DOI: 10.2478/awutm-2013-0017

VERSITA

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LI, 2, (2013), 73– 87

On Controllability of Fuzzy Dynamical Matrix Lyapunov Systems

M.S.N.Murty, G. Suresh Kumar, B.V. Appa Rao, and K.A.S.N.V. Prasad

Abstract. In this paper, we provide a way to combine matrix Lyapunov systems with fuzzy sets to form a new system called fuzzy dynamical matrix Lyapunov system and obtain a sufficient condition for the controllability of this system.

AMS Subject Classification (2010). 93B05, 93C42, 34A07
Keywords. Fuzzy matrix Lyapunov system, Kronecker product, fundamental matrix, controllability.

1 Introduction

The importance of control theory in applied mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, artificial satellites etc., are well known. The main aim being to compel or control a given system to behave in some desired fashion. In the present day world we require systems being controlled automatically without direct human intervention.

Matrix Lyapunov type systems arise in a number of areas of applied mathematics such as dynamic programming, optimal filters, quantum mechanics, and systems engineering. Now we focus our attention to the first order fuzzy matrix Lyapunov system modelled by

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t)U(t), \;\; X(0) = X_0, \;\; t > 0, \qquad (1.1)$$

$$Y(t) = C(t)X(t) + D(t)U(t), \tag{1.2}$$

where $U(t)$ is a $n \times n$ fuzzy input matrix called fuzzy control and $Y(t)$ is a $n \times n$ fuzzy output matrix. Here $A(t), B(t), F(t), C(t)$, and $D(t)$ are matrices of order $n \times n$, whose elements are continuous functions of t on $J = [0, T] \subset R$ $(T > 0)$.

The existence and uniqueness theory for two point boundary value problems associated with matrix Lyapunov systems was obtained by Murty and Rao [9]. Further, controllability, observability, and realizability concepts for matrix Lyapunov systems were studied by Murty, Rao and Suresh Kumar [10]. Recently, Dubey and Georege [6] obtained sufficient conditions for the controllability of semi-linear matrix Lyapunov systems. The controllability criteria for fuzzy dynamical control systems was studied by Ding and Kandel [5]. Controllability and Observability criteria for Fuzzy dynamical matrix Lyapunov systems were studied by Murty and Suresh Kumar [11] with the use of fuzzy rule base. Moreover, Murty and Suresh Kumar [12] also studied the observability of fuzzy dynamical matrix Lyapunov systems without using the fuzzy rule base. This paper deals with obtaing sufficient conditions for the existence of controllability for fuzzy dynamical matrix Lyapunov systems with fuzzy set as input.

The paper is well organized as follows. In section 2 we present some basic definitions and results relating to fuzzy sets and Kronecker product of matrices. Further, we obtain general solution of the system (1.1), when $U(t)$ is a crisp continuous matrix. In section 3 we formulate a new fuzzy system by combining matrix Lyapunov systems with fuzzy sets called fuzzy dynamical matrix Lyapunov system, which can be regarded as a new approach to intelligent control, and also find its solution set. In section 4 we obtain a sufficient condition for the controllability of fuzzy dynamical matrix Lyapunov system without using fuzzy rule base and highlight the main theorem with a suitable example.

This paper generalizes some of the results of Ding and Kandel [5] to matrix Lyapunov systems.

2 Preliminaries

Let $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of R^n. Define the addition and scalar multiplication in $P_k(R^n)$ as usual.

Radstrom [14] states that $P_k(R^n)$ is a commutative semi-group under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in R$ and

$A, B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \inf\{\epsilon : A \subset N(B,\epsilon), B \subset N(A,\epsilon)\},$$

where

$$N(A,\epsilon) = \{x \in R^n : ||x-y|| < \epsilon, for \;\; some \;\; y \in A\}.$$

Definition 2.1. *A set valued function $F : J \to P_k(R^n)$ is said to be measurable if it satisfies any one of the following equivalent conditions;*

1. *for all $u \in R^n$, $t \to d_{F(t)}(u) = \inf_{v \in F(t)} \|u - v\|$ is measurable,*
2. *$GrF = \{(t,u) \in J \times R^n : u \in F(t)\} \in \Sigma \times \beta(R^n)$, where $\Sigma, \beta(R^n)$ are Borel σ-field of J and R^n respectively (Graph measurability),*
3. *there exists a sequence $\{f_n(.)\}_{n\geq 1}$ of measurable functions such that $F(t) = \overline{\{f_n(.)\}_{n\geq 1}}$, for all $t \in J$ (Castaing's representation)*[13].

