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On some Concepts of (a, b, c)-Trichotomy for Noninvertible Linear Discrete-Time Systems in Banach Spaces

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Abstract. This paper considers three general trichotomy concepts for noninvertible linear discrete-time systems in Banach spaces. Characterizations of these concepts are obtained from the point of view of the projections sequences. Some illustrative examples are given in order to prove that these concepts are distinct.

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1 Introduction

The notion of trichotomy plays a central role in the qualitative theory of discrete -time systems, which has an impressive development. As a natural generalization of the dichotomy property, concepts of trichotomy have been introduced by R. S. Sacker and G. R. Sell in [13] and S. Elaydi and O. Hajek in [4]. The case of discrete-time systems was considered by S. Elaydi and K. Janglajew in [5]. In the last decades a substantial part of the trichotomy theory was dedicated to the extension of the methods used in dichotomy theory to the trichotomy case (see [1], [2], [3], [6], [7], [8], [9], [10], [11], [12]).

This paper presents three general trichotomy concepts ((a,b,c)- *trichotomy, strong* (a,b,c)- *trichotomy and weak* (a,b,c)-*trichotomy)* for non-invertible discrete linear systems. Connections between these concepts are given. These concepts contain as particular cases some uniform and nonuniform concepts of exponential or polynomial trichotomies.

2 Preliminaries

Let X be a real or complex Banach space and let *B(X)* be the space of all bounded linear operators on X. The norms on X and on *B(X)* will be denoted by $\|\cdot\|$. We also denote by I and O the identity operator and respectively the null operator on X, Δ the set of all pairs of natural numbers (m,n) with $m \geq n$, $T = \Delta \times X$.

Let (A_n) be a sequence in *B(X)*. We consider the linear discrete-time system

$$x_{n+1} = A_n x_n,\ n \in \mathbb{N}. \tag{$\mathcal{A}$}$$

Every solution $x = (x_n)$ of the system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n$$

for all $(m,n) \in \Delta$, where $A : \Delta \to B(X)$ is defined by

$$A_m^n := \begin{cases} A_{m-1}...A_n, & \text{if } m > n \\ I, & \text{if } m = n \end{cases}$$

Remark 2.1. We have that

$$A_m^n A_n^p = A_m^p$$

for all (m,n) and $(n,p) \in \Delta$.

Definition 2.1. A sequence (P_n) is called a **projections sequence on X,** if

$$P_n^2 = P_n$$

for every $n \in \mathbb{N}$.

A projections sequence (P_n) with the property

$$P_{n+1}A_n = A_n P_n$$

for all $n \in \mathbb{N}$, is called **invariant** for the system $(\mathcal{A})$.

Remark 2.2. The projections sequence (P_n) is invariant for $(\mathcal{A})$ if and only if

$$A_m^n P_n = P_m A_m^n$$

for all $(m, n) \in \Delta$.

Definition 2.2. A projections sequence (P_n) is called **strongly invariant** for the system $(\mathcal{A})$, if it is invariant for $(\mathcal{A})$ and for every $(m, n) \in \Delta$ the restriction of A_m^n on $RangeP_n$ is an isomorphism from $RangeP_n$ to $RangeP_m$.

Remark 2.3. If the projections sequence (P_n) is strongly invariant for the system $(\mathcal{A})$ then there exists $B : \Delta \to \mathcal{B}(X)$ such that

$b_1)$ $A_m^n B_m^n P_m = P_m$;

$b_2)$ $B_m^n A_m^n P_n = P_n$;

$b_3)$ $B_m^n P_m = P_n B_m^n P_m$;

for all $(m, n) \in \Delta$.

Definition 2.3. Three projections sequences (P_n^1),(P_n^2),(P_n^3) are called **supplementary** if

$s_1)$ $P_n^1 + P_n^2 + P_n^3 = I$, for every $n \in \mathbb{N}$;

$s_2)$ $P_n^i \cdot P_n^j = O$, for all $n \in \mathbb{N}$ and all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

If $\mathcal{P} = \{(P_n^1), (P_n^2), (P_n^3)\}$ is a family of three supplementary projections sequences which are invariant for $(\mathcal{A})$ then we say that the pair $(\mathcal{A}, \mathcal{P})$ is a **trichotomic pair**.

