots \Psi_n].$$

Definition 2.3. *(*[2]*) A function* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ *is said to be* $\Psi-$ *bounded on* $\mathbb{R}_+$ *if* Ψf *is bounded on* $\mathbb{R}_+$ *(i.e.* $\sup_{t \geq 0} \parallel \Psi(t) f(t) \parallel < +\infty$).

Below, we extend this definition for matrix functions.

Definition 2.4. *([4]) A matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is said to be $\Psi-$ bounded on $\mathbb{R}_+$ if the matrix function ΨM is bounded on $\mathbb{R}_+$ (i.e. $\sup\limits_{t\geq 0} \mid \Psi(t)M(t) \mid< +\infty$).*

Now, we shall assume that A, B and F are continuous $n \times n$ - matrices on $\mathbb{R}_+$.

By a solution of (1.1), we mean a continuously differentiable $n \times n -$ matrix function X satisfying the equation (1.1) for all $t \geq 0$.

The following lemmas play a vital role in the proof of the main results.

Lemma 2.4. *([4]) The matrix function $X(t)$ is a solution of (1) if and only if the vector valued function $x(t) = Vec(X(t))$ is a solution of the differential system*

$$x' = (I_n \otimes A(t) + B^T(t) \otimes I_n)x + f(t), \tag{2.1}$$

where $f(t) = \mathcal{V}ec(F(t))$.

Proof. See Lemma 7, [4]. □

Definition 2.5. *The above system (2.1) is called "corresponding Kronecker product system associated with (1.1)".*

Lemma 2.5. *([4]) The matrix function $M(t)$ is $\Psi-$ bounded on $\mathbb{R}_+$ if and only if the vector function $\mathcal{V}ec(M(t))$ is $I_n \otimes \Psi$ $-$ bounded on $\mathbb{R}_+$.*

Proof. See Lemma 5, [4]. □

The next Lemma is Lemma 1 of [7]. Because the proof is incomplete, we presented it with a complete proof in [4], as Lemma 6.

Lemma 2.6. *([4]) Let $X(t)$ and $Y(t)$ be the fundamental matrices for the equations*

$$X'(t) = A(t)X(t) \tag{2.2}$$

and

$$Y'(t) = Y(t)B(t) \tag{2.3}$$

respectively.

Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the system

$$z'(t) = (I_n \otimes A(t) + B^T(t) \otimes I_n)z(t). \tag{2.4}$$

If, in addition, $X(0) = I_n$ and $Y(0) = I_n$, then $Z(0) = I_{n^2}$.

Proof. See Lemma 6, [4]. □

Now, let $Z(t)$ the above fundamental matrix for the system (2.4) with $Z(0) = I_{n^2}$.

Let $\widetilde{X}_1$ denote the subspace of $\mathbb{R}^{n^2}$ consisting of all vectors which are values of $I_n \otimes \Psi-$ bounded solutions of (2.4) on $\mathbb{R}_+$ for $t = 0$ and let $\widetilde{X}_2$ an arbitrary fixed subspace of $\mathbb{R}^{n^2}$, supplementary to $\widetilde{X}_1$. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ denote the corresponding projections of $\mathbb{R}^{n^2}$ onto $\widetilde{X}_1$, $\widetilde{X}_2$ respectively.

Finally, we remind two theorems which will be used in the proofs of our main results.

Theorem 2.7. *([3]) If A is a continuous $d \times d$ real matrix on $\mathbb{R}_+$ then, the system $x' = A(t)x + f(t)$ has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi-$ bounded function f on $\mathbb{R}_+$ if and only if for the fundamental matrix $Y(t)$ of the system $x' = A(t)x$ there exists a positive constant K such that, for $t \geq 0$,*

$$\begin{aligned} &\int_0^t \mid \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \mid ds + \\ &+\int_t^\infty \mid \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \mid ds \leq K. \end{aligned} \tag{2.5}$$

Theorem 2.8. *([3]) Suppose that:*

1°. The fundamental matrix Y(t) of the system $x' = A(t)x$ satisfies the condition (2.5) for all $t \geq 0$, where K is a positive constant;

2°. The continuous and $\Psi-$ bounded function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is such that

$$\lim_{t\to+\infty} \parallel \Psi(t)f(t) \parallel = 0.$$

Then, every $\Psi-$ bounded solution x of the system $x' = A(t)x + f(t)$ is such that

$$\lim_{t\to+\infty} \parallel \Psi(t)x(t) \parallel = 0.$$

Remark 2.2. In these theorems, P_1 and P_2 are supplementary projections as $\widetilde{P}_1$ and $\widetilde{P}_2$, for the system $x' = A(t)x$.