We denote by S_F^1 the set of all selections of $F(.)$ that belong to the Lebesgue Bochner space $L^1_{R^n}(J)$. i.e.

$$S_F^1 = \{f(.) \in L^1_{R^n}(J) : f(t) \in F(t) a.e.\}.$$

We present the Aumann's integral as follows;

$$(A)\int_J F(t)dt = \{\int_J f(t)dt, f(.) \in S_F^1\}.$$

We say that $F : J \to P_k(R^n)$ is integrably bounded if it is measurable and there exists a function $h : J \to R$, $h \in L^1_{R^n}(J)$ such that $\|u\| \leq h(t)$, $u \in F(t)$. From [2], we know that if F is closed valued measurable multifunction, then $\int_J F(t)dt$ is convex in R^n. Furthermore, if F is bounded integrable then $\int_J F(t)dt$ is compact in R^n. Let

$$E^n = \{u : R^n \to [0,1]/u \text{ satisfies (i)-(iv) below}\},$$

where (i) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$; (ii) u is fuzzy convex, i.e. for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semi-continuous; (iv) $[u]^0 = \overline{\{x \in R^n / u(x) > 0\}}$ is compact. For $0 < \alpha \leq 1$, the α-level set is denoted and defined by $[u]^\alpha = \{x \in R^n / u(x) \geq \alpha\}$. Then from (i)-(iv) it follows that $[u]^\alpha \in P_k(R^n)$, for all $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$ by

$$D(u, v) = \sup\{d([u]^\alpha, [v]^\alpha)/\alpha \in [0, 1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. It is easy to show that D is a metric in E^n and using results of [[4], [14]], we see that (E^n, D) is a complete metric space, but not locally compact. Moreover, the distance D verifies that $D(u+w, v+w) = D(u, v)$, $u, v, w \in E^n$, $D(\lambda u, \lambda v) = |\lambda| D(u, v)$, $u, v \in E^n, \lambda \in R$, $D(u + w, v + z) \leq D(u, v) + D(w, z)$, $u, v, w, z \in E^n$.

We note that (E^n, D) is not a vector space. But it can be imbedded isomorphically as a cone in a Banach space [14].

Regarding fundamentals of differentiability and integrability of fuzzy functions we refer to O.Kaleva [7], Lakshmikantham and Mohapatra [8].

In the sequel we need the following representation theorem.

Theorem 2.1. [13] *If $u \in E^n$, then*

1. *$[u]^\alpha \in P_k(R^n)$, for all $0 \leq \alpha \leq 1$.*
2. *$[u]^{\alpha_2} \subset [u]^{\alpha_1}$, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.*
3. *If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}$.*

Conversely, if $\{A^\alpha : 0 \leq \alpha \leq 1\}$ is a family of subsets of R^n satisfying (1)-(3), then there exists a $u \in E^n$ such that $[u]^\alpha = A^\alpha$, for $0 < \alpha \leq 1$ and $[u]^0 = \overline{\bigcup_{0 \leq \alpha \leq 1}} A^\alpha \subset A^0$.

A fuzzy set valued mapping $F : J \to E^n$ is called fuzzy integrably bounded if $F_0(t)$ is integrably bounded.