Definition 2.4. A family $\mathcal{P} = \{(P_n^1), (P_n^2), (P_n^3)\}$ of three supplementary projections sequences is called **compatible** with the system $(\mathcal{A})$ if

$c_1)$ (P_n^1) is invariant for $(\mathcal{A})$;

$c_2)$ (P_n^2) and (P_n^3) are strongly invariant for $(\mathcal{A})$.

3 (a, b, c)- trichotomy

Let $(\mathcal{A}, \mathcal{P})$ be a trichotomic pair and let $a = (a_n), b = (b_n)$ and $c = (c_n)$ be three nondecreasing sequences of positive real numbers with $a_n \geq 1, b_n \geq 1$ and $c_n \geq 1$ for all $n \in \mathbb{N}$.

Definition 3.1. We say that the pair $(\mathcal{A}, \mathcal{P})$ is **(a,b,c)-trichotomic** if there exists $N \geq 1$ such that

$t_1)$ $a_m \|A_m^n P_n^1 x\| \leq N a_n b_n \|P_n^1 x\|$;

$t_2)$ $a_m \|P_n^2 x\| \leq N a_n b_m \|A_m^n P_n^2 x\|$;

$t_3)$ $c_n \|A_m^n P_n^3 x\| \leq N b_n c_m \|P_n^3 x\|$;

$t_4)$ $c_n \|P_n^3 x\| \leq N b_m c_m \|A_m^n P_n^3 x\|$;

for all $(m, n, x) \in T$.

In the particular case when the sequence (b_n) is constant, we say that $(\mathcal{A}, \mathcal{P})$ is **uniformly (a,c)-trichotomic**.

Remark 3.1. As particular cases of $(a, b, c) - trichotomy$ we observe that

i) if $a_n = e^{\alpha n}$ and $c_n = e^{\beta n}$ with $\alpha, \beta > 0$ then we recover the notion of **nonuniform exponential trichotomy** and in particular when the sequence (b_n) is constant we obtain the classical notion of **nonuniform exponential trichotomy**;

ii) if $a_n = (n+1)^\alpha$ and $c_n = (n+1)^\beta$ with $\alpha, \beta > 0$ then we recover the notion of **nonuniform polynomial trichotomy** and in particular when the sequence (b_n) is constant we obtain the notion of **uniform polynomial trichotomy**;

iii) if $P_n^3 = O$ for every $n \in \mathbb{N}$, then we recover the property of **(a,b) dichotomy nonuniform exponential dichotomy** (for $a_n = e^{\alpha n}$ with $\alpha > 0$), **uniform exponential dichotomy** (for $a_n = e^{\alpha n}$ and (b_n) constant), **nonuniform polynomial dichotomy** (for $a_n = (n+1)^\alpha$ with $\alpha > 0$) and **uniform polynomial dichotomy** (when $a_n = (n+1)^\alpha$ and (b_n) constant.)

A characterization of (a,b,c)-trichotomy is given by

Theorem 3.1. *Let $(\mathcal{A}, \mathcal{P})$ be a trichotomic pair with the property that $(\mathcal{P})$ is compatible with $(\mathcal{A})$. Then the pair $(\mathcal{A}, \mathcal{P})$ is $(a, b, c) - trichotomic$ if and only if there exists a constant $N \geq 1$ such that*

$t_1')$ $a_m \|A_m^n P_n^1 x\| \leq N a_n b_n \|P_n^1 x\|$;

$t_2')$ $a_m \|B_m^n P_m^2 x\| \leq N a_n b_m \|P_m^2 x\|$;

$t_3')$ $c_n \|A_m^n P_n^3 x\| \leq N b_n c_m \|P_n^3 x\|$;

t_4') $c_n\|B_m^n P_m^3 x\| \leq N b_m c_m \|P_m^3 x\|$;

for all $(m, n, x) \in T$.

