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Study of Fixed Point Theorem for Common Limit Range Property and Application to Functional Equations

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Abstract. The aim of our paper is to use common limit range property for two pairs of mappings deriving common fixed point results under a generalized altering distance function. Some examples are given to exhibit different type of situation which shows the requirements of conditions of our results. At the end the existence and uniqueness of solutions for certain system of functional equations arising in dynamic programming with the help of a common fixed point theorem is presented.

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1 Introduction and Preliminaries

The existence, uniqueness, and iterative approximations of solutions for several classes of functional equations arising in dynamic programming were studied by a lot of researchers. Bellman [7] first studied the existence of solutions for some classes of functional equations arising in dynamic programming. Bellman and Lee [8] pointed out that the basic form of the functional equations in dynamic programming is as follows:

$$q(x) = \sup_{y \in D}\{G(x, y, q(\tau(x, y)))\}, \quad x \in W, \tag{1.1}$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U as well as V are Banach spaces.
In 1984, Bhakta and Mitra [9] obtained some existence theorem for the following functional equation which arises in multistage decision process related to dynamic programming

$$q(x) = \sup_{y \in D}\{g(x, y) + G(x, y, q(\tau(x, y)))\}, \quad x \in W, \tag{1.2}$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U as well as V are Banach spaces.
Thereafter many work have been in this direction and obtain existence and uniqueness results of solution and common solution for some functional equations and systems of functional equations in dynamic programming with the use of fixed point results. For detail see [33, 34, 38–41] and the references therein.
The Banach Contraction Principle is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions—modifying the basic contractive condition or changing the ambiental space. This celebrated theorem can be stated as follow.

Theorem 1.1. [6]. *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq kd(x, y), \ \forall\, x, y \in X, \tag{1.3}$$

where k is a constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in X$.

Inequality (1.3) implies continuity of T. A natural question is whether we can find contractive conditions which will imply existence of a fixed point in a complete metric space but will not imply continuity.
In 1968, Kannan [29] constructed a contractive condition, like that of Banach, possessed a unique fixed point, which could be obtained by starting at any point x_0 in the space, and using function iteration defined by $x_{n+1} = Tx_n$ (also called Picard iteration). However, unlike the Banach condition, there exist discontinuous functions satisfying the definition of Kannan, although

such mappings are continuous at the fixed point. Following the appearance of [29] many authors created contractive conditions not requiring continuity of the mapping, see [46]. Today fixed point literature of contractive mappings contains many such papers. One survey of a number of these conditions appears in [47]. However, sometimes one may come across situations where the full force of metric requirements are not used in the proofs of certain metrical fixed point theorems. Motivated by this fact, several authors obtained fixed point and common fixed point results in symmetric and semi-metric space.
There is in the literature a great number of generalizations of the Banach contraction principle (see [36] and references cited therein). In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. There are control functions which alter the distance between two points in a metric space. In this direction, Khan et al. [31] addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

Definition 1.1. *(altering distance function [31]).* $\varphi : [0, +\infty) \to [0, +\infty)$ *is called an altering distance function if the following properties are satisfied:*

(a) φ *is continuous and non-decreasing, and*

(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Theorem 1.2. [31]. *Let* (X, d) *be a complete metric space, let* φ *be an altering distance function, and let* $T : X \to X$ *be a self-mapping which satisfies the following inequality:*

$$\varphi(d(Tx, Ty)) \leq c\varphi(d(x, y)) \tag{1.4}$$

for all $x, y \in X$ *and for some* $0 < c < 1$. *Then,* T *has a unique fixed point.*

Putting $\varphi(t) = t$ in the previous theorem, (1.4) reduces to (1.3).
Rhoades [48] extended Theorem 1.1 by introducing weakly contractive mapping in complete metric spaces.

Definition 1.2. *(weakly contractive mapping [48]). Let* X *be a metric space. A mapping* $T : X \to X$ *is called weakly contractive if and only if:*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \ \forall\, x, y \in X, \tag{1.5}$$

where φ *is an altering distance function.*

Theorem 1.3. [48, Theorem 2] . *Let (X, d) be a complete metric space. If $T : X \to X$ is a weakly contractive mapping, then T has a unique fixed point.*

Note that Alber et al. [3] assumed an additional condition on φ which is $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$. But Rhoades [48] obtained the result noted in Theorem 1.3 without using this particular assumption. If one takes $\varphi(t) = (1 - k)t$, where $0 < k < 1$, then (1.5) reduces to (1.2). One of the main generalizations of the well-known Banach principle is the following theorem established by Boyd and Wong [10]. In their theorem it is assumed that $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right (that is, $r_n \to r \geq 0$ implies $\limsup_{n\to\infty} \psi(r_n) \leq \psi(r)$).

Theorem 1.4. [10] *Let (X, d) be a complete metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)) \text{ for each } x, y \in X, \tag{1.6}$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then T has a unique fixed point x^*, and $\{T^n(x)\}$ converges to x^* for each $x \in X$.

Similarly, Reich [45] presented the following:

Theorem 1.5. [45] *Let (X, d) be a complete metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for each } x, y \in X,\ x \neq y, \tag{1.7}$$

where $\beta : [0, \infty) \to [0, 1)$ and $\lim_{t\to r+} \sup \beta(t) < 1$ for all $0 < r < +\infty$. Then T has a fixed point x^*.