Definition 2.2. *Let $F : J \to E^n$ be a fuzzy integrably bounded mapping. The fuzzy integral of F over J denoted by $\int_J F(t)dt$, is defined level-set-wise by*

$$\left[\int_J F(t)dt\right]^\alpha = (A) \int_J F_\alpha(t)dt, \;\; 0 < \alpha \leq 1.$$

Let $F : J \times E^n \to E^n$, consider the fuzzy differential equation

$$u' = F(t, u), \;\; u(0) = u_0. \tag{2.1}$$

Definition 2.3. *A mapping $u : J \to E^n$ is a fuzzy weak solution to* (2.1) *if it is continuous and satisfies the integral equation*

$$u(t) = u_0 + \int_0^t F(s, u(s))ds, \ \ \forall t \in J.$$

If F is continuous, then this weak solution also satisfies (2.1) and we call it fuzzy strong solution to (2.1).

Definition 2.4. [1] *Let $A \in C^{m\times n}$ and $B \in C^{p\times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $C^{mp\times nq}$.

Definition 2.5. [1] *Let $A = [a_{ij}] \in C^{m\times n}$, we denote*

$$\hat{A} = Vec\, A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.n} \end{bmatrix}, \ \ where \ \ A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix} \ \ (1 \le j \le n).$$

Regarding properties and rules for Kronecker product of matrices we refer to Murty and Suresh Kumar [1].

Now by applying the Vec operator to the matrix Lyapunov system (1.1) satisfying (1.2) and using the above properties, we have

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \ \ \hat{X}(0) = \hat{X}_0, \tag{2.2}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t), \tag{2.3}$$

where $G(t) = (B^* \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $\hat{X}$= Vec $X(t)$, $\hat{U}$= Vec $U(t)$ are column matrices of order n^2.

The corresponding linear homogeneous system of (2.2) is

$$\hat{X}'(t) = G(t)\hat{X}(t), \ \ \hat{X}(0) = \hat{X}_0. \tag{2.4}$$

Lemma 2.1. *Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems*

$$X'(t) = A(t)X(t), \;\; X(0) = I_n, \tag{2.5}$$

and

$$[X^*(t)]' = B^*(t)X^*(t), \;\; X(0) = I_n \tag{2.6}$$

respectively. Then the matrix $\psi(t) \otimes \phi(t)$ is a fundamental matrix of (2.4) *and the solution of* (2.4) *is $\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0$.*

Proof. For proof, we refer to Lemma 1 of [11]. □

Theorem 2.2. *Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems* (2.5) *and* (2.6)*, then the unique solution of* (2.2) *is*

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes \hat{F}(s))\hat{U}(s)ds. \tag{2.7}$$

Proof. For proof, we refer to Theorem 1 of [11]. □

3 Formation of fuzzy dynamical Lyapunov systems

In this section we show that the following system

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \hat{X}(0) = \hat{X}_0 \tag{3.1}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t), \tag{3.2}$$

determines a fuzzy system, when $\hat{U}(t)$ is a fuzzy set.
Fix $0 < \alpha \leq 1$, let $[\hat{U}(t)]^\alpha$ be the α-level set of $\hat{U}(t)$. For any positive number T, Consider differential inclusions

$$\hat{X}'_\alpha(t) \in G(t)\hat{X}_\alpha(t) + (I_n \otimes F(t))[\hat{U}(t)]^\alpha, \;\; t \in [0, T] \tag{3.3}$$

$$\hat{X}(0) = \hat{X}_0. \tag{3.4}$$

Let $\hat{X}^\alpha$ be the solution set of inclusions (3.3) and (3.4).
Claim (i). $[\hat{X}(t)]^\alpha \in P_k(R^{n^2})$, for every $t \in [0, T]$.
First, we prove that $\hat{X}^\alpha$ is nonempty, compact and convex in $C[[0, T], R^{n^2}]$. Since $[\hat{U}(t)]^\alpha$ has measurable selection, we have $\hat{X}^\alpha$ is nonempty. Let

$$K = \max_{t\in[0,T]} \|\phi(t)\|, \quad L = \max_{t\in[0,T]} \|\psi(t)\|$$

$$M = \max\{\|u(t)\| : u(t) \in [\hat{U}(t)]^\alpha, t \in [0, T]\}$$

and $N = \max_{t\in[0,T]} \|F(t)\|$. If for any $\hat{X} \in \hat{X}^\alpha$, then there is a selection $u(t) \in [\hat{U}(t)]^\alpha$ such that

