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On (h, k)-Dichotomy and (h, k)-Trichotomy of Noninvertible Evolution Operators in Banach Spaces

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Abstract. The paper considers some concepts of (h, k)-dichotomy and (h, k)-trichotomy for noninvertible evolution operators in Banach spaces. A characterization of the (h, k)-trichotomy of an evolution operator in terms of (h, k)-dichotomy for two associated evolution operators is given. As applications of this result, characterizations for nonuniform exponential trichotomy and nonuniform polynomial trichotomy are obtained.

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1 Introduction

In the qualitative theory of nonautonomous dynamical systems an important role is played by the dichotomy and trichotomy properties. These concepts were studied in an extensive manner from the point of view of uniform, nonuniform, exponential and polynomial behaviors (see, for example: [1], [2], [3], [5], [8], [9], [11], [13], [15], [16], [17], [18], [19]).

As natural generalizations of the above behaviors are successfully modeled by the properties of (h, k)-dichotomy and (h, k)-trichotomy. These concepts were studied in a large number of papers containing many interesting results. For more recent works we refer the reader to [4], [6], [7], [10], [12], [14], [20], [21] and the references therein.

In this paper we introduce two general concepts of (h,k)-dichotomy respectively (h,k)-trichotomy for dynamical systems defined by an evolution operator in a Banach space. We consider simultaneously the general cases of nonautonomous, noninvertible and nonuniform dynamics with arbitrary growth rates. These concepts include as particular cases some uniform (nonuniform) exponential or polynomial dichotomy respectively trichotomy.

Our main aim is to prove the equivalence between (h,k)-trichotomy of an evolution operator U and (h,k)-dichotomy of two evolution operators associated to U.

As consequences, we obtain characterizations of nonuniform exponential trichotomy and nonuniform polynomial trichotomy for differential equations in Banach spaces.

2 Evolution operators and families of projections

Let X be a real or complex Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $||\cdot||$.

We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}^2 \text{ with } t \geq s \geq 0\} \text{ and } T = \Delta \times X.$$

Definition 2.1. An operator valued $U : \Delta \to \mathcal{B}(X)$ is called an *evolution operator* on X if

(e_1) $U(t,t) = I$ (the identity operator on X) for every $t \geq 0$;

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all (t,s) and $(s,t_0) \in \Delta$.

Definition 2.2. An operator valued $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a *family of projections* if

$$P(t)^2 = P(t) \quad \text{for every} \quad t \geq 0.$$

Definition 2.3. Given an evolution operator $U : \Delta \to \mathcal{B}(X)$, we say that a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is *invariant for U,* if

$$U(t,s)P(s) = P(t)U(t,s) \quad \text{for all} \quad (t,s) \in \Delta.$$

In the dichotomy theory are used families of two projections. In this aim we introduce

Definition 2.4. If $P_1, P_2 : \mathbb{R}_+ \to \mathcal{B}(X)$ are two families of projections on X, then we say that $\mathcal{P} = \{P_1, P_2\}$ is

(i) *orthogonal,* if $P_1(t) + P_2(t) = I$ for every $t \geq 0$;

(ii) *compatible* for the evolution operator $U : \Delta \to \mathcal{B}(X)$, if $\mathcal{P}$ is orthogonal and P_1, P_2 are invariant for U.

In the trichotomy theory are used families of three projections. For these families we give

Definition 2.5. Let $P_1, P_2, P_3 : \mathbb{R}_+ \to \mathcal{B}(X)$ be three families of projections on X. We say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is

(i) *orthogonal,* if

   (o_1) $P_1(t) + P_2(t) + P_3(t) = I$ for every $t \geq 0$;

   (o_2) $P_i(t)P_j(t) = P_j(t)P_i(t) = 0$ for all $t \geq 0$ and $i, j \in \{1, 2, 3\}$ with $i \neq j$;

   (o_3) $||P_i(t)x + P_j(t)x||^2 = ||P_i(t)x||^2 + ||P_j(t)x||^2$ for all $t \geq 0, x \in X$ and $i, j \in \{1, 2, 3\}$ with $i \neq j$;

(ii) *compatible* with the evolution operator $U : \Delta \to \mathcal{B}(X)$, if $\mathcal{P}$ is orthogonal and P_1, P_2 and P_3 are invariant for U.

Definition 2.6. Let $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$ be compatible with the evolution operator U. We say that $\mathcal{P}^+$ and $\mathcal{P}^-$ are *supplementary* if

(s_1) $P_1^+(t)P_2^-(t) = P_2^-(t)P_1^+(t) = 0$,

(s_2) $P_2^+(t)P_1^-(t) = P_1^-(t)P_2^+(t) = P_2^+(t) - P_2^-(t) = P_1^-(t) - P_1^+(t)$,

(s_3) $||P_1^+(t)x + P_2^-(t)x||^2 = ||P_1^+(t)x||^2 + ||P_2^-(t)x||^2$,

(s_4) $||P_2^+(t)x - P_2^-(t)x||^2 = ||P_2^+(t)x||^2 - ||P_2^-(t)x||^2$,

(s_5) $||P_1^-(t)x - P_1^+(t)x||^2 = ||P_1^-(t)x||^2 - ||P_1^+(t)x||^2$

for all $t \geq 0$ and $x \in X$.

