DOI: 10.2478/awutm-2013-0003

Analele Universităţii de Vest,
Timişoara
Seria Matematică – Informatică
LI, 1, (2013), 37– 45

Radius Problems for Certain Classes of Analytic Functions

Basem Aref Frasin and Jin-Lin Liu

Abstract. Let $\mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ be a subclass of analytic functions $f(z)$ which satisfies $\frac{f(z)}{z} \neq 0 \ \ (z \in \mathcal{U})$ and

$$\left|\beta_1 \left(\frac{z}{f(z)}\right)'' + \beta_2 \left(\frac{z}{f(z)}\right)''' + \beta_3 \left(\frac{z}{f(z)}\right)''''\right| \leq \lambda \qquad (z \in \mathcal{U}),$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$. The object of the present paper is to obtain radius problems of $\frac{1}{\delta} f(\delta z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ if $f(z)$ satisfies $\mathrm{Re}\left\{\frac{1-z(\frac{z}{f(z)})''}{1+(\frac{z}{f(z)})'}\right\} > \alpha$.

AMS Subject Classification (2000). 30C45.
Keywords. Analytic functions, univalent functions, Cauchy-Schwarz inequality, radius problems.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{H}(\alpha)$ be the subclass of analytic functions $f(z)$ defined by

$$\mathrm{Re}\left\{\frac{1 - z(\frac{z}{f(z)})''}{1 + (\frac{z}{f(z)})'}\right\} > \alpha \qquad (0 \leq \alpha < 1; z \in \mathcal{U}), \tag{1.2}$$

where $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$, $b_n = |b_n|\, e^{i(n-1)\theta}$.

For a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ if it satisfies $\frac{f(z)}{z} \neq 0 \ \ (z \in \mathcal{U})$ and

$$\left|\beta_1 \left(\frac{z}{f(z)}\right)'' + \beta_2 \left(\frac{z}{f(z)}\right)''' + \beta_3 \left(\frac{z}{f(z)}\right)''''\right| \leq \lambda \qquad (z \in \mathcal{U}), \tag{1.3}$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$. Very recently, Kobashi et al. [3] have studied the class $\mathcal{P}(\beta_1, \beta_2, 0; \lambda)$ (see also, [2]) and Kobashi et al. [1] have considered the class $\mathcal{P}_4(\lambda) = \mathcal{P}(0, 0, 1; \lambda)$ defined by

$$\left|\left(\frac{z}{f(z)}\right)''''\right| \leq \lambda \qquad (z \in \mathcal{U}). \tag{1.4}$$

Let us consider the function $f_\gamma(z)$ given by $f_\gamma(z) = \frac{z}{(1-z)^\gamma} \quad (\gamma \geq 0)$.Then,we observe that

$$\frac{f_\gamma(z)}{z} = \frac{1}{(1-z)^\gamma} \neq 0 \qquad (z \in \mathcal{U}),$$

and

$$\begin{aligned}
& \left|\beta_1 \left(\frac{z}{f_\gamma(z)}\right)'' + \beta_2 \left(\frac{z}{f_\gamma(z)}\right)''' + \beta_3 \left(\frac{z}{f_\gamma(z)}\right)''''\right| \\
= \ & \left|\beta_1 \gamma(\gamma-1)(1-z)^{\gamma-2} + \beta_2 \gamma(\gamma-1)(\gamma-2)(1-z)^{\gamma-3}\right. \\
& \left. + \beta_3 \gamma(\gamma-1)(\gamma-2)(\gamma-3)(1-z)^{\gamma-4}\right| \\
< \ & \left|\beta_1 \gamma(\gamma-1)(2)^{\gamma-2} + \beta_2 \gamma(\gamma-1)(\gamma-2)(2)^{\gamma-3}\right. \\
& \left. + \beta_3 \gamma(\gamma-1)(\gamma-2)(\gamma-3)(2)^{\gamma-4}\right|
\end{aligned}$$

Therefore, if $\gamma = 2$, then

$$\left|\beta_1 \left(\frac{z}{f_2(z)}\right)'' + \beta_2 \left(\frac{z}{f_2(z)}\right)''' + \beta_3 \left(\frac{z}{f_2(z)}\right)''''\right| < 2\,|\beta_1|\,.$$