3 The main result

In this section we present the main result of our paper in connection with the existence of $\Psi-$ bounded solutions for the nonhomogeneous Lyapunov matrix differential equation (1.1).

Theorem 3.1. *Let A and B be continuous $n \times n$ real matrix function on $\mathbb{R}_+$ and let X and Y be the fundamental matrices of the homogeneous linear equations (2.2) and (2.3) respectively for which $X(0) = Y(0) = I_n$.*

Then, the equation (1.1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi-$ bounded matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ if and only if there exists a positive constant K such that, for all $t \geq 0$,

$$\int_0^t \mid \ (Y^T(t) \otimes (\Psi(t)X(t)))\widetilde{P}_1((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))) \mid ds + \tag{3.1}$$
$$+\int_t^\infty \mid \ (Y^T(t) \otimes (\Psi(t)X(t)))\widetilde{P}_2((Y^T(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))) \mid ds \leq K.$$

Proof. First, we prove the "only if" part.

Suppose that the equation (1.1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi-$ bounded matrix function F : $\mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$.

Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ be a continuous and $I_n \otimes \Psi-$ bounded function on $\mathbb{R}_+$. From Lemma 2.5, it follows that the matrix function $F(t) = \mathcal{V}ec^{-1}(f(t))$ is continuous and $\Psi-$ bounded on $\mathbb{R}_+$. From the hypothesis, the equation

$$X' = A(t)X + XB(t) + \mathcal{V}ec^{-1}(f(t))$$

has at least one $\Psi-$ bounded solution $X(t)$ on $\mathbb{R}_+$.

From Lemma 2.4 and Lemma 2.5, it follows that the vector valued function $x(t) = \mathcal{V}ec(X(t))$ is a $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of the differential system

$$x' = (I_n \otimes A(t) + B^T(t) \otimes I_n)x + f(t).$$

Thus, this system has at least one $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $I_n \otimes \Psi-$ bounded function f on $\mathbb{R}_+$.

From Theorem 2.7, for a fundamental matrix $Z(t)$ of (2.4), there exists a positive constant K such that

$$\int_0^t \mid \ (I_n \otimes \Psi(t))Z(t)\widetilde{P}_1 Z^{-1}(s)(I_n \otimes \Psi(s))^{-1} \mid ds +$$
$$\int_t^\infty \mid \ (I_n \otimes \Psi(t))Z(t)\widetilde{P}_2 Z^{-1}(s)(I_n \otimes \Psi(s))^{-1} \mid ds \leq K,$$

for all $t \geq 0$.

By Lemma 2.6, we have $Z(t) = Y^T(t) \otimes X(t)$. Now, a calculation shows that (3.1) holds.

Now, we prove the "if" part.

Suppose that (3.1) holds for some $K > 0$ and for all $t \geq 0$.

Let $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ a continuous and $\Psi-$ bounded matrix function on $\mathbb{R}_+$.

From Lemma 2.5, it follows that the vector valued function $f(t) = \mathcal{V}ec(F(t))$ is continuous and $I_n \otimes \Psi-$ bounded function on $\mathbb{R}_+$.

From this, (3.1), Lemma 2.6 and Theorem 2.7, it follows that the differential system

$$x'(t) = (I_n \otimes A(t) + B^T(t) \otimes I_n)x(t) + f(t)$$

has at least one $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$. Let $x(t)$ be this solution.

From Lemma 2.4 and Lemma 2.5, it follows that the matrix function $X(t) = \mathcal{V}ec^{-1}(x(t))$ is a $\Psi-$ bounded solution on $\mathbb{R}_+$ of the equation (1.1) (because $F(t) = \mathcal{V}ec^{-1}(f(t))$).

Thus, the differential equation (1.1) has at least one $\Psi-$ bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi-$ bounded matrix function F on $\mathbb{R}_+$.

The proof is now complete. $\square$

Remark 3.1. Theorem 3.1 generalizes Theorem 1, [3].

Indeed, in the particular case $B(t) = O_n$, we have $Y = I_n$ and then $Z(t) = I_n \otimes X(t)$. If, in addition

$$F(t) = \begin{pmatrix} f_1(t) & f_1(t) & \cdots & f_1(t) \\ f_2(t) & f_2(t) & \cdots & f_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(t) & f_n(t) & \cdots & f_n(t) \end{pmatrix},$$

it is easy to see that the solutions of (1) are

$$X(t) = \begin{pmatrix} x_1(t) & x_1(t) & \cdots & x_1(t) \\ x_2(t) & x_2(t) & \cdots & x_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n(t) & x_n(t) & \cdots & x_n(t) \end{pmatrix},$$

where $x = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the solution of the system

$$x'(t) = A(t)x(t) + f(t),$$

with $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T$.