A first connection between these concepts is given by

Proposition 2.1. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator U then*

$$\mathcal{P}^+ = \{P_1^+, P_2^+\} \text{ and } \mathcal{P}^- = \{P_1^-, P_2^-\},$$

where

$$P_1^+ = P_1,\ P_2^+ = P_2 + P_3,\ P_1^- = P_1 + P_3,\ P_2^- = P_2,$$

are supplementary.

Proof. It is obvious that if $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with U then $\mathcal{P}^+$ and $\mathcal{P}^-$ are also compatible with U.

Moreover, we have

$$(s_1) \qquad P_1^+ P_2^- = P_1 P_2 = 0 = P_2 P_1 = P_2^- P_1^+$$

and

$$(s_2) \qquad P_2^+ P_1^- = P_1^- P_2^+ = P_2^+ - P_2^- = P_1^- - P_1^+ = P_3.$$

The following conditions (s_3), (s_4) and (s_5) from Definition 2.6 are verified because

$$\begin{aligned}(s_3) \quad & ||P_1^+(t)x + P_2^-(t)x||^2 = ||P_1(t)x + P_2(t)x||^2 \\ & = ||P_1(t)x||^2 + ||P_2(t)x||^2 = ||P_1^+(t)x||^2 + ||P_2^-(t)x||^2,\end{aligned}$$

$$\begin{aligned}(s_4) \quad & ||P_2^+(t)x - P_2^-(t)x||^2 = ||P_3(t)x||^2 = ||P_2(t)x + P_3(t)x||^2 \\ & - ||P_2(t)x||^2 = ||P_2^+(t)x||^2 - ||P_2^-(t)x||^2,\end{aligned}$$

$$\begin{aligned}(s_5) \quad & ||P_1^-(t)x - P_1^+(t)x||^2 = ||P_3(t)x||^2 = ||P_1(t)x + P_3(t)x||^2 \\ & - ||P_1(t)x||^2 = ||P_1^-(t)x||^2 - ||P_1^+(t)x||^2\end{aligned}$$

for all $(t, s, x) \in T$. □

A converse of Proposition 2.1 is given by

Proposition 2.2. *If $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$ are supplementary then $\mathcal{P} = \{P_1, P_2, P_3\}$, where*

$$P_1 = P_1^+, \quad P_2 = P_2^-, \quad P_3 = P_1^- P_2^+$$

is compatible with the evolution operator U.

Proof. If $\mathcal{P}^+$ and $\mathcal{P}^-$ are supplementary then

(o_1)
$$P_1+P_2+P_3 = P_1^+ + P_2^- + P_1^-P_2^+ = P_1^+ + P_2^- + P_2^+ - P_2^- = \\ = P_1^+ + P_2^+ = I,$$

(o_2)
$$P_1P_2 = P_1^+P_2^- = P_2^-P_1^+ = P_2P_1 = 0,$$
$$P_1P_3 = P_1^+P_1^-P_2^+ = P_1^+P_2^+P_1^- = 0 = P_1^-P_2^+P_1^+ = P_3P_1$$
and
$$P_2P_3 = P_2^-P_2^+P_1^- = 0 = P_1^-P_2^+P_2^- = P_3P_2,$$

(o_3)
$$||P_1(t)x + P_2(t)x||^2 = ||P_1^+(t)x + P_2^-(t)x||^2 \\ = ||P_1^+(t)x||^2 + ||P_2^-(t)x||^2 = ||P_1(t)x||^2 + ||P_2(t)x||^2,$$
$$||P_1(t)x + P_3(t)x||^2 = ||P_1^-(t)x||^2 = ||P_1^+(t)x||^2 + ||P_1^-(t)x||^2 \\ - ||P_1^+(t)x||^2 = ||P_1^+(t)x||^2 + ||P_1^-(t)x - P_1^+(t)x||^2 \\ = ||P_1(t)x||^2 + ||P_3(t)x||^2,$$
$$||P_2(t)x + P_3(t)x||^2 = ||P_2^+(t)x||^2 = ||P_2^-(t)x||^2 \\ + ||P_2^+(t)x - P_2^-(t)x||^2 = ||P_2(t)x||^2 + ||P_3(t)x||^2$$

for all $(t,s,x) \in T$.