This implies that $f_2(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda > 2\,|\beta_1|$. If $\gamma = 3$, then we have

$$\left|\beta_1\left(\frac{z}{f_3(z)}\right)'' + \beta_2\left(\frac{z}{f_3(z)}\right)''' + \beta_3\left(\frac{z}{f_3(z)}\right)''''\right| < 6(2|\beta_1| + |\beta_2|).$$

Thus, $f_3(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda > 6(2|\beta_1| + |\beta_2|)$. Further, if $\gamma = 4$, then we have

$$\left|\beta_1\left(\frac{z}{f_4(z)}\right)'' + \beta_2\left(\frac{z}{f_4(z)}\right)''' + \beta_3\left(\frac{z}{f_4(z)}\right)''''\right| < 8(6|\beta_1| + 6|\beta_2| + 3|\beta_3|).$$

Therefore, $f_4(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda > 8(6|\beta_1| + 6|\beta_2| + 3|\beta_3|)$.
If we consider the function $f(z)$ given by $f(z) = \frac{z}{\sum_{k=0}^{n} z^k}$, then

$$\begin{aligned}
&\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)''' + \beta_3\left(\frac{z}{f(z)}\right)''''\right| \\
< \; & |\beta_1|\sum_{k=2}^{n} k(k-1) + |\beta_2|\sum_{k=3}^{n} k(k-1)(k-2) + |\beta_3|\sum_{k=4}^{n} k(k-1)(k-2)(k-3) \\
= \; & n(n+1)\left[\frac{|\beta_2|(n-1)}{3} + \frac{|\beta_3|(n-1)(n-2)}{4} + \frac{|\beta_3|(n-1)(n-2)(n-3)}{5}\right].
\end{aligned}$$

Therefore, $f(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$, with

$$\lambda = n(n+1)\left[\frac{|\beta_2|(n-1)}{3} + \frac{|\beta_3|(n-1)(n-2)}{4} + \frac{|\beta_3|(n-1)(n-2)(n-3)}{5}\right].$$

2 Radius problem for the class $\mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$

To obtain the radius problem for the class $\mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$, we need the following lemmas.

Lemma 2.1. *Let $f(z) \in \mathcal{A}$ and $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0$ $(z \in \mathcal{U})$. If $f(z)$ satisfies*

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|b_n| \leq \lambda \tag{2.1}$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$, then $f(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$.

Proof. We observe that

$$\begin{aligned}
& \left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)''' + \beta_3\left(\frac{z}{f(z)}\right)''''\right| \\
= & \left|\beta_1\sum_{n=2}^{\infty} n(n-1)b_n z^{n-2} + \beta_2\sum_{n=2}^{\infty} n(n-1)(n-2)b_n z^{n-3}\right. \\
& \left. +\beta_3\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)b_n z^{n-4}\right| \\
\leq & \ |\beta_1|\sum_{n=2}^{\infty} n(n-1)\,|b_n| + |\beta_2|\sum_{n=3}^{\infty} n(n-1)(n-2)\,|b_n| + \\
& |\beta_3|\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)\,|b_n| \\
= & \sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|b_n|
\end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (2.1), then $f(z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$. ∎

Lemma 2.2. *If $f(z) \in \mathcal{H}(\alpha)$;$0 \leq \alpha < \frac{1}{1+|b_1|}$ and $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$, $b_n = |b_n|\, e^{i(n-1)\theta}$, then*

$$\sum_{n=2}^{\infty} n(n-1+\alpha)\,|b_n| \leq 1 - \alpha - \alpha\,|b_1|\,. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{H}(\alpha)$ and $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$, $b_n = |b_n|\, e^{i(n-1)\theta}(n = 1, 2, \cdots)$. Then,

$$\begin{aligned}
\operatorname{Re}\left\{\frac{1 - z(\frac{z}{f(z)})''}{1 + (\frac{z}{f(z)})'}\right\} &= \operatorname{Re}\left(\frac{1 - \sum\limits_{n=2}^{\infty} n(n-1)b_n z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} n b_n z^{n-1}}\right) \\
&= \operatorname{Re}\left(\frac{1 - \sum\limits_{n=2}^{\infty} n(n-1)\,|b_n|\, e^{i(n-1)\theta} z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} n\,|b_n|\, e^{i(n-1)\theta} z^{n-1}}\right) > \alpha \qquad (z \in \mathcal{U}).
\end{aligned}$$