Proof. It is sufficient to prove that $(t_2) \Leftrightarrow (t_2')$ and $(t_4) \Leftrightarrow (t_4')$. For to prove that $(t_2) \Rightarrow (t_2')$ we observe that

$$\begin{aligned} a_m\|B_m^n P_m^2 x\| &= a_m\|P_n^2 B_m^n P_m^2 x\| \\ &\leq N a_n b_m \|A_m^n P_n^2 B_m^n P_m^2 x\| \\ &\leq N a_n b_m \|A_m^n B_m^n P_m^2 x\| \\ &= N a_n b_m \|P_m^2 x\| \end{aligned}$$

for all $(m, n, x) \in T$.
Similarly, for to prove the implication $(t_2') \Rightarrow (t_2)$ we observe that

$$\begin{aligned} a_m\|P_n^2 x\| &= a_m\|B_m^n P_m^2 A_m^n P_n^2 x\| \\ &\leq N a_n b_m \|P_m^2 A_m^n P_n^2 x\| \\ &= N a_n b_m \|A_m^n P_n^2 x\| \end{aligned}$$

for all $(m, n, x) \in T$.
$(t_4) \Rightarrow (t_4')$. If we suppose that (t_4) is satisfied then

$$\begin{aligned} c_n\|B_m^n P_m^3 x\| &= c_n\|P_n^3 B_m^n P_m^3 x\| \\ &\leq N b_m c_m \|A_m^n P_n^3 B_m^n P_m^3 x\| \\ &= N b_m c_m \|A_m^n B_m^n P_m^3 x\| \\ &= N b_m c_m \|P_m^3 x\| \end{aligned}$$

for all $(m, n, x) \in T$ and hence the inequality (t_4') in holds. $(t_4') \Rightarrow (t_4)$. If (t_4') holds then

$$\begin{aligned} c_n\|P_n^3 x\| &= c_n\|B_m^n A_m^n P_n^3 x\| \\ &\leq N b_m c_m \|P_m^3 A_m^n P_n^3 x\| \\ &= N b_m c_m \|A_m^n P_n^3 x\| \end{aligned}$$

for every $(m, n, x) \in T$ and hence the inequality (t_4) is verified. $\square$

4 Strong (a, b, c)- trichotomy

Let $(\mathcal{A}, \mathcal{P})$ be a trichotomic pair and let $a = (a_n), b = (b_n)$ and $c = (c_n)$ be three nondecreasing sequences of positive real numbers with $a_n \geq 1, b_n \geq 1$ and $c_n \geq 1$ for all $n \in \mathbb{N}$.

Definition 4.1. We say that the pair $(\mathcal{A}, \mathcal{P})$ is **stongly (a,b,c)-trichotomic** if there exists $N \geq 1$ such that

$st_1)$ $a_m \|A_m^n P_n^1 x\| \leq N a_n b_n \|x\|$;

$st_2)$ $a_m \|x\| \leq N a_n b_m \|A_m^n P_n^2 x\|$;

$st_3)$ $c_n \|A_m^n P_n^3 x\| \leq N b_n c_m \|x\|$;

$st_4)$ $c_n \|x\| \leq N b_m c_m \|A_m^n P_n^3 x\|$;

for all $(m, n, x) \in T$.

Proposition 4.1. *If the pair $(\mathcal{A}, \mathcal{P})$ is strongly (a,b,c)-trichotomic then it is also (a,b,c)- trichotomic.*
Proof. *If $(\mathcal{A}, \mathcal{P})$ is strongly (a,b,c)-trichotomic then by substituting x by $P_n^1 x$ in (st_1), x by $P_n^2 x$ in (st_2), respectively x by $P_n^3 x$ in (st_3) and (st_4) then we obtain that $(t_1), (t_2), (t_3)$ and (t_4) are satisfied and hence $(\mathcal{A}, \mathcal{P})$ is $(a, b, c) -$ trichotomic.*

Remark 4.1. The converse of the implication from Proposition 4.1 is not generally true. (see Example 6.3)

Definition 4.2. Let $\mathcal{P} = \{(P_n^1), (P_n^2), (P_n^3)\}$ be a family of three supplementary projections sequences and let $b = (b_n)$ be a nondecreasing sequence of positive real numbers with $b_n \geq 1$.

We say that $\mathcal{P}$ **is b-bounded**, if there exists $M \geq 1$ such that

$$\|P_n^j\| \leq M b_n$$

for all $n \in \mathbb{N}$ and all $j \in \{1, 2, 3\}$.