It results that $\mathcal{P}$ is orthogonal. If $\mathcal{P}^+$ and $\mathcal{P}^-$ are supplementary, then $P_1^{\pm}$, $P_2^{\pm}$ are invariant for U. It follows that $P_1 = P_1^+$, $P_2 = P_3^-$ and $P_3 = P_1^- - P_1^+$ are invariant for U, i.e. $\mathcal{P}$ is invariant for U.

Finally, we obtain that $\mathcal{P}$ is compatible with U. □

3 (h,k)-dichotomy

We say that an increasing function $h : \mathbb{R}_+ \to [1,+\infty)$ is a *growth rate*, if $\lim\limits_{t\to\infty} h(t) = +\infty$.

Let $h,k : \mathbb{R}_+ \to [1,+\infty)$ be two growth rates.

Definition 3.1. We say that the evolution operator $U : \Delta \to \mathcal{B}(X)$ is (h,k)-*dichotomic,* if there are two families of projections $P_1, P_2 : \mathbb{R}_+ \to \mathcal{B}(X)$ and three constants $N \geq 1,\ d > 0$ and $\varepsilon \geq 0$ such that

(d_0) $\mathcal{P} = \{P_1, P_2\}$ is compatible with U;

(d_1) $h(t)^d||U(t,s)P_1(s)x|| \leq Nh(s)^dk(s)^\varepsilon||P_1(s)x||$;

(d_2) $h(t)^d||P_2(s)x|| \leq Nh(s)^d k(t)^\varepsilon ||U(t,s)P_2(s)x||$

for all $(t, s, x) \in T$.

If $\varepsilon = 0$ or when the function k is constant, then we say that the evolution operator U is *uniformly h-dichotomic.*

The constants N, d and ε are called *dichotomy constants.*

Remark 3.1. As particular cases of (h, k)-dichotomy we remark that

(i) if $h(t) = k(t) = e^t$ for all $t \geq 0$, then we recover the notion of *nonuniform exponential dichotomy* and in particular when the function k is constant or $\varepsilon = 0$, we obtain the classical notion of *uniform exponential dichotomy*;

(ii) if $h(t) = k(t) = t + 1$ for all $t \geq 0$, then we obtain the property of *nonuniform polynomial dichotomy* and in particular when $\varepsilon = 0$ or the function k is constant, we recover the classical notion of *uniform polynomial dichotomy.*

The following example shows that for every two growth rates $h, k : \mathbb{R}_+ \to [1, +\infty)$ and for all constants $N \geq 1$, $d > 0$ and $\varepsilon \geq 0$ there exists an evolution operator U and a family of projections $\mathcal{P} = \{P_1, P_2\}$ compatible with U such that U is (h, k)-dichotomic with respect to $\mathcal{P}$ and with the dichotomic constants N, d and ε.

Example 3.1. On $X = \mathbb{R}^3$ endowed with the norm

$$||(x_1, x_2, x_3)|| = |x_1| + |x_2| + |x_3|$$

we consider the family $\mathcal{P} = \{P_1, P_2\}$, where $P_1, P_2 : \mathbb{R}^3 \to \mathcal{B}(X)$ are defined by

$$P_1(t)(x_1, x_2, x_3) = (x_1, 0, 0)$$

respectively

$$P_2(t)(x_1, x_2, x_3) = (0, x_2, x_3)$$

for all $t \geq 0$ and $x = (x_1, x_2, x_3) \in X$.

Given the growth rates $h, k : \mathbb{R}_+ \to [1, +\infty)$ and the constants $N \geq 1$, $d > 0$ and $\varepsilon \geq 0$, we consider the evolution operator $U : \Delta \to \mathcal{B}(X)$, defined by

$$U(t, s)(x) = \left(\left(\frac{h(s)}{h(t)}\right)^d \left(\frac{k(s)}{k(t)}\right)^\varepsilon x_1, \left(\frac{h(t)}{h(s)}\right)^{2d} \left(\frac{k(s)}{k(t)}\right)^\varepsilon x_2,\right.$$

$$\left(\frac{h(t)}{h(s)}\right)^{2d}\left(\frac{k(s)}{k(t)}\right)^{\varepsilon} x_3\Bigg)$$

for all $(t,s) \in \Delta,\ x = (x_1, x_2, x_3) \in X$.

It is easy to verify that the projections family $\mathcal{P} = \{P_1, P_2\}$ is compatible with U.

Moreover, we have that

$$(d_1) \quad h(t)^d ||U(t,s)P_1(s)x|| = h(s)^d \left(\frac{k(s)}{k(t)}\right)^{\varepsilon} ||P_1(s)x|| \\ \leq N h(s)^d k(s)^{\varepsilon} ||P_1(s)x||$$

and

$$\begin{aligned}(d_2) \quad h(t)^d ||P_2(s)x|| &= h(t)^d(|x_2| + |x_3|) \leq h(t)^d k(s)^{\varepsilon}(|x_2| + |x_3|) \\ &= h(t)^d k(s)^{\varepsilon} \left(\frac{h(s)}{h(t)}\right)^{2d} \left(\frac{k(t)}{k(s)}\right)^{\varepsilon} ||U(t,s)P_2(s)x|| \\ &= h(s)^d k(t)^{\varepsilon} \left(\frac{h(s)}{h(t)}\right)^{d} ||U(t,s)P_2(s)x|| \\ &\leq N h(s)^d k(t)^{\varepsilon} ||U(t,s)P_2(s)x||\end{aligned}$$

for all $(t,s) \in \Delta$ and all $x = (x_1, x_2, x_3) \in X$.