Weak contractions are also closely related to maps of Boyd and Wong [10] and the Reich type ones [45]. Namely, if ϕ is a lower semi-continuous function from the right then $\psi(t) = t - \varphi(t)$ is an upper semi-continuous function from the right, and moreover, (1.5) turns into (1.6). Therefore the weak contraction is of Boyd and Wong type. And if we define $\beta(t) = 1 - \frac{\varphi(t)}{t}$ for $t > 0$ and $\beta(0) = 0$, then (1.5) turns into (1.7). Therefore the weak contraction becomes a Reich type one.
Dutta and Choudhury in [14] obtained the following generalization of Theorems 1.2 and 1.3.

Theorem 1.6. [14]. *Let (X, d) be a complete metric space and $T : X \to X$ satisfying:*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{1.8}$$

for all $x, y \in X$, where ψ and φ are altering distance functions. Then T has a unique fixed point.

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [2, 12, 13, 36, 42, 52]. In [12], Choudhury introduced the concept of a generalized altering distance function for three variables.

Definition 1.3. [12]. *A function $\varphi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to be a generalized altering distance function if and only if*

(i) φ is continuous,

(ii) φ is non-decreasing in all the three variables,

(iii) $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0$.

$\mathfrak{F}_3$ will denote the set of all functions ψ satisfying conditions (i)–(iii).

The following are examples of generalized altering distance functions with three variables.

Example 1.1. (a) $\psi(t_1, t_2, t_3) = k \max\{t_1, t_2, t_3\}, \; k > 0$;

(b) $\psi(t_1, t_2, t_3) = \frac{\max\{t_1,t_2,t_3\}}{1+\max\{t_1,t_2,t_3\}}$;

(c) $\psi(t_1, t_2, t_3) = t_1^p + t_2^q + t_3^r, \; p, q, r \geq 1$.

In [12], Choudhury proved the following common fixed point theorem using altering distances for three variables.

Theorem 1.7. [12]. *Let (X, d) be a complete metric space and $S, T : X \to X$ two self mappings such that the following inequality is satisfied:*

$$\Phi_1(d(Sx, Ty)) \leq \psi_1(d(x, y), d(x, Sx), d(y, Ty)) - \psi_2(d(x, y), d(x, Sx), d(y, Ty)) \tag{1.9}$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \mathfrak{F}_3$ and $\Phi_1(x) = \psi_1(x, x, x)$. Then S and T have a common fixed point.

In particular, Abbas and Ali [2] proved a fixed point theorem for two mappings satisfying a generalized (ψ, φ)-weak contractive condition in a complete metric space. Abbas and Ali [2] proved a common fixed point theorem for any even number of self mappings in a complete metric space and also generalized the earlier mentioned results.

There exist a lot of generalizations of metric spaces which showed themselves useful in obtaining more powerful fixed point and common fixed point results.

Symmetric spaces are among the most important ones, since very often not the full power of metric requirements are needed in proving these results. The notion of symmetric space goes back to Wilson [54].

Definition 1.4. *A symmetric on a nonempty set X is a function $d : X\times X \to [0,+\infty)$ satisfying the following conditions:*

1. *$d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*

Example 1.2. The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) = \{\, \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \,\}$ together with $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x,y) := \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}, \text{ where } x = \{x_n\},\ \ y = \{y_n\} \in l_p(\mathbb{R})$$

is a symmetric space.

Let d be a symmetric on X. For $x \in X$ and $\epsilon > 0$, let $\mathcal{B}(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a semimetric if for each $x \in X$ and each $\epsilon > 0$, $\mathcal{B}(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. A symmetric (resp., semimetric) space (X, d) is a topological space whose topology $\tau(d)$ is induced by symmetric (resp., semimetric) d. The difference between a symmetric and a metric is engineered by the triangle inequality. Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point theorems some additional axioms are required. The following axioms, which are available, e.g., in [4, 5, 11, 15, 19, 54] are relevant to this presentation.

From now on (X, d) stands for a symmetric space, whereas Y denotes an arbitrary non-empty set. Then

(W_3) [54] given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$;

(W_4) [54] given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$;

(HE) [4] given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$;

$(1C)$ [15] a symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ where $\{x_n\}$ is a sequence in X and $x, y \in X$;

(CC) [15] a symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Here, it can be observed that $(W_4) \implies (W_3)$ and $(1C) \implies (W_3)$ but the converse implications are not true. In general, all other possible implications amongst (W_3), $(1C)$ and (HE) are not true. However, (CC) implies all the remaining four conditions, namely (W_3), (W_4), (HE) and $(1C)$. For detailed description, we refer to an interesting note by Cho et al. [11] which contains some illustrative examples. Employing these axioms, several authors proved common fixed point theorems in the framework of symmetric and semi-metric spaces. For detail see [1, 4, 5, 11, 15–17, 19, 24, 30, 35, 53, 54] and references therein.

We note that if (X, d) is a cone metric space over a normal cone and $D = \|d\|$ then (X, D) is a symmetric space which satisfies axiom (CC) but is not in general a metric space (see [25]). Fixed point results in cone symmetric spaces were obtained in [43].

During the late 20th century, metrical common fixed point theory saw a trend of investigation which moved around commuting nature of two maps. Several conditions were introduced, including weak commutativity (Sessa [49]), compatibility (Jungck [26]), weak compatibility (Jungck and Rhoades [27]) and many others, and a lot of respective common fixed point results were obtained. A survey of these notions and relationship among them can be seen in [28].

We recall that two mappings $A, S : X \to X$ are called weakly compatible if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$.