$$\begin{aligned}\hat{X}(t) &= (\psi(t) \otimes \phi(t))\hat{X}_0 \\ &\quad + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\|\hat{X}(t)\| &\le \|(\psi(t) \otimes \phi(t))\hat{X}_0\| \\ &\quad + \int_0^t \|(\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u(s)\|ds \\ &\le \|\psi(t)\|\|\phi(t)\|\|\hat{X}_0\| \\ &\quad + \int_0^t \|\psi(t-s)\|\|\phi(t-s)\|\|F(s)\|\|u(s)\|ds \\ &\le KL\|\hat{X}_0\| + KLNM.\end{aligned}$$

Thus $\hat{X}^\alpha$ is bounded. For any $t_1, t_2 \in [0, T]$

$$\begin{aligned}\hat{X}(t_1) - \hat{X}(t_2) &= (\psi(t_1) \otimes \phi(t_1))\hat{X}_0 \\ &\quad + \int_0^{t_1} (\psi(t_1 - s) \otimes \phi(t_1 - s))(I_n \otimes F(s))u(s)ds \\ &\quad - (\psi(t_2) \otimes \phi(t_2))\hat{X}_0 \\ &\quad - \int_0^{t_2} (\psi(t_2 - s) \otimes \phi(t_2 - s))(I_n \otimes F(s))u(s)ds.\end{aligned}$$

Therefore

$$\begin{aligned}&\|\hat{X}(t_1) - \hat{X}(t_2)\| \le \|(\psi(t_1) \otimes \phi(t_1)) - (\psi(t_2) \otimes \phi(t_2))\|\|\hat{X}_0\| \\ &\quad + \int_{t_2}^{t_1} \|(\psi(t_1 - s) \otimes \phi(t_1 - s))(I_n \otimes F(s))u(s)\|ds \\ &\quad + \int_0^{t_2} \|[(\psi(t_1 - s) \otimes \phi(t_1 - s)) - (\psi(t_2 - s) \otimes \phi(t_2 - s))] \\ &\qquad\qquad (I_n \otimes F(s))u(s)\|ds \\ &\le \|(\psi(t_1) \otimes \phi(t_1)) - (\psi(t_2) \otimes \phi(t_2))\|\|\hat{X}_0\| + KLNM|t_1 - t_2| \\ &\quad + MN \int_0^T \|(\psi(t_1 - s) \otimes \phi(t_1 - s)) - (\psi(t_2 - s) \otimes \phi(t_2 - s))\|ds.\end{aligned}$$

Since $\phi(t)$ and $\psi(t)$ are uniformly continuous on $[0,T]$, $\hat{X}$ is equi-continuous. Thus $\hat{X}^\alpha$ is relatively compact. If $\hat{X}^\alpha$ is closed, then it is compact.
Let $\hat{X}_k \in \hat{X}^\alpha$ and $\hat{X}_k \to \hat{X}$. For each $\hat{X}_k$, there is a $u_k \in [\hat{U}(t)]^\alpha$ such that

$$\hat{X}_k(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u_k(s)ds. \quad (3.5)$$

Since $u_k \in [\hat{U}(t)]^\alpha$, it is closed, then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ converging weakly to $u \in [\hat{U}(t)]^\alpha$. From Mazur's theorem [3], there exists a sequence of numbers $\lambda_j > 0$, $\sum \lambda_j = 1$ such that $\sum \lambda_j u_{k_j}$ converges strongly to u. Thus from (3.5), we have

$$\begin{aligned}\sum \lambda_j \hat{X}_{k_j}(t) &= \sum \lambda_j (\psi(t) \otimes \phi(t))\hat{X}_0 \\ &+ \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s)) \sum \lambda_j u_{k_j}(s)ds. \quad (3.6)\end{aligned}$$

From Fatou's lemma, taking the limit as $j \to \infty$ on both sides of (3.6), we have