Finally, it results that U is (h,k)-dichotomic with respect to $\mathcal{P} = \{P_1, P_2\}$ and with the dichotomic constants $N,\ d$ and ε.

4 (h,k)-trichotomy

A natural generalization of the (h,k)-dichotomy property is introduced by

Definition 4.1. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is called (h,k)-*trichotomic*, if there are three families of projections $P_1, P_2, P_3 : \mathbb{R}_+ \to \mathcal{B}(X)$ and four constants $N \geq 1,\ \alpha > 0,\ \beta \geq 0$ and $\varepsilon \geq 0$ such that

(t_0) $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with U;

(t_1) $h(t)^{\alpha} ||U(t,s)P_1(s)x|| \leq N h(s)^{\alpha} k(s)^{\varepsilon} ||P_1(s)x||$;

(t_2) $h(t)^{\alpha} ||P_2(s)x|| \leq N h(s)^{\alpha} k(t)^{\varepsilon} ||U(t,s)P_2(s)x||$;

(t_3) $h(s)^{\beta} ||U(t,s)P_3(s)x|| \leq N h(t)^{\beta} k(s)^{\varepsilon} ||P_3(s)x||$;

(t_4) $h(s)^\beta ||P_3(s)x|| \leq Nh(t)^\beta k(t)^\varepsilon ||U(t,s)P_3(s)x||$

for all $(t,s,x) \in T$.

In the particular case when $\varepsilon = 0$ or k is a constant function, we say that U is *uniformly h-trichotomic.*

The constants N, α, β and ε are called *trichotomy constants.*

Remark 4.1. As particular cases of (h,k)-trichotomy we have that

(i) if $h(t) = k(t) = e^t$ for all $t \geq 0$, then we obtain the property of *nonuniform exponential trichotomy* and in particular when $\varepsilon = 0$ or the function k is constant we recover the classical notion of *uniform exponential trichotomy*;

(ii) if $h(t) = k(t) = t+1$ for all $t \geq 0$, then we recover the notion of *nonuniform polynomial trichotomy* respectively the property of *uniform polynomial trichotomy* (when $\varepsilon = 0$ or k is constant);

(iii) if $P_3(t) = 0$ for every $t \geq 0$ we obtain the notion of (h,k)-dichotomy, i.e. the (h,k)-dichotomy is a particular case of (h,k)-trichotomy.

It is obvious that if U is uniformly h-trichotomic then it is (h,k)-trichotomic for every growth rate k.

The converse is not valid, phenomenon illustrated by

Example 4.1. On $X = \mathbb{R}^3$ endowed with the norm

$$||(x_1,x_2,x_3)|| = \max\{|x_1|,|x_2|,|x_3|\}$$

we consider the families of projections $P_1, P_2, P_3 : \mathbb{R}^3 \to \mathcal{B}(X)$ defined by

$$P_1(t)(x_1,x_2,x_3) = (x_1,0,0),$$

$$P_2(t)(x_1,x_2,x_3) = (0,x_2,0),$$

$$P_3(t)(x_1,x_2,x_3) = (0,0,x_3)$$

for all $t \geq 0$ and $x = (x_1,x_2,x_3) \in X$.

Given the growth rates $h,k : \mathbb{R}_+ \to [1,+\infty)$ and the constants $N \geq 1$, $\alpha > 0$, $\beta \geq 0$ and $\varepsilon \geq 0$, we consider the evolution operator $U : \Delta \to \mathcal{B}(X)$, defined by

$$U(t,s)(x_1,x_2,x_3) = \left(\frac{h(s)}{h(t)}\right)^\alpha \frac{k(s)^{\varepsilon \sin^2 s}}{k(t)^{\varepsilon \sin^2 t}} P_1(s)x + \left(\frac{h(t)}{h(s)}\right)^\alpha \left(\frac{k(s)}{k(t)}\right)^\varepsilon P_2(s)x$$

$$+\left(\frac{h(t)}{h(s)}\right)^{\beta}\left(\frac{k(s)}{k(t)}\right)^{\varepsilon}P_3(s)x$$

for all $(t,s,x)\in T$.