In this case, the condition (3.1) becomes the condition (2.5).

Thus, Theorem generalizes the result from [3].

We prove finally a theorem in which we will see that the asymptotic behavior of solutions of (1.1) is completely determined by the asymptotic behavior of $F(t)$ as $t \longrightarrow \infty$.

Theorem 3.2. *Suppose that:*

1). The fundamental matrices $X(t)$ and $Y(t)$ of (2.2) and (2.3) respectively ($X(0) = Y(0) = I_n$) satisfy the condition (3.1) for some $K > 0$ and for all $t \geq 0$;

2). The continuous matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ satisfies the condition

$$\lim_{t\to\infty} \mid \Psi(t)F(t) \mid \ = 0.$$

Then, every $\Psi-$ bounded solution $X(t)$ of (1.1) satisfies the condition

$$\lim_{t\to\infty} \mid \Psi(t)X(t) \mid \ = 0.$$

Proof. Let $X(t)$ be a $\Psi -$ bounded solution of (1.1). From Lemma 2.4 and Lemma 2.5, it follows that the function $x(t) = \mathcal{V}ec(X(t))$ is a $I_n \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of the differential system

$$x' = (I_n \otimes A(t) + B^T(t) \otimes I_n)x + f(t),$$

where $f(t) = \mathcal{V}ec(F(t))$.

Also, from the proof of Lemma 2.5, we have

$$\| (I_n \otimes \Psi(t)) \cdot f(t) \|_{\mathbb{R}^{n^2}} \ \leq \ \mid \Psi(t)F(t) \mid, \ t \geq 0$$

and then,

$$\lim_{t\to\infty} \| (I_n \otimes \Psi(t)) \cdot f(t) \|_{\mathbb{R}^{n^2}} = 0.$$

From the Theorem 2.8, it follows that

$$\lim_{t\to\infty} \| (I_n \otimes \Psi(t)) \cdot x(t) \|_{\mathbb{R}^{n^2}} = 0.$$

Now, from the proof of Lemma 2.5 again, we have

$$\mid \Psi(t)X(t) \mid \ \leq \ n \| (I_n \otimes \Psi(t)) \cdot x(t) \|_{\mathbb{R}^{n^2}}, \ t \geq 0$$

and then

$$\lim_{t\to\infty} \mid \Psi(t)X(t) \mid \ = 0.$$

The proof is now complete. □

Remark 3.2. Theorem 3.2 generalizes Theorem 2.2, [3].

Remark 3.3. Note that if we do not have $\lim\limits_{t\to\infty} \mid \Psi(t)F(t) \mid = 0$, then, the $\Psi-$ bounded solution $X(t)$ may be such that $\mid \Psi(t)X(t) \mid \nrightarrow 0$ as $t \to \infty$.

This is shown by the next

Example 3.1. Consider the linear equation (1.1) with

$$A(t) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},\ B(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},\ \text{and } F(t) = \begin{pmatrix} e^{4t} & 0 \\ e^{-2t} & 0 \end{pmatrix}.$$

The fundamental matrices for the equations (2.2) and (2.3) are

$$X(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix},\ Y(t) = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

respectively.

Consider

$$\Psi(t) = \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

It is easy to see that the condition of Theorem 3.2 is satisfied with

$$\widetilde{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \widetilde{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $K = 2$.

On the other hand, we have $\mid \Psi(t)F(t) \mid = 1$, for all $t \geq 0$.

The solutions of the equation (1.1) are

$$X(t) = \begin{pmatrix} c_1e^{3t} + e^{4t} & c_3 \\ c_2e^{-t} - e^{-2t} & c_4e^{-4t} \end{pmatrix},\ c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

The $\Psi-$ bounded solutions of the equation (1.1) are

$$X(t) = \begin{pmatrix} c_1e^{3t} + e^{4t} & c_3 \\ -e^{-2t} & c_4e^{-4t} \end{pmatrix},\ c_1, c_3, c_4 \in \mathbb{R}.$$

It is easy to see that for every $\Psi-$ bounded solution of (1.1) we have

$$\lim_{t\to\infty} \mid \Psi(t)X(t) \mid = 1.$$

Note that the asymptotic properties of the components of the solutions are not the same. On the other hand, we see that the asymptotic properties of the components of the solutions are the same, via matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

Remark 3.4. This Example shows that the hypothesis 2 of Theorem 3.2 is an essential condition for the conclusion of the theorem.

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