In the study of common fixed points of compatible-type mappings we often require assumption of completeness of the space or continuity of mappings involved besides some contractive condition, but the study of fixed points of non-compatible mappings can be extended to the class of non-expansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or completeness of the space. Aamri and El Moutawakil [1] generalized the concept of non-compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition. Although (E.A) property is a generalization of the concept of non-compatible maps, yet it requires either completeness of the whole

space or some of the range spaces or continuity of maps. Most recently, Liu et al. [32] defined a common (E.A) property for two pairs of mappings.
As a further generalization, new notion of CLRg property, recently given by Sintunavarat and Kuman [50], does not impose such conditions. The importance of CLRg property is that it ensures that one does not require the closedness of range of subspaces. Recently, Imdad et al. [20] extended the notion of common limit range property to two pairs of self mappings which further relaxes the requirement on closedness of the subspaces. Since then, a number of fixed point theorems have been established by several researchers in different settings under common limit range property. We refer the reader to [21, 23, 30] and references therein.
Now we give definitions of the mentioned properties for non-self mappings.

Definition 1.5. *Let Y be an arbitrary set, (X, d) be a symmetric space and let A, B, S, T be mappings from Y into X. Then*

1. *the pair (A, S) is said to satisfy the property (E.A)* [1] *if there exists a sequence $\{x_n\}$ in Y such that*
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$
for some $z \in X$;

2. *the pairs (A, S) and (B, T) are said to share the common property (E.A)* [32], *if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that*
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$
for some $z \in X$;

3. *the pair (A, S) is said to have the common limit range property with respect to the mapping S (denoted by (CLR_S))* [50] *if there exists a sequence $\{x_n\}$ in Y such that*
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$
where $z \in S(Y)$;

4. *the pairs (A, S) and (B, T) are said to have the common limit range property (with respect to mappings S and T)* [20], *often denoted by (CLR_{ST}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that*
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$
where $z \in S(Y) \cap T(Y)$.

Remark 1.1. 1. If we set $A = B$ and $S = T$, then condition (4) reduces to condition (3).

2. Evidently, (CLR_{ST}) property implies the common property (E.A) but not conversely.

In this paper, attempt is made to find existence and uniqueness of solutions for certain system of functional equations arising in dynamic programming through the help of a common fixed point theorem satisfying a generalized altering distance function for two pairs of non-self weakly compatible mappings enjoying common limit range property in symmetric spaces. We furnish examples to demonstrate the validity of the results and to highlight the realized improvements in our results over the corresponding relevant results in the existing literature. We extend our main result to four finite families of mappings in symmetric spaces using the notion of pairwise commuting mappings. Our results generalizing the results given in the paper [2, 12, 14, 44] and many others.

2 Common fixed point theorems for two pairs of mappings

The attempted improvements in this paper are the following.
(i) The results are proved in symmetric spaces.
(ii) The condition on containment of ranges amongst the involved mappings is relaxed.
(iii) Continuity requirements of all the involved mappings are completely relaxed.
(iv) The (E.A) property is replaced by $(CLR_{S,T})$ property which is the most general among all existing weak commutativity concepts.
(v) The condition on completeness of the whole space is relaxed.
Now we state and prove our main result.

Theorem 2.1. *Let (X, d) be a symmetric space where d satisfies the conditions $(1C)$ and (HE). Let Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose A, B, S, T satisfy the following condition:*

$$\Psi_1(d(Ax, By)) \leq \psi_1(d(Ax, Sx), d(By, Ty), d(Sx, Ty)) - \psi_2(d(Ax, Sx), d(By, Ty), d(Sx, Ty)). \tag{2.1}$$

for some $\psi_1, \psi_2 \in \mathfrak{F}_3$ and all $x, y \in Y$. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each.

Moreover, if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) share the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(Y) \cap T(Y)$. Since $z \in S(Y)$, there exists a point $u \in Y$ such that $Su = z$. Putting $x = u$ and $y = y_n$ in condition (2.1), we get

$$\begin{aligned}\Psi_1(d(Au, By_n)) \leq \psi_1(&d(Au, Su), d(By_n, Ty_n), d(Su, Ty_n)) \\ &- \psi_2(d(Au, Su), d(By_n, Ty_n), d(Su, Ty_n)).\end{aligned} \tag{2.2}$$

Passing the upper limit as $n \to \infty$ in condition (2.2) and using properties $(1C)$ and (HE), we have

$$\begin{aligned}\Psi_1(d(Au, z)) &\leq \psi_1(d(Au, z), 0, 0) - \psi_2(d(Au, z), 0, 0) \\ &\leq \Psi_1(d(Au, z)) - \psi_2(d(Au, z), 0, 0)\end{aligned} \tag{2.3}$$

and it follows easily that $\psi_2(d(Au, z), 0, 0) = 0$ and thus $Au = z$. Therefore $Au = Su = z$, which shows that u is a coincidence point of the pair (A, S). As $z \in T(Y)$, there exists a point $v \in Y$ such that $Tv = z$. Putting $x = u$ and $y = v$ in condition (2.1), we have

$$\begin{aligned}\Psi_1(d(z, Bv)) &= \Psi_1(d(Au, Bv)) \\ &\leq \psi_1(d(Au, Su), d(Bv, Tv), d(Su, Tv)) \\ &\quad - \psi_1(d(Au, Su), d(Bv, Tv), d(Su, Tv)) \\ &= \psi_1(d(z, z), d(Bv, z), d(z, z)) - \psi_2(d(z, z), d(Bv, z), d(z, z)) \\ &= \psi_1(0, d(Bv, z), 0) - \psi_2(0, d(Bv, z), 0) \\ &\leq \Psi_1(d(Bv, z)) - \psi_2(0, d(Bv, z), 0)\end{aligned} \tag{2.4}$$