Proposition 4.2. *If the pair $(\mathcal{A}, \mathcal{P})$ is strongly (a,b,c)-trichotomic then $\mathcal{P}$ is b-bounded.*
Proof. *If $(\mathcal{A}, \mathcal{P})$ is strongly (a,b,c)-trichotomic then for $m = n$ in the inequalities (st_1) and (st_3) we obtain $\|P_n^1\| \leq N b_n$, $\|P_n^3\| \leq N b_n$ for all $n \in \mathbb{N}$. Then $\|P_n^2\| \leq 1 + \|P_n^1\| + \|P_n^3\| \leq 3N b_n$ for every $n \in \mathbb{N}$.*
Finally it results that $\mathcal{P}$ is b-bounded.

5 Weak (a, b, c)- trichotomy

Let $(\mathcal{A}, \mathcal{P})$ be a trichotomic pair and let $a = (a_n), b = (b_n)$ and $c = (c_n)$ be three nondecreasing sequences of positive real numbers with $a_n \geq 1, b_n \geq 1$ and $c_n \geq 1$ for all $n \in \mathbb{N}$.

Definition 5.1. We say that the pair $(\mathcal{A}, \mathcal{P})$ is **weakly (a,b,c)-trichotomic** if there exists $N \geq 1$ such that

$wt_1)$ $a_m \|A_m^n P_n^1\| \leq N a_n b_n \|P_n^1\|$;

$wt_2)$ $a_m \|P_n^2\| \leq N a_n b_m \|A_m^n P_n^2\|$;

$wt_3)$ $c_n \|A_m^n P_n^3\| \leq N b_n c_m \|P_n^3\|$;

$wt_4)$ $c_n \|P_n^3\| \leq N b_m c_m \|A_m^n P_n^3\|$;

for all $(m, n) \in \Delta$.

Proposition 5.1. *If the pair $(\mathcal{A}, \mathcal{P})$ is $(a, b, c) - trichotomic$ then it is also weakly (a,b,c)- trichotomic.*

Proof. It follows from Definition 3.1 by taking the supremum with respect to $\|x\| \leq 1$. □

Corollary 5.1. *If $(\mathcal{A}, \mathcal{P})$ is strongly (a,b,c)- trichotomic then it also weakly (a,b,c)-trichotomic.*
Proof. *It is a consequence of the Proposition 4.1 and 5.1.*

A characterization of weak (a,b,c)-trichotomy is given by

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{P})$ be a trichotomic pair with the property that $(\mathcal{P})$ is compatible with $(\mathcal{A}$. If the pair $(\mathcal{A}, \mathcal{P})$ is $weakly(a, b, c) - trichotomic$ then there exists a constant $N \geq 1$ such that*

$wt_1^{'})$ $a_m \|A_m^n P_n^1\| \leq N a_n b_n \|P_n^1\|$;

$wt_2^{'})$ $a_m \|B_m^n P_m^2\| \leq N a_n b_m \|P_m^2\|$;

$wt_3^{'})$ $c_n \|A_m^n P_n^3\| \leq N b_n c_m \|P_n^3\|$;

$wt_4^{'})$ $c_n \|B_m^n P_m^3\| \leq N b_m c_m \|P_m^3\|$;

for all $(m, n) \in \Delta$.

Proof. It is similar to the proof of Theorem 3.1. □

6 Examples

Firstly, we present a pair $(\mathcal{A}, \mathcal{P})$ which is $(a, b, c) - trichotomic.$

Example 6.1. Let $\mathcal{P} = \{(P_n^1), (P_n^2), (P_n^3)\}$ be a family of three supplementary projections sequences with the following properties

(*i*) $P_m^2 P_n^1 = P_m^2 P_n^3 = O$ and $P_m^2 P_n^2 = P_m^2$ for all $(m, n) \in \mathbb{N}^2$;

(*ii*) $\|P_n^2 x\| \leq \|P_m^2 x\|$ for every $(m, n, x) \in T$.

For an example of such family see Example 6.2.
Let $a = (a_n), b = (b_n)$ and $c = (c_n)$ be three nondecreasing sequences of positive real numbers with $a_n \geq 1, b_n \geq 1$ and $c_n \geq 1$ for all $n \in \mathbb{N}$.