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))u(s)ds. \quad (3.7)$$

Thus $\hat{X}(t) \in \hat{X}^\alpha$, and hence $\hat{X}^\alpha$ is closed.
Let $\hat{X}_1$, $\hat{X}_2 \in \hat{X}^\alpha$, then there exist $u_1, u_2 \in [\hat{U}(t)]^\alpha$ such that

$$\hat{X}_1'(t) = G(t)\hat{X}_1(t) + (I_n \otimes F(t))u_1(t)$$

and

$$\hat{X}_2'(t) = G(t)\hat{X}_2(t) + (I_n \otimes F(t))u_2(t).$$

Let $\hat{X} = \lambda\hat{X}_1(t) + (1-\lambda)\hat{X}_2(t)$, $0 \le \lambda \le 1$, then

$$\begin{aligned}\hat{X}' &= \lambda\hat{X}_1'(t) + (1-\lambda)\hat{X}_2'(t) \\ &= \lambda \Big(G(t)\hat{X}_1(t) + (I_n \otimes F(t))u_1(t)\Big) \\ &\quad + (1-\lambda) \Big(G(t)\hat{X}_2(t) + (I_n \otimes F(t))u_2(t)\Big) \\ &= G(t) \Big[\lambda\hat{X}_1(t) + (1-\lambda)\hat{X}_2(t)\Big] \\ &\quad + (I_n \otimes F(t)) \left[\lambda u_1(t) + (1-\lambda)u_2(t)\right].\end{aligned}$$

Since $[\hat{U}(t)]^\alpha$ is convex, $\lambda u_1(t) + (1-\lambda)u_2(t) \in [\hat{U}(t)]^\alpha$, we have

$$\hat{X}'(t) \in G(t)\hat{X}(t) + (I_n \otimes F(t))[\hat{U}(t)]^\alpha.$$

i.e $\hat{X} \in \hat{X}^\alpha$. Thus $\hat{X}^\alpha$ is convex.
From Arzela-Ascoli theorem, we know that $[\hat{X}(t)]^\alpha$ is compact in R^{n^2} for every $t \in [0,T]$. Also it is obvious that $[\hat{X}(t)]^\alpha$ is convex in R^{n^2}. Thus we have $[\hat{X}(t)]^\alpha \in P_k(R^{n^2})$ for every $t \in [0,T]$. Hence the claim.
Claim (ii). $[\hat{X}(t)]^{\alpha_2} \subset [\hat{X}(t)]^{\alpha_1}$, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Since $[\hat{U}(t)]^{\alpha_2} \subset [\hat{U}(t)]^{\alpha_1}$, we have $S^1_{[\hat{U}(t)]^{\alpha_2}} \subset S^1_{[\hat{U}(t)]^{\alpha_1}}$ and the following inclusion

$$\begin{aligned}\hat{X}'_{\alpha_2}(t) &\in G(t)\hat{X}_{\alpha_2} + (I_n \otimes F(t))[\hat{U}(t)]^{\alpha_2}\\ &\subset G(t)\hat{X}_{\alpha_1} + (I_n \otimes F(t))[\hat{U}(t)]^{\alpha_1}.\end{aligned}$$

Therefore

$$\begin{aligned}\hat{X}_{\alpha_2}(t) &\in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{[\hat{U}(s)]^{\alpha_2}} ds\\ &\subset (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{[\hat{U}(s)]^{\alpha_1}} ds.\end{aligned}$$

Thus $\hat{X}^{\alpha_2} \subset \hat{X}^{\alpha_1}$, and hence $[\hat{X}(t)]^{\alpha_2} \subset [\hat{X}(t)]^{\alpha_1}$. Hence the claim.
Claim (iii). If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $\hat{X}^\alpha(t) = \bigcap_{k\geq 1} \hat{X}^{\alpha_k}(t)$.
Since $\hat{U}(t)$ is fuzzy set, we have $[\hat{U}(t)]^\alpha = \bigcap_{k\geq 1}[\hat{U}(t)]^{\alpha_k}$ and then $S^1_{[\hat{U}(t)]^\alpha} = S^1_{\bigcap_{k\geq 1}[\hat{U}(t)]^{\alpha_k}}$. Therefore

$$\begin{aligned}\hat{X}'_\alpha(t) &\in G(t)\hat{X}_\alpha + (I_n \otimes F(t))[\hat{U}(t)]^\alpha\\ &= G(t)\hat{X}_\alpha + (I_n \otimes F(t))[\hat{U}(t)]^{\alpha_k}\\ &\subset G(t)\hat{X}_\alpha + (I_n \otimes F(t))[\hat{U}(t)]^{\alpha_k},\ \forall\ k = 1, 2, ...\end{aligned}$$