which holds unless $\psi_2(0, d(Bv, z), 0) = 0$ and so $z = Bv$. Thus, $Bv = Tv = z$, which shows that v is a coincidence point of the pair (B, T).
Suppose now that $Y = X$. Since the pairs (A, S) and (B, T) are weakly compatible, $Au = Su$ and $Bv = Tv$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$. Putting $x = z$ and $y = v$ in condition (2.1), we have

$$
\begin{aligned}
\Psi_1(d(Az,z)) &= \Psi_1(d(Az,Bv)) \\
&\le \psi_1(d(Az,Sz),d(Bv,Tv),d(Sz,Tv)) \\
&\quad - \psi_2(d(Az,Sz),d(Bv,Tv),d(Sz,Tv)) \\
&= \psi_1(d(Az,Az),d(z,z),d(Az,z)) - \\
&\psi_2(d(Az,Az),d(z,z),d(Az,z)) \\
&= \psi_1(0,0,d(Az,z)) - \psi_2(0,0,d(Az,z)) \\
&\le \Psi_1(d(Az,z)) - \psi_2(0,0,d(Az,z)).
\end{aligned} \tag{2.5}
$$

It follows that $\psi_2(0,0,d(Az,z)) = 0$ and thus $z = Az = Sz$. Therefore z is a common fixed point of the pair (A,S). Putting $x = u$ and $y = z$ in condition (2.1), we have

$$
\begin{aligned}
\Psi_1(d(z,Bz)) &= \Psi_1(d(Au,Bz)) \\
&\le \psi_1(d(Au,Su),d(Bz,Tz),d(Su,Tz)) \\
&\quad - \psi_2(d(Au,Su),d(Bz,Tz),d(Su,Tz)) \\
&= \psi_1(d(z,z),d(Bz,Bz),d(z,Bz)) - \\
&\psi_2(d(z,z),d(Bz,Bz),d(z,Bz)) \\
&= \psi_1(0,0,d(z,Bz)) - \psi_2(0,0,d(z,Bz)) \\
&\le \Psi_1(d(z,Bz)) - \psi_2(0,0,d(z,Bz)).
\end{aligned} \tag{2.6}
$$

From (2.6), we obtain

$$\Psi_1(d(z,Bz)) \le \Psi_1(d(z,Bz)) - \psi_2(0,0,d(z,Bz)).$$

and then $\psi_2(0,0,d(z,Bz)) = 0$, that is, $z = Bz$. Therefore $Bz = Tz = z$ and we can conclude that z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point is an easy consequence of condition (2.1) and so, to avoid repetition, we omit the details. □

Remark 2.1. If we take

$$\psi_1(t_1,t_2,t_3) = \max\{t_1,t_2,t_3\} \text{ and } \psi_2(t_1,t_2,t_3) = (1-k)\max\{t_1,t_2,t_3\},$$

for $k \in (0,1)$ then $\Psi_1(t) = t$ for all $t \ge 0$, and the contractive condition 2.1 in Theorem 2.1 becomes

$$d(Ax,By) \le k \max\left\{ d(Ax,Sy), d(By,Ty), d(Sx,Ty) \right\}.$$

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions may be derived from the above theorems. For example, if we consider

$$\begin{aligned}\psi_1(x,y,z) &= k_1x^q + k_2y^q + k_3z^q,\\ \psi_2(x,y,z) &= (1-k)[k_1x^q + k_2y^q + k_3z^q],\end{aligned}$$

where $q > 0$ and $0 < k = k_1 + k_2 + k_3 < 1$, we obtain the following results.

The next result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let (X, d) be a symmetric space, where d satisfies the conditions $(1C)$ and (HE) and let Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$ such that*

$$(d(Ax, By))^q \leq k_1(d(Ax, Sy))^q + k_2(d(By, Ty))^q + k_3(d(Sx, Ty))^q, \quad (2.7)$$

for all $x, y \in X$ where $q > 0$ and $0 < k_1 + k_2 + k_3 < 1$. If the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Remark 2.2. Other fixed point results may also be obtained under specific choices of ψ_1 and ψ_2.

The following proposition will help us to get further results.

Proposition 2.3. *Let (X, d) be a symmetric space where d satisfies the condition (CC) while Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the following hypotheses hold:*

1. *the pair (A, S) satisfies the (CLR_S) property $\big[$resp., the pair (B, T) satisfies the (CLR_T) property$\big]$;*
2. *$A(Y) \subset T(Y)$ $\big[$resp., $B(Y) \subset S(Y)\big]$;*
3. *$T(Y)$ $\big[$resp., $S(Y)\big]$ is a closed subset of X;*
4. *$\{By_n\}$ converges for every sequence $\{y_n\}$ in Y such that $\{Ty_n\}$ converges $\big[$resp., $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in Y such that $\{Sx_n\}$ converges$\big]$;*

5. *for some $\psi_1, \psi_2 \in \mathfrak{F}_3$ and all $x, y \in Y$*

$$\Psi_1(d(Ax, By)) \leq \psi_1(d(Ax, Sx), d(By, Ty), d(Sx, Ty)) - \psi_2(d(Ax, Sx), d(By, Ty), d(Sx, Ty)). \tag{2.8}$$

Then the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. If the pair (A, S) satisfy the (CLR_S) property, there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(Y)$. By (2), $A(Y) \subset T(Y)$ (where $T(Y)$ is a closed subset of X) and for each $\{x_n\} \subset Y$ there corresponds a sequence $\{y_n\} \subset Y$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z,$$

where $z \in S(Y) \cap T(Y)$. Thus, we have

$$\lim_{n\to+\infty} d(Ax_n, z) = \lim_{n\to+\infty} d(Sx_n, z) = \lim_{n\to+\infty} d(Ty_n, z) = 0.$$