We observe that $\mathcal{P}=\{P_1,P_2,P_3\}$ is compatible with U and

$$(t_1)\quad h(t)^{\alpha}||U(t,s)P_1(s)x|| = h(s)^{\alpha}\frac{k(s)^{\varepsilon\sin^2 s}}{k(t)^{\varepsilon\sin^2 t}}||P_1(s)x|| \leq Nh(s)^{\alpha}k(s)^{\varepsilon}||P_1(s)x||,$$

$$(t_2)\quad h(t)^{\alpha}||P_2(s)x|| = h(s)^{\alpha}\left(\frac{k(t)}{k(s)}\right)^{\varepsilon}||U(t,s)P_2(s)x|| \leq Nh(s)^{\alpha}k(t)^{\varepsilon}||U(t,s)P_2(s)x||,$$

$$(t_3)\quad h(s)^{\beta}||U(t,s)P_3(s)x|| = h(t)^{\beta}\left(\frac{k(s)}{k(t)}\right)^{\varepsilon}||P_3(s)x|| \leq Nh(t)^{\beta}k(s)^{\varepsilon}||P_3(s)x||,$$

$$\begin{aligned}(t_4)\quad h(s)^{\beta}||P_3(s)x|| &= h(s)^{\beta}\left(\frac{h(s)}{h(t)}\right)^{\beta}\left(\frac{k(t)}{k(s)}\right)^{\varepsilon}||U(t,s)P_3(s)x||\\ &= h(t)^{\beta}\left(\frac{h(s)}{h(t)}\right)^{2\beta}\left(\frac{k(t)}{k(s)}\right)^{\varepsilon}||U(t,s)P_3(s)x||\\ &\leq Nh(t)^{\beta}k(t)^{\varepsilon}||U(t,s)P_3(s)x||,\end{aligned}$$

for all $(t,s,x)\in T$.

Thus U is (h,k)-trichotomic with the trichotomy constants $N,\ \alpha,\ \beta,\ \varepsilon$.

If we suppose that U is uniformly h-trichotomic, there is $N\geq 1$ such that

$$k(s)^{\varepsilon\sin^2 s}\leq Nk(t)^{\varepsilon\sin^2 t}\quad\text{for every } (t,s)\in\Delta.$$

For $t=(2n+1)\pi,\ s=(2n+1)\frac{\pi}{2}$ and $n\to+\infty$ in the above relation, we obtain a contradiction.

5 The main results

Let $h,k:\mathbb{R}_+\to[1,+\infty)$ be two growth rates and let $U:\Delta\to\mathcal{B}(X)$ be an evolution operator on the Banach space X.

For every $b > 0$ we associate to the evolution operator U the following two evolution operators $B_h^+, B_h^- : \Delta \to \mathcal{B}(X)$ defined by

$$B_h^+(t,s) = \left(\frac{h(t)}{h(s)}\right)^b U(t,s)$$

and respectively

$$B_h^-(t,s) = \left(\frac{h(t)}{h(s)}\right)^{-b} U(t,s)$$

for all $(t,s) \in \Delta$.

If $P_1, P_2, P_3 : \mathbb{R}_+ \to \mathcal{B}(X)$ are three families of projections on X with the property that $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator U then we shall denote by

$$\mathcal{P}^+ = \{P_1^+, P_2^+\} \text{ and } \mathcal{P}^- = \{P_1^-, P_2^-\}$$

the supplementary families defined in Proposition 2.1.

A first result which states that a particular case for (h,k)-trichotomy of the evolution operator U implies (h,k)-dichotomy of B_h^+ and B_h^- for a some constant $b > 0$.

Thus we prove

Theorem 5.1. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of projections which is compatible with the evolution operator U. If U is (h,k)-trichotomic with respect to $\mathcal{P}$ and the trichotomic constants $N \geq 1,\ \alpha > \beta \geq 0$ and $\varepsilon \geq 0$ then there exist*

- (i) *two supplementary families $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$;*
- (ii) *two constants $b \in (\beta, \alpha)$ and $d \in (0, b]$ such that*
  - $(ii)^+$ *B_h^+ is (h,k)-dichotomic with respect to $\mathcal{P}^+$ and the dichotomic constants $N,\ d,\ \varepsilon$;*
  - $(ii)^-$ *B_h^- is (h,k)-dichotomic with respect to $\mathcal{P}^-$ and the dichotomic constants $N,\ d,\ \varepsilon$;*

Proof. Suppose that U is (h,k)-trichotomic with respect to families of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ and the trichotomic constants $N \geq 1,\ \alpha > \beta \geq 0$ and $\varepsilon \geq 0$.

Let $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ respectively $\mathcal{P}^- = \{P_1^-, P_2^-\}$ be the supplementary families of projections given by Proposition 2.1.

If we denote

$$b = \frac{\alpha + \beta}{2} \quad \text{and} \quad d = \frac{\alpha - \beta}{2}$$

then $b\in(\beta,\alpha),\ d\in(0,b]$ and $d=\alpha-b=b-\beta$.