Thus we have $\hat{X}^\alpha \subset \hat{X}^{\alpha_k}$, $k = 1, 2, ...$, which implies that

$$\hat{X}^\alpha \subset \bigcap_{k\geq 1} \hat{X}^{\alpha_k}. \tag{3.8}$$

Let $\hat{X}$ be the solution to the inclusion

$$\hat{X}^{\alpha_k}(t) \in G(t)\hat{X}_{\alpha_k} + (I_n \otimes F(t))[\hat{U}(t)]^{\alpha_k}, k \geq 1.$$

Then

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{[\hat{U}(s)]^{\alpha_k}} ds$$

it follows that

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s)) \bigcap_{k\geq 1} S^1_{[\hat{U}(s)]^{\alpha_k}} ds$$

$$= (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s)) S^1_{[\hat{U}(s)]^{\alpha}} ds.$$

This implies that $\hat{X} \in \hat{X}^{\alpha}$. Therefore

$$\bigcap_{k\geq 1} \hat{X}^{\alpha_k} \subset \hat{X}^{\alpha} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\hat{X}^{\alpha} = \bigcap_{k\geq 1} \hat{X}^{\alpha_k}$$

and hence

$$\hat{X}^{\alpha}(t) = \bigcap_{k\geq 1} \hat{X}^{\alpha_k}(t).$$

From Claims (i)-(iii) and applying Theorem 2.1, there exists $\hat{X}(t) \in E^{n^2}$ on $[0, T]$ such that $\hat{X}^{\alpha}(t)$ is a solution to the differential inclusion (3.3) and (3.4). Hence the system (3.1), (3.2) is a fuzzy dynamical Lyapunov system, it can be expressed as

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \hat{X}(0) = \{\hat{X}_0\}, \tag{3.10}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t). \tag{3.11}$$

The solution of the fuzzy dynamical system (3.10), (3.11) is given by

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s) ds. \tag{3.12}$$

4 Controllability of fuzzy dynamical Lyapunov systems

In this section we study the controllability of the fuzzy system (3.10) satisfying (3.11).

Definition 4.1. *The fuzzy system (3.10), (3.11) is said to be completely controllable if for any t_0, any initial state $\hat{X}(t_0) = \hat{X}_0$ and any given final state $\hat{X}_f$ there exists a finite time $t_1 > t_0$ and a control $\hat{U}(t)$, $t_0 \leq t \leq t_1$, such that $\hat{X}(t_1) = \hat{X}_f$.*

Lemma 4.1. *If F is a fuzzy set, then $\int_0^T F dt = TF$.*

Proof. For proof, we refer to Lemma 2 of [11]. □

Lemma 4.2. *Let P, Q be two fuzzy sets and $h(t)$ be a non zero continuous function on $[0, T]$, satisfying*

$$\int_0^T h(t) P dt = \int_0^T h(t) Q dt,$$

then $P = Q$.

Proof. For proof, we refer to Lemma 3 of [11]. □

Theorem 4.1. *The fuzzy system (3.10), (3.11) is completely controllable if the $n^2 \times n^2$ symmetric controllability matrix*

$$W(0,T) = \int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))(I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^* dt. \tag{4.1}$$

is nonsingular. Furthermore, the fuzzy control $\hat{U}(t)$ transfer the state of the system from $\hat{X}(0) = \hat{X}_0$ to a fuzzy state $\hat{X}(T) = \hat{X}_f$, can be chosen as

$\hat{U}(t) = \frac{1}{T}(I_n \otimes F(t))^{-1}(\psi(T-t) \otimes \phi(T-t))^{-1}\hat{X}_f - (I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^* W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0.$