Therefore, by (HE) we have

$$\lim_{n\to\infty} d(Ax_n, Sx_n) = 0 \text{ and } \lim_{n\to\infty} d(Sx_n, Ty_n) = 0.$$

By (4), the sequence $\{By_n\}$ converges; we need to show that $By_n \to z$ as $n \to \infty$. By (CC), we get $\lim_{n\to\infty} d(Ax_n, By_n) = d(z, \lim_{n\to\infty} By_n)$, $\lim_{n\to\infty} d(Sx_n, By_n) = d(z, \lim_{n\to\infty} By_n)$ and $\lim_{n\to\infty} d(By_n, Ty_n) = d(\lim_{n\to\infty} By_n, z)$. Putting $x = x_n$ and $y = y_n$ in condition (2.8), we get

$$\Psi_1(d(Ax_n, By_n)) \leq \psi_1(d(Ax_n, Sx_n), d(By_n, Ty_n), d(Sx_n, Ty_n)) - \Psi_2(d(Ax_n, Sx_n), d(By_n, Ty_n), d(Sx_n, Ty_n)). \tag{2.9}$$

Passing the upper limit as $n \to \infty$ in inequality (2.9), we have

$$\Psi_1(d(z, \lim_{n\to\infty} By_n)) \leq \psi_1(0, d(\lim_{n\to\infty} By_n, z), 0) - \psi_2(0, d(\lim_{n\to\infty} By_n, z), 0). \tag{2.10}$$

Hence, inequality (2.10) implies

$$\Psi_1(d(z, \lim_{n\to\infty} By_n)) \leq \Psi_1(d(z, \lim_{n\to\infty} By_n)) - \psi_2(0, d(\lim_{n\to\infty} By_n, z), 0),$$

that is, $\psi_2(0, d(\lim_{n\to\infty} By_n, z), 0) \leq 0$. Thus, $\psi_2(0, d(\lim_{n\to\infty} By_n, z), 0) = 0$ and by the properties of function $\psi_2 \in \mathfrak{F}_3$, we have $d(z, \lim_{n\to\infty} By_n) = 0$. Hence $By_n \to z$ as $n \to \infty$ which shows that the pairs (A, S) and (B, T) share the (CLR_{ST}) property. □

The converse of Proposition 2.3 is not true. For a counterexample, see [20, Example 3.5].
If we replace (IC) and (HE) property by (CC) property in Theorem 2.1, we have following result.

Theorem 2.4. *Let (X, d) be a symmetric space, where d satisfies the condition (CC), and let Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the conditions* (1)–(5) *of Proposition 2.3 hold. Then (A, S) and (B, T) have a coincidence point each.*
Moreover if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Following Proposition 2.3, the pairs (A, S) and (B, T) share the (CLR_{ST}) property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(Y) \cap T(Y)$. The rest of the proof runs on the lines of the proof of Theorem 2.1, therefore the details are omitted. □

Obviously, if the pairs (A, S) and (B, T) satisfy the common property (E.A), and, at the same time, $S(Y)$ and $T(Y)$ are closed subsets of X, then the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Hence, we have the following variant of Theorem 2.1.

Theorem 2.5. *Let (X, d) be a symmetric space, where d satisfies the conditions $(1C)$ and (HE), and let Y be an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the inequality* (2.8) *and the following hypotheses hold:*

1. *the pairs (A, S) and (B, T) satisfy the common property (E.A);*
2. *$S(Y)$ and $T(Y)$ are closed subsets of X.*

Then (A, S) and (B, T) have a coincidence point each. Moreover, if $Y = X$, then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. If the pairs (A, S) and (B, T) share the common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$. Since $S(Y)$ is closed, $\lim_{n\to\infty} Sx_n = z = Su$ for some $u \in Y$. Also, since $T(Y)$ is closed, then $\lim_{n\to\infty} Ty_n = z = Tv$ for some $v \in Y$. The rest of the proof runs on the lines of the proof of Theorem 2.1, therefore such details are omitted. □

Next, we state two more variants of our results, which can be proved on the lines of the proofs of Theorems 2.4 and 2.5.

Corollary 2.6. *The conclusions of Theorem 2.5 remain true if condition* (2) *is replaced by the following:*

(2′) $\overline{A(Y)} \subset T(Y)$ *and* $\overline{B(Y)} \subset S(Y)$,

where $\overline{A(Y)}$ *and* $\overline{B(Y)}$ *denote the closure of ranges of the mappings* A *and* B.

Corollary 2.7. *The conclusions of Theorem 2.5 remain true if the condition* (2) *is replaced by the following:*

(2″) $A(Y)$ *and* $B(Y)$ *are closed subsets of* X, *and* $A(Y) \subset T(Y)$, $B(Y) \subset S(Y)$.

By choosing A, B, S and T suitably in Theorem 2.1, we can deduce some corollaries for a pair as well as for a triple of self mappings. Here, as a sample, we give the following natural result for a pair of self mappings.

Corollary 2.8. *Let* (X, d) *be a symmetric space, where* d *satisfies the conditions* $(1C)$ *and* (HE), *and let* Y *be an arbitrary non-empty set with* $A, S : Y \to X$. *Suppose that*

1. *the pair* (A, S) *enjoys the* (CLR_S) *property;*
2. *for some* $\psi_1, \psi_2 \in \mathfrak{F}_3$ *and all* $x, y \in Y$

$$\begin{aligned}\Psi_1(d(Ax, Ay)) \leq\ & \psi_1(d(Ax, Sx), d(Ay, Sy), d(Sx, Sy)) \\ & - \psi_2(d(Ax, Sx), d(Ay, Sy), d(Sx, Sy)).\end{aligned}$$

Then the pair (A, S) *has a coincidence point. Moreover, if* $Y = X$, *then* A *and* S *have a unique common fixed point provided the pair* (A, S) *is weakly compatible.*

By choosing A, B, S and T suitably in Theorem 2.1, we can deduce some corollaries for a pair as well as for a triple of self mappings. Since the formulations of these results are similar to those in [20, 23], we omit the details here. We just state the following result for four families of mappings.