$(d_1)^+$ We observe that by condition (t_1) from Definition 4.1 it results that

$$\begin{aligned}h(t)^d||B_h^+(t,s)P_1^+(s)x|| &= h(t)^{d+b}h(s)^{-b}||U(t,s)P_1(s)x||\\ &\le Nh(t)^{d+b-\alpha}h(s)^{\alpha-b}k(s)^{\varepsilon}||P_1(s)x||\\ &= Nh(s)^dk(s)^{\varepsilon}||P_1^+(s)x||\end{aligned}$$

for all $(t,s,x)\in T$.

$(d_2)^+$ Similarly, by the condition (t_2) and (t_4) from Definition 4.1 we obtain

$$\begin{aligned}&h(t)^{2d}||P_2^+(s)x||^2 = h(t)^{2d}(||P_2(s)x||^2+||P_3(s)x||^2)\\ &\quad\le N^2h(t)^{2d}k(t)^{2\varepsilon}(h(s)^{2\alpha}h(t)^{-2\alpha}||U(t,s)P_2(s)x||^2\\ &\quad+h(t)^{2\beta}h(s)^{-2\beta}||U(t,s)P_3(s)x||^2)\\ &\quad= N^2h(t)^{2d}k(t)^{2\varepsilon}(h(s)^{2\alpha+2b}h(t)^{-2\alpha-2b}||B_h^+(t,s)P_2(s)x||^2\\ &\quad+h(t)^{2\beta-2b}h(s)^{2b-2\beta}||B_h^+(t,s)P_3(s)x||^2)\\ &\quad= N^2h(s)^{2\alpha-2b}k(t)^{2\varepsilon}(h(s)^{4b}h(t)^{-4b}||B_h^+(t,s)P_2(s)x||^2\\ &\quad+h(t)^{2\alpha+2\beta-4b}h(s)^{4b-2\alpha-2\beta}||B_h^+(t,s)P_3(s)x||^2)\\ &\quad= N^2h(s)^{2d}k(t)^{2\varepsilon}(h(s)^{4b}h(t)^{-4b}||B_h^+(t,s)P_2(s)x||^2\\ &\quad+||B_h^+(t,s)P_3(s)x||^2)\le N^2h(s)^{2d}k(t)^{2\varepsilon}(||P_2(t)B_h^+(t,s)x||^2\\ &\quad+||P_3(t)B_h^+(t,s)x||^2)=N^2h(s)^{2d}k(t)^{2\varepsilon}||P_2^+(t)B_h^+(t,s)x||^2\\ &\quad= N^2h(s)^{2d}k(t)^{2\varepsilon}||B_h^+(t,s)P_2^+(s)x||^2\end{aligned}$$

and hence

$$h(t)^d||P_2^+(s)x||\le Nh(s)^dk(t)^{\varepsilon}||B_h^+(t,s)P_2^+(s)x||$$

for all $(t,s,x)\in T$.

The properties $(d_1)^+$ and $(d_2)^+$ show that B_h^+ is (h,k)-dichotomic with respect to $\mathcal{P}^+$ and with the dichotomic constants $N,\ d$ and ε.

$(d_1)^-$ Similarly, as in the proof of $(d_2)^+$, the conditions (t_1) and (t_3)

from Definition 4.1 imply that

$$\begin{aligned}
h(t)^{2d}||B_h^-(t,s)P_1^-(s)x||^2 &= h(t)^{2d-2b}h(s)^{2b}||U(t,s)(P_1(s)x+P_3(s)x)||^2\\
&= h(t)^{2d-2b}h(s)^{2b}||P_1(t)U(t,s)x+P_3(t)U(t,s)x||^2\\
&= h(t)^{2d-2b}h(s)^{2b}(||P_1(t)U(t,s)x||^2+||P_3(t)U(t,s)x||^2)\\
&= h(t)^{2d-2b}h(s)^{2b}(||U(t,s)P_1(s)x||^2+||U(t,s)P_3(s)x||^2)\\
&\leq N^2h(t)^{2d-2b}h(s)^{2b}k(s)^{2\varepsilon}(h(t)^{-2\alpha}h(s)^{2\alpha}||P_1(s)x||^2\\
&+ h(t)^{2\beta}h(s)^{-2\beta}||P_3(s)x||^2)\\
&= N^2h(t)^{2d+2b-2\alpha}h(s)^{-2b+2\alpha}k(s)^{2\varepsilon}(h(t)^{-4b}h(s)^{4b}||P_1(s)x||^2\\
&+ h(t)^{2\beta-4b+2\alpha}h(s)^{4b-2\beta-2\alpha}||P_3(s)x||^2)\\
&\leq N^2h(s)^{2d}k(s)^{2\varepsilon}(||P_1(s)x||^2+||P_3(s)x||^2)\\
&= N^2h(s)^{2d}k(s)^{2\varepsilon}||P_1^-(s)x||^2
\end{aligned}$$

and hence

$$h(t)^d||B_h^-(t,s)P_1^-(s)x|| \leq Nh(s)^dk(s)^\varepsilon||P_1^-(s)x||$$

for all $(t,s,x)\in T$.