Proof. Suppose that the symmetric controllability matrix $W(0,T)$ is nonsingular. Therefore $W^{-1}(0,T)$ exists. Multiplying $W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0$ on both sides of (4.1), we have

$$(\psi(T) \otimes \phi(T))\hat{X}_0 = \int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))(I_n \otimes F(t))^* (\psi(T-t) \otimes \phi(T-t))^* W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0 dt. \tag{4.2}$$

If the fuzzy control $\hat{U}(t)$ transfer the state of the system from $\hat{X}_0$ to $\hat{X}_f$ over $[0, T]$, then from (3.12), we get

$$\hat{X}(T) = \hat{X}_f = (\psi(T) \otimes \phi(T))\hat{X}_0 + \int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))\hat{U}(t) dt. \tag{4.3}$$

Using Lemma 4.1, $\hat{X}_f$ can be written as

$$\hat{X}_f = \frac{1}{T}\int_0^T \hat{X}_f dt = \frac{1}{T}\int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))$$

$$(I_n \otimes F(t))^{-1}(\psi(T-t) \otimes \phi(T-t))^{-1}\hat{X}_f dt. \qquad (4.4)$$

From (4.3) and (4.4), we have

$$\frac{1}{T}\int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}\hat{X}_f dt$$

$$= \int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))(I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^*$$

$$W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0 dt + \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\hat{U}(t)dt.$$

It follows that

$$\int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))\hat{U}(t)dt$$

$$= \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\{\frac{1}{T}(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}$$

$$\hat{X}_f - (I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^* W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0\}dt$$

By using Lemma 4.2, we get

$$\hat{U}(t) = \frac{1}{T}(I_n \otimes F(t))^{-1}(\psi(T-t) \otimes \phi(T-t))^{-1}\hat{X}_f$$

$$-(I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^* W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0.$$

$\square$

Remark 4.1. The nonsingularity of the symmetric controllability matrix $W(0,T)$ in Theorem 4.1 is only a sufficient condition but not necessary.

Example 4.1. Consider the fuzzy dynamical matrix Lyapunov system (1.1) satisfying (1.2) with

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T = 1, \text{ and } X_0 = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 1 \end{bmatrix}.$$

Also assume that α-level sets of the final state

$$X_f^\alpha = \begin{bmatrix} [\alpha+1, 2] & [\alpha-1, 1-\alpha] \\ [2\alpha+1, 3] & [0, -1.5(\alpha-1)] \end{bmatrix}.$$

Then the fundamental matrices of (2.5) and (2.6) are

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ \psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Now the fundamental matrix of (2.4) is

$$\psi(t) \otimes \phi(t) = \begin{bmatrix} e^t & te^t & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^t & te^t \\ 0 & 0 & 0 & e^t \end{bmatrix} = e^t \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider

$$(\psi(1-t)\otimes\phi(1-t))(I_n\otimes F(t))(I_n\otimes F(t))^*(\psi(1-t)\otimes\phi(1-t))^*$$

$$= e^{1-t} \begin{bmatrix} 1 & 1-t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-t \\ 0 & 0 & 0 & 1 \end{bmatrix} e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{1-t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-t & 1 \end{bmatrix} = e^2 \begin{bmatrix} t^2-2t+2 & 1-t & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & t^2-2t+2 & 1-t \\ 0 & 0 & 1-t & 1 \end{bmatrix}.$$

Therefore

$$W(0,1) = \int_0^1 e^2 \begin{bmatrix} t^2-2t+2 & 1-t & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & t^2-2t+2 & 1-t \\ 0 & 0 & 1-t & 1 \end{bmatrix} dt$$

$$= e^2 \begin{bmatrix} \frac{4}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

Clearly, it is nonsingular.