Corollary 2.9. *Let (X,d) be a symmetric space, where d satisfies the conditions $(1C)$ and (HE), and let Y be an arbitrary non-empty set. Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of mappings from Y to X, where $A = A_1A_2\cdots A_m$, $B = B_1B_2\cdots B_n$, $S = S_1S_2\cdots S_p$ and $T = T_1T_2\cdots T_q$ satisfy condition (2.8) of Lemma 2.3, and the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property. Then (A,S) and (B,T) have a point of coincidence each. Moreover if $Y = X$, then $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ have a unique common fixed point provided the pairs of families $(\{A_i\},\{S_k\})$ and $(\{B_r\},\{T_h\})$ commute pairwise, where $i \in \{1,2,\dots,m\}$, $k \in \{1,2,\dots,p\}$, $r \in \{1,2,\dots,n\}$ and $h \in \{1,2,\dots,q\}$.*

Proof. The proof can be completed on the lines of a theorem of Imdad et al. [20, Theorem 2.2]. □

Now, we indicate that Corollary 2.9 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample, we derive the following theorem by setting one family to be of two members while the remaining three of single members.

Corollary 2.10. *Let (X,d) be a symmetric space, where d satisfies the conditions $(1C)$ and (HE), and let Y be an arbitrary non-empty set with $A,B,R,S,T : Y \to X$. Suppose that the following hypotheses hold:*

1. *the pairs (A,SR) and (B,T) share the $(CLR_{(SR)(T)})$ property;*
2. *for some $\psi_1,\psi_2 \in \mathcal{F}_3$ and all $x,y \in Y$*

$$\begin{aligned}\Psi_1(d(Ax,By)) \le\ & \psi_1(d(Ax,SRx), d(By,Ty), d(SRx,Ty)) \\ & - \psi_2(d(Ax,SRx), d(By,Ty), d(SRx,Ty)). \quad (2.11)\end{aligned}$$

Then (A,SR) and (B,T) have a coincidence point each. Moreover, if $Y = X$, then A,B,R,S and T have a unique common fixed point provided both the pairs (A,SR) and (B,T) commute pairwise, that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$.

Similarly, one can derive a common fixed point theorem for six mappings by setting two families of two members while the rest two of single members, and so on.
By setting $A_1 = A_2 = \cdots = A_m = A$, $B_1 = B_2 = \cdots = B_p = B$, $S_1 = S_2 = \cdots = S_n = S$ and $T_1 = T_2 = \cdots = T_q = T$ in Corollary 2.9, one deduces the following result.

Corollary 2.11. *Let A,B,S and T be self mappings of a symmetric space (X,d) satisfying the conditions $(1C)$ and (HE). Suppose that, for fixed positive integers m,n,p,q,*

1. *the pairs* (A^m, S^p) *and* (B^n, T^q) *share the* $(CLR_{S^pT^q})$ *property;*

2. *for some* $\psi_1, \psi_2 \in \mathfrak{F}_3$ *and all* $x, y \in Y$

$$\begin{aligned}\Psi_1(d(A^m x, B^n y)) \leq \psi_1(d(A^m x, S^p x), d(B^n y, T^q y), d(S^p x, T^q y)) \\ - \psi_2(d(A^m x, S^p x), d(B^n y, T^q y), d(S^p x, T^q y)).\end{aligned} \tag{2.12}$$

Then A, B, S *and* T *have a unique common fixed point provided* $AS = SA$ *and* $BT = TB$.

Remark 2.3. Corollary 2.11 is a slight but partial generalization of Theorem 2.1 as the commutativity requirements (that is, $AS = SA$ and $BT = TB$) in this corollary are relatively stronger as compared to weak compatibility in Theorem 2.1.

3 Illustrative examples

Now we furnish examples demonstrating the validity of the hypotheses of Theorem 2.1. This example is inspire due to Imdad et al. [20].

Example 3.1. Let $Y = [1, 10) \subset [1, +\infty) = X$ be endowed with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in Y$ which also satisfies $(1C)$ and (HE). Consider the mappings $A, B, S, T : Y \to X$ given by

$$Ax = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 10), \\ 8 & \text{if } 1 < x \leq 3, \end{cases} \qquad Bx = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 10), \\ 5 & \text{if } 1 < x \leq 3, \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } 1 < x \leq 3, \\ \frac{4x-5}{7}, & \text{if } 3 < x < 10, \end{cases} \qquad Tx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } 1 < x \leq 3, \\ \frac{x+4}{7} & \text{if } 3 < x < 10. \end{cases}$$

Then we have $A(Y) = \{1, 8\} \nsubseteq [1, 2) \cup \{10\} = T(Y)$ and $B(Y) = \{1, 5\} \nsubseteq [1, 5) \cup \{10\} = S(Y)$. Consider two sequences $\{x_n\} = \{1\}$, $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$. Then the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Indeed we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1,$$

where $1 \in S(Y) \cap T(Y)$; however, $S(Y)$ and $T(Y)$ are not closed subsets of X.