Consider the linear discrete-time system $(\mathcal{A})$ generated by sequence

$$A_n = \frac{a_n}{a_{n+1}} P_n^1 + \frac{a_{n+1}}{a_n} P_{n+1}^2 + \frac{c_n}{c_{n+1}} P_n^3.$$

Then $(\mathcal{A}, \mathcal{P})$ is a trichotomic pair with

$$A_m^n = \frac{a_n}{a_m} P_n^1 + \frac{a_m}{a_n} P_m^2 + \frac{c_n}{c_m} P_n^3$$

$$A_m^n P_n^1 = \frac{a_n}{a_m} P_n^1, A_m^n P_n^2 = \frac{a_m}{a_n} P_m^2, A_m^n P_n^3 = \frac{c_n}{c_m} P_n^3.$$

Moreover we have that

$t_1)$ $a_m \|A_m^n P_n^1 x\| = a_n \|P_n^1 x\| \leq a_n b_n \|P_n^1 x\|$;

$t_2)$ $a_m \|P_n^2 x\| \leq a_m \|P_m^2 x\| = a_n \|A_m^n P_n^2 x\| \leq a_n b_m \|A_m^n P_n^2 x\|$;

$t_3)$ $c_n \|A_m^n P_n^3 x\| \leq c_n \|P_n^3 x\| \leq b_n c_m \|P_n^3 x\|$;

$t_4)$ $c_n \|P_n^3 x\| = c_m \|A_m^n P_n^3 x\| \leq b_m c_m \|A_m^n P_n^3 x\|$;

for all $(m, n, x) \in T$.
Thus $(\mathcal{A}, \mathcal{P})$ is uniformly $(a, c) - trichotomic$ and hence it is also $(a, b, c) - trichotomic$ for every nondecreasing sequence (b_n).
This example shows that for every triplet of sequences (a, b, c) there exists a trichotomic pair $(\mathcal{A}, \mathcal{P})$ which is $(a, b, c) - trichotomic.$

An example of a $strong(a, b, c) - trichotomic$ pair is presented in

Example 6.2. On $X = \mathbb{R}^3$ endowed with the norm

$$\|(x_1, x_2, x_3)\| = max\{|x_1|, |x_2|, |x_3|\}.$$

We consider the family $\mathcal{P} = \{(P_n^1), (P_n^2), (P_n^3)\}$ of three supplementary projections sequences defined by

$$P_n^1(x_1, x_2, x_3) = (x_1 + x_2 b_n, 0, 0)$$

$$P_n^2(x_1, x_2, x_3) = (-x_2 b_n, x_2, 0)$$

$$P_n^3(x_1, x_2, x_3) = (0, 0, x_3).$$

Where b_n is a nondecreasing real sequence with $b_n \geq 1$.
It is easy to see that $\mathcal{P}$ satisfies the properties (i) and (ii) from Example 6.1.
Moreover $\|P_n^j\| \leq 2b_n$ for all $n \in \mathbb{N}$ and all $j \in \{1, 2, 3\}$.
Thus $\mathcal{P}$ is *b-bounded*.
If we consider the system $(\mathcal{A})$ defined in previous example then $(\mathcal{A}, \mathcal{P})$ satisfies the inequalities $(st_1), (st_2), (st_3)$ and (st_4) and hence $(\mathcal{A}, \mathcal{P})$ is *strongly* $(a, b, c)-$ *trichotomic*.

An example of a $(a, b, c)-$*trichotomic* pair which is not *strongly* $(a, b, c)-$ *trichotomic* is given in

Example 6.3. On $X = \mathbb{R}^3$ with the same norm as in the previous example we consider the family $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ given by

$$P_n^1(x_1, x_2, x_3) = (x_1 + x_2 b_n^2, 0, 0)$$

$$P_n^2(x_1, x_2, x_3) = (-x_2 b_n^2, x_2, 0)$$

$$P_n^3(x_1, x_2, x_3) = (0, 0, x_3)$$

and the linear discrete-time system $(\mathcal{A})$ generated by the sequence

$$A_n = \frac{1}{e} P_n^1 + e P_n^2 + \frac{1}{e} P_n^3.$$

For $a_n = b_n = c_n = e^n$ we have that the pair $(\mathcal{A}, \mathcal{P})$ is $(a, b, c) -$ *trichotomic* and because (P_n^2) is not *b-bounded* it follows that $(\mathcal{A}, \mathcal{P})$ is not *strongly-invariant*.

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