Thus from Theorem 4.1,the input $\hat{U}(t)$ can be chosen by the following α-level sets.

$$\hat{U}^{\alpha}(t) = e^{-t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^{t-1} \begin{bmatrix} 1 & t-1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t-1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} [\alpha+1,2] \\ [2\alpha+1,3] \\ [\alpha-1,1-\alpha] \\ [0,-1.5(\alpha-1)] \end{bmatrix}$$

$$-e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^{1-t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-t & 1 \end{bmatrix} \frac{12}{13} e^{-2}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{4}{3} \end{bmatrix} e \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$= e^{-1} \begin{bmatrix} [(3t-2)+\alpha, 2(t-1)\alpha+(t+1)] \\ [2\alpha+1, 3] \\ [(\alpha-1)(2.5-1.5t), 1-\alpha] \\ [0, -1.5(\alpha-1)] \end{bmatrix} - \frac{12}{13} \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{4}(5t-9) \\ \frac{3}{2} \\ \frac{11-9t}{6} \end{bmatrix}.$$

References

[1] **Alexander Graham**, *Kronecker products and matrix calculus with applications*, Ellis Horwood Ltd., England, 1981

[2] **R.J.Aumann**, Integrals of set-valued functions, *Journal of Mathematical Analysis and Applications*, **12**, (1965), 1–12

[3] **J.B.Conway**, *A course in functional analysis*, Springer-Verlag, New York, 1990

[4] **G.Debreu**, Integration of correspondence, *in: Proc. Fifth Berkeley Symp. Math. Statist. Probab. Part 1(Univ. California Press, Berkeley, CA)*, **2**, (1967), 351–372

[5] **Z.Ding and A.Kandel**, On the controllability of fuzzy dynamical systems (I), *The Journal of Fuzzy Mathematics*, **18/1**, (2000), 203–214

[6] **B. Dubey and R.K. George**, Controllability of semi-linear matrix Lyapunov systems, *Electronic Journal of Differential Equations*, **2013, No.42**, (2013), 1–12

[7] **O.Kaleva**, Fuzzy differential equations, *Fuzzy sets and Systems*, **24**, (1987), 301–317

[8] **V.Lakshmikantham and R.Mohapatra**, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, London, 2003

[9] **M.S.N.Murty and B.V.Appa Rao**, Two point boundary value problems for matrix Lyapunov systems, *Journal of the Indian Mathematical society*, **73/1**, (2006), 1–7

[10] **M.S.N.Murty B.V.Appa Rao and G.Suresh Kumar**, Controllability, Observability, and Realizability of matrix Lyapunov systems, *Bulletin of the Korean Mathematical Society*, **43/1**, (2006), 149–159

[11] **M.S.N.Murty and G.Suresh Kumar**, On Controllability and Observability of Fuzzy Dynamical Matrix Lyapunov Systems, *Advances in Fuzzy Systems*, **2008**, (2008), 1–16

[12] **M.S.N.Murty and G.Suresh Kumar**, On Observability of Fuzzy Dynamical Matrix Lyapunov Systems, *Kyungpook Mathematical Journal*, **48**, (2008), 473–486

[13] **C.V.Negoita and D.A.Ralescu**, *Applications of fuzzy sets to systems analysis*, Willey, New York, 1975

[14] **M.Radstrom**, An embedding theorem for space of convex sets, *Proceedings of American Mathematical Society*, **3**, (1952), 165–169

M.S.N.Murty

Department of Mathematics, Achary Nagarjuna University
Nagrajuna Nagar, Guntur, Andhra Pradesh, India.

E-mail: drmsn2002@gmail.com

G. Suresh Kumar

Department of Mathematics, Koneru Lakshmaiah Educational Foundation (K L University)
Vaddeswaram, Guntur, Andhra Pradesh, India.

E-mail: drgsk006@kluniversity.in

B.V. Appa Rao

Department of Mathematics, Koneru Lakshmaiah Educational Foundation
(K L University)
Vaddeswaram, Guntur, Andhra Pradesh, India.

E-mail: bvardr2010@kluniversity.in

K.A.S.N.V. Prasad

Department of Applied Mathematics, Krishna University P.G Centre
Nuzvid-521201, Krishna Dt., Andhra Pradesh, India.

E-mail: prasad.krosuri@gmail.com

Received: 30.09.2013

Accepted: 4.12.2013