Now, define functions $\psi_1, \psi_2 : [0, +\infty)^3 \to [0, +\infty)$ by

$$\psi_1(t, u, v) = \max\{t, u, v\}, \quad \psi_2(t, u, v) = \frac{1}{8} \max\{t, u, v\} \text{ for all } t, u, v \geq 0.$$

Clearly ψ_1 and ψ_2 are generalized altering distance functions and $\Psi_1(t) = t$ for all $t \geq 0$.
Now, we will check the inequality (2.7). We distinguish the following possible cases:
- If $x = y = 1$, then we get $d(Ax, By) = 0$ and (2.7) is trivially satisfied;
- if $x = 1$, $y \in (1, 3]$, then we get $d(Ax, By) = 16$, and (2.7) reduces to $16 \leq \frac{7}{8} \cdot 81 \approx 70$;
- if $x = 1$, $y \in (3, 10)$, then we get $d(Ax, By) = 0$ and (2.7) is trivially satisfied;
- if $x \in (1, 3]$, $y = 1$, then we get $d(Ax, By) = 49$, so (2.7) reduces to $49 \leq \frac{7}{8} \cdot 81 \approx 70$;
- if $x, y \in (1, 3]$, then we get $d(Ax, By) = 9$, so (2.7) reduces to $9 \leq \frac{7}{8} \cdot 9 \approx 22$;
- if $x \in (1, 3]$, $y \in (3, 10)$, then we get $d(Ax, By) = 49$, so (2.7) reduces to $49 \leq \frac{7}{8} \cdot 64 = 56$;
- if $x \in (3, 10)$, $y = 1$, then we get $d(Ax, By) = 0$ and (2.7) is trivially satisfied;
- if $x \in (3, 10)$, $y \in (1, 3]$, then we get $d(Ax, By) = 16$, so (2.7) reduces to $16 \leq \frac{7}{8} \cdot 81 \approx 70$;
- if $x, y \in (3, 10)$, then we get $d(Ax, By) = 0$ and (2.7) is trivially satisfied.

Thus, all the conditions of Theorem 2.1 (more precisely, Corollary 2.2) are satisfied (except $Y = X$), and 1 is a unique common fixed point of the pairs (A, S) and (B, T). Also, all the involved mappings are discontinuous at their unique common fixed point 1.

Next example highlights the non-closeness of ranges of S and T in X in the Corollary 2.6 and 2.7.

Example 3.2. Let $Y = [1, 15) \subset [1, +\infty) = X$ be equipped endowed with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in Y$ which also satisfies $(1C)$ and (HE). Consider the mappings $A, B, S, T : Y \to X$ by

$$Ax = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 15), \\ 10 & \text{if } 1 < x \leq 3, \end{cases} \qquad Bx = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 15), \\ 4 & \text{if } 1 < x \leq 3, \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } x = 1, \\ 4 & \text{if } 1 < x \leq 3, \\ \frac{x+1}{4}, & \text{if } 3 < x < 15, \end{cases} \qquad Tx = \begin{cases} 1 & \text{if } x = 1, \\ 10 + x & \text{if } 1 < x \leq 3, \\ x - 2 & \text{if } 3 < x < 15. \end{cases}$$

Then we have $A(Y) = \{1, 10\} \subseteq [1, 13] = T(Y)$ and $B(Y) = \{1, 4\} \subseteq [1, 4] = S(Y)$. Consider two sequences $\big($or $\{x_n\} = \{1\}$, $\{y_n\} = \big\{3 + \frac{1}{n}\big\}_{n\in\mathbb{N}}\big)$, Then the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Indeed we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1,$$

where $1 \in S(Y) \cap T(Y)$.
Notice that there is no ψ_1 and ψ_2 altering distance functions which satisfy the condition (2.8) of Theorem 2.1. For example $x \in (1, 3], y = 1$,

$$81 \nleq \psi_1(36, 0, 9) - \psi_2(36, 0, 9).$$

Thus, all the conditions of Theorem 2.1 (also the Corollary 2.2) are satisfied, except $Y = X$, but 1 is a unique common fixed point of the pairs (A, S) and (B, T). Here, it is worth noting that Theorem 2.1 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of Y. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

Now we furnish examples demonstrating the validity of condition (2.8) of Theorem 2.1 is only a necessary condition but not sufficient.

Example 3.3. Consider $X = [2, 20]$ equipped with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in Y$ which also satisfies $(1C)$ and (HE). Consider the mappings $A, B, S, T : Y \to X$ by

$$Ax = Bx = \begin{cases} 2 & \text{if } x = 2, \\ 7 & \text{if } 2 < x \leq 5, \\ 2, & \text{if } x < 5, \end{cases} \quad Sx = Tx = \begin{cases} 2 & \text{if } x = 2, \\ 7 & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3} & \text{if } x < 5. \end{cases}$$

Then the pair (A, S) satisfies all the conditions of Theorem 2.1 and has a coincidence at $x = 2$ which also remains a common fixed point of the pair. It is notice that condition (2.8) is not satisfies (e.g. $x \in (2, 5]$ and $y = 2$). This confirms that condition (2.8) of Theorem 2.1 is only a necessary condition but not sufficient.