$(d_2)^-$ From the condition (t_2) from Definition 4.1 we obtain

$$\begin{aligned}
h(t)^d||P_2^-(s)x|| = h(t)^d||P_2(s)x|| &\leq Nh(t)^{d-\alpha}h(s)^\alpha k(t)^\varepsilon||U(t,s)P_2(s)x||\\
&= Nh(t)^{d-\alpha+b}h(s)^{\alpha-b}k(t)^\varepsilon||B_h^-(t,s)P_2^-(s)x||\\
&= Nh(s)^dk(t)^\varepsilon||B_h^-(t,s)P_2^-(s)x||
\end{aligned}$$

for all $(t,s,x)\in T$.

Thus, we obtain that and B_h^- is (h,k)-dichotomic with respect to $\mathcal{P}^-$ and with the dichotomic constants $N,\ d$ and ε. □

A converse of the previous theorem is

Theorem 5.2. *Suppose that there are:*

- (i) *two supplementary families of projections* $\mathcal{P}^+=\{P_1^+,P_2^+\}$ *and* $\mathcal{P}^-=\{P_1^-,P_2^-\}$ *which are compatible with the evolution operator* U;
- (ii) *four constants* $N\geq 1,\ b>0,\ d\in(0,b]$ *and* $\varepsilon\geq 0$ *such that*
  - $(ii)^+$ B_h^+ *is* (h,k)*-dichotomic with respect to* $\mathcal{P}^+$ *and the dichotomic constants* $N,\ d,\ \varepsilon$;
  - $(ii)^-$ B_h^- *is* (h,k)*-dichotomic with respect to* $\mathcal{P}^-$ *and the dichotomic constants* $N,\ d,\ \varepsilon$.

Then there exist two constants $\alpha > \beta \geq 0$ and a family of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ which is compatible with U such that U is (h,k)-trichotomic with respect to $\mathcal{P}$ and the trichotomic constants $N,\ \alpha,\ \beta,\ \varepsilon$.

Proof. If b and d are the constants from hypothesis then we shall denote

$$\alpha = b + d \quad \text{and} \quad \beta = b - d.$$

Then $\alpha > \beta,\ b \in (\beta, \alpha)$ and $d = \alpha - b = b - \beta$.

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be the family of projections given by Proposition 2.2.

(t_1) From $(ii)^+$ and $(d_1)^+$ it results that

$$\begin{aligned} h(t)^\alpha ||U(t,s)P_1(s)x|| &= h(t)^{\alpha-b}h(s)^b||B_h^+(t,s)P_1^+(s)x|| \\ &\leq Nh(t)^{\alpha-b-d}h(s)^{b+d}||P_1^+(s)x|| \\ &\leq Nh(s)^\alpha k(s)^\varepsilon ||P_1(s)x|| \end{aligned}$$

for all $(t,s,x) \in T$.

(t_2) Similarly, from $(ii)^-$ and $(d_2)^-$ we obtain

$$\begin{aligned} h(t)^\alpha ||P_2(s)x|| &= h(t)^\alpha ||P_2^-(s)x|| \leq Nh(t)^{\alpha-d}h(s)^d k(t)^\varepsilon ||B_h^-(t,s)P_2^-(s)x|| \\ &= Nh(t)^{\alpha-d-b}h(s)^{b+d}k(t)^\varepsilon ||U(t,s)P_2(s)x|| \\ &= Nh(s)^\alpha k(t)^\varepsilon ||U(t,s)P_2(s)x|| \end{aligned}$$

for all $(t,s,x) \in T$.

(t_3) The conditions $(ii)^-$ and $(d_1)^-$ imply

$$\begin{aligned} h(s)^\beta ||U(t,s)P_3(s)x|| &= h(t)^b h(s)^{\beta-b}||B_h^-(t,s)P_1^-(s)P_2^+(s)x|| \\ &\leq Nh(t)^{b-d}h(s)^{\beta-b+d}k(s)^\varepsilon ||P_3(s)x|| \\ &= Nh(t)^\beta k(s)^\varepsilon ||P_3(s)x|| \end{aligned}$$

for all $(t,s,x) \in T$.

(t_4) Similarly, from $(ii)^+$ and $(d_2)^+$ we obtain

$$\begin{aligned} h(s)^\beta ||P_3(s)x|| &= h(s)^\beta ||P_2^+(s)P_1^-(s)x|| \\ &\leq Nh(t)^{-d}h(s)^{d+\beta}k(t)^\varepsilon ||B_h^+(t,s)P_2^+(s)P_1^-(s)x|| \\ &= Nh(t)^{b-d}h(s)^{d+\beta-b}k(t)^\varepsilon ||U(t,s)P_3(s)x|| \\ &= Nh(t)^\beta k(t)^\varepsilon ||U(t,s)P_3(s)x|| \end{aligned}$$

for all $(t,s,x) \in T$.