4 Applications to existence theorems for functional equations arising in dynamic programming

Consider a multistage process is reduced to the system of functional equations

$$q_i(x) = \sup_{y\in D}\{g(x, y) + G_i(x, y, q_i(\tau(x, y)))\}, \quad x \in W, \tag{4.1}$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U as well as V are Banach spaces, $i \in \{1, 2, 3\}$.
The purpose of this section is to give an existence and uniqueness of solutions for a certain system of functional equations (4.1) arising in dynamic programming using Corollary 2.2.
Let $B(W)$ be the set of all bounded real-valued functions on W and, for an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W} |h(x)|$. Clearly, $(B(W), \|\cdot\|)$ endowed with the metric d defined by

$$d(h, k) = \sup_{x\in W} |h(x) - k(x)|$$

for all $h, k \in B(W)$, is a Banach space. Now, the convergence in the space $B(W)$ with respect to $\|\cdot\|$ is uniform. Therefore, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then the sequence $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded. Therefore $h^* \in B(W)$.
We consider the operators $T_i : B(W) \to B(W)$ given by

$$T_i(h)(x) = \sup_{y\in D}\{g(x, y) + G_i(x, y, h_i(\tau(x, y)))\}, \tag{4.2}$$

for $h \in B(W)$, $x \in W$, where $i \in \{1, 2, 3\}$; these mappings are well-defined if the functions g and G_i are bounded. Also, define

$$M(h, k) = k_1[d(T_1(h), T_3(h))]^2 + k_2[d(T_2(k), T_3(k))]^2 + k_3[d(T_3(h), T_3(k))]^2,$$

for all $h, k \in B(W)$.
Now, we are equipped to state and prove the following result.

Theorem 4.1. *Let $T_i : B(W) \to B(W)$ be given by (4.2), where $i \in \{1, 2, 3\}$. Suppose that the following hypotheses hold:*

(1) there exist $k_1, k_2, k_3 \geq 0$ such that

$$|G_1(x, y, h(x)) - G_2(x, y, k(x))| \\ \leq [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}}$$

and $k_1 + k_2 + k_3 < 1$, for all $x \in W$, $y \in D$;

(2) $g : W \times D \to \mathbb{R}$ and $G_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions, where $i \in \{1, 2, 3\}$;

(3) there exist sequences $\{h_n\}$ in $B(W)$ and $h^* \in B(W)$ such that

$$\lim_{n\to\infty} T_1(h_n) = \lim_{n\to\infty} T_2(h_n) = \lim_{n\to\infty} T_3(h_n) = h^*;$$

(4) $T_1T_3(h) = T_3T_1(h)$, *whenever* $T_1(h) = T_3(h)$, *for some* $h \in B(W)$*;*

(5) $T_2T_3(k) = T_3T_2(k)$, *whenever* $T_2(k) = T_3(k)$, *for some* $k \in B(W)$*.*

Then the system of functional equations (4.1) *has a unique bounded solution.*

Proof. Consider the symmetric $d_s : B(W) \times B(W) \to [0, +\infty)$ given by $d_s(h, k) = |h - k|^2$, for all $h, k \in B(W)$. Notice that the conditions $(1C)$ and (HE) hold trivially. By hypothesis (3) the pairs (T_1, T_3) and (T_2, T_3) share the common limit range property with respect to T_3. Now, let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$. Then there exist $y_1, y_2 \in D$ such that

$$T_1(h_1)(x) < g(x, y_1) + G_1(x, y_1, h_1(\tau(x, y_1))) + \lambda, \tag{4.3}$$

$$T_2(h_2)(x) < g(x, y_2) + G_2(x, y_2, h_2(\tau(x, y_2))) + \lambda, \tag{4.4}$$

$$T_1(h_1)(x) \geq g(x, y_2) + G_1(x, y_2, h_1(\tau(x, y_2))), \tag{4.5}$$

$$T_2(h_2)(x) \geq g(x, y_1) + G_2(x, y_1, h_2(\tau(x, y_1))). \tag{4.6}$$

Next, by using (4.3) and (4.6), we obtain

$$\begin{aligned} T_1(h_1)(x) - T_2(h_2)(x) &< G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1))) + \lambda \\ &\leq |G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1)))| + \lambda \\ &\leq [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + \\ &\quad k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}} + \lambda \end{aligned}$$

and so we have

$$\begin{aligned} T_1(h_1)(x) - T_2(h_2)(x) < [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + \\ k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}} + \lambda. \end{aligned} \tag{4.7}$$

Analogously, by using (4.4) and (4.5), we get

$$\begin{aligned} T_2(h_2)(x) - T_1(h_1)(x) < [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + \\ k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}} + \lambda. \end{aligned} \tag{4.8}$$

Finally, from (4.8) and (4.9), we deduce

$$\begin{aligned} |T_1(h_1)(x) - T_2(h_2)(x)| < [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + \\ k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}} + \lambda, \end{aligned} \tag{4.9}$$

or, equivalently,

$$d(T_1(h_1), T_2(h_2)) \leq [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}} + \lambda.$$

Notice that the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, therefore we obtain immediately that

$$d(T_1(h_1), T_2(h_2)) \leq [k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + k_3|T_3(h) - T_3(k)|^2]^{\frac{1}{2}},$$

or, equivalently,

$$\begin{aligned} d_s(T_1(h_1), T_2(h_2)) \leq k_1|T_1(h) - T_3(h)|^2 + k_2|T_2(k) - T_3(k)|^2 + \\ k_3|T_3(h) - T_3(k)|^2 = k_1 d_s(T_1(h), T_3(h)) + \\ k_2 d_s(T_2(k), T_3(k)) + k_3 d_s(T_3(h), T_3(k)) \end{aligned}$$

where $k_1 + k_2 + k_3 < 1$. Then, putting $A = T_1$, $B = T_2$ and $S = T = T_3$, Corollary 2.2 is satisfied with $q = 1$ and $Y = X = B(W)$. Moreover, in view of the hypotheses (4) and (5), the pairs (T_1, T_3) and (T_2, T_3) are weakly compatible, and so T_1, T_2 and T_3 have a unique common fixed point, that is, the system of functional equations (4.1) has a unique bounded solution. □

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