Finally, it results that the evolution operator U is (h,k)-trichotomic with respect to $\mathcal{P}$ and the trichotomic constants $N,\ \alpha,\ \beta,\ \varepsilon$. $\square$

The main result of this paper is given by

Theorem 5.3. *An evolution operator $U : \Delta \to \mathcal{B}(X)$ is (h,k)-trichotomic with respect to the family of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ and the trichotomic constants $N \geq 1,\ \alpha > \beta \geq 0$ and $\varepsilon \geq 0$ if and only if there are $b \in (0,\alpha),\ d \in (0,b]$ and two supplementary families of projections $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$ compatible with U such that the associated evolution operators B_h^+, respectively B_h^- are (h,k)-dichotomic with the dichotomic constants $N,\ d,\ \varepsilon$ and the families of projections $\mathcal{P}^+$, respectively $\mathcal{P}^-$.*

Proof. It follows from Theorem 5.1 and Theorem 5.2. □

An important particular case, is when the evolution operator U is generated by a nonautonomous linear differential equation

$$(A) \qquad \dot{x}(t) = A(t)x(t),$$

where $A : \mathbb{R}_+ \to \mathcal{B}(X)$.

In this case, we say that the equation (A) admits a nonuniform (h,k)-dichotomy (respectively (h,k)-trichotomy) if the associated evolution operator U is (h,k)-dichotomic (respectively (h,k)-trichotomic).

In the particular cases

$$h(t) = k(t) = e^t \quad \text{respectively} \quad h(t) = k(t) = \frac{1}{t+1}$$

for all $t \geq 0$ we obtain the concepts of nonuniform exponential dichotomy and nonuniform exponential trichotomy, respectively nonuniform polynomial dichotomy and nonuniform polynomial trichotomy.

If $h(t) = e^t$ for all $t \geq 0$ and U is generated by (A) then

$$B_h^+(t,s) = e^{b(t-s)}U(t,s)$$

respectively

$$B_h^-(t,s) = e^{-b(t-s)}U(t,s)$$

are generated by the differential equations

$$(B_e^+) \quad \dot{x}(t) = B_e^+(t)x(t)$$

respectively

$$(B_e^-) \quad \dot{x}(t) = B_e^-(t)x(t),$$

where

$$B_e^+(t) = A(t) + bI \quad \text{respectively} \quad B_e^-(t) = A(t) - bI.$$

As an immediate consequence of Theorem 5.3 we obtain

Corollary 5.1. The differential equation (A) admits a nonuniform exponential trichotomy with the trichotomic constants $N \geq 1,\ \alpha > \beta \geq 0$ and $\varepsilon \geq 0$ if and only if there are $b \in (0, \alpha),\ d \in (0, b]$ and two supplementary families of projections $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$ such that the equations (B_e^+), respectively (B_e^-) admit a nonuniform exponential dichotomy with the dichotomic constants $N,\ d,\ \varepsilon$ and the families of projections $\mathcal{P}^+$ respectively $\mathcal{P}^-$.

Proof. It results from Theorem 5.3. □

Similarly, if $h(t) = \dfrac{1}{t+1}$ for all $t \geq 0$ and U is generated by the differential equation (A) then

$$B_h^+(t,s) = \left(\frac{t+1}{s+1}\right)^b U(t,s)$$

respectively

$$B_h^-(t,s) = \left(\frac{t+1}{s+1}\right)^{-b} U(t,s)$$

are generated by the differential equations

$$(B_p^+) \quad \dot{x}(t) = B_p^+(t)x(t)$$

respectively

$$(B_p^-) \quad \dot{x}(t) = B_p^-(t)x(t),$$

where

$$B_p^+(t) = A(t) + \frac{bI}{t+1} \quad \text{respectively} \quad B_p^-(t) = A(t) - \frac{bI}{t+1}.$$

For this case we obtain

Corollary 5.2. The differential equation (A) admits a nonuniform polynomial trichotomy with the trichotomy constants $N \geq 1,\ \alpha > \beta \geq 0$ and $\varepsilon \geq 0$ if and only if there are $b \in (0, \alpha),\ d \in (0, b]$ and two supplementary families of projections $\mathcal{P}^+ = \{P_1^+, P_2^+\}$ and $\mathcal{P}^- = \{P_1^-, P_2^-\}$ such that the equations (B_p^+) respectively (B_p^-) admit a nonuniform polynomial dichotomy with the dichotomy constants $N,\ d,\ \varepsilon$ and the families of projections $\mathcal{P}^+$ respectively $\mathcal{P}^-$.

Proof. It follows from Theorem 5.3. □

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