If we consider a point $z = |z| e^{-i\theta}$, then we have

$$\frac{1 - \sum\limits_{n=2}^{\infty} n(n-1) |b_n| |z|^{n-1}}{1 + |b_1| + \sum\limits_{n=2}^{\infty} n |b_n| |z|^{n-1}} > \alpha.$$

Letting $|z| \to 1^-$, we obtain the inequality (2.2), and this completes the proof. ∎

Remark 2.1. If $f(z) \in \mathcal{H}(\alpha)$; $0 \leq \alpha < \frac{1}{1+|b_1|}$, then the inequality

$$\sum_{n=2}^{\infty} n(n-1+\alpha) |b_n| \leq 1 - \alpha - \alpha |b_1|$$

implies that

$$\sum_{n=2}^{\infty} n(n-1) |b_n|^2 \leq 1 - \alpha - \alpha |b_1| .$$

Applying the above lemmas, we derive the following theorem.

Theorem 2.3. *If* $f(z) \in \mathcal{H}(\alpha)$; $0 \leq \alpha < \frac{1}{1+|b_1|}$ *and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$, *then the function* $\frac{1}{\delta} f(\delta z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$ *for* $0 < |\delta| \leq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\begin{aligned} & |\beta_1| \frac{|\delta|^2 \sqrt{2(1-\alpha-\alpha|b_1|)}}{(1-|\delta|^2)^{3/2}} + |\beta_2| \frac{|\delta|^3 \sqrt{6(1+3|\delta|^2)(1-\alpha-\alpha|b_1|-2|b_2|^2)}}{(1-|\delta|^2)^{5/2}} \\ & |\beta_3| \frac{|\delta|^4 \sqrt{48(1+8|\delta|^2+6|\delta|^4)(1-\alpha-\alpha|b_1|-2|b_2|^2-6|b_3|^2)}}{\left(1-|\delta|^2\right)^{7/2}} \\ = \ & \lambda. \end{aligned}$$

in $0 < |\delta| < 1$.

Proof. Since $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \neq 0$ $(z \in \mathcal{U})$ for $f(z) \in \mathcal{H}(\alpha)$, we see that

$$\frac{z}{\frac{1}{\delta} f(\delta z)} = 1 + \sum_{n=1}^{\infty} \delta^n b_n z^n$$

for $0 < |\delta| < 1$. Thus, to show that $\frac{1}{\delta} f(\delta z) \in \mathcal{P}(\beta_1, \beta_2, \beta_3; \lambda)$, from Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |\delta|^n |b_n| \leq \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^n |b_n| \\
\leq\ & |\beta_1| \left(\sum_{n=2}^{\infty} n(n-1)|\delta|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} n(n-1)|b_n|^2\right)^{\frac{1}{2}} \\
&+ |\beta_2| \left(\sum_{n=3}^{\infty} n(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty} n(n-1)|b_n|^2\right)^{\frac{1}{2}} \\
&+ |\beta_3| \left(\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=4}^{\infty} n(n-1)|b_n|^2\right)^{\frac{1}{2}} \\
\leq\ & |\beta_1| \left(\sum_{n=2}^{\infty} n(n-1)|\delta|^{2n}\right)^{\frac{1}{2}} \sqrt{1-\alpha-\alpha|b_1|} \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
+ \quad & |\beta_2| \left(\sum_{n=3}^{\infty} n(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}} \sqrt{1-\alpha-\alpha|b_1|-2|b_2|^2} \\
+ \quad & |\beta_3| \left(\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}} \sqrt{1-\alpha-\alpha|b_1|-2|b_2|^2-6|b_3|^2}.
\end{aligned}$$

We note that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad (|x|<1),$$

thus, we have

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}. \qquad (2.4)$$

Since

$$\sum_{n=3}^{\infty}(n-2)x^{n-1} = x^2\left(\sum_{n=3}^{\infty}(n-2)x^{n-3}\right) = x^2\left(\sum_{n=3}^{\infty}x^{n-2}\right)' = \frac{x^2}{(1-x)^2},$$

we see that

$$\sum_{n=3}^{\infty}(n-1)(n-2)^2x^n = x^3\left(\frac{x^2}{(1-x)^2}\right)'' = \frac{2x^3+4x^4}{(1-x)^4}.$$

and thus, we obtain

$$\sum_{n=3}^{\infty} n(n-1)(n-2)^2 x^n = \frac{6x^3(1+3x)}{(1-x)^5}. \tag{2.5}$$

Furthermore, we have

$$\begin{aligned}\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n &= x^4\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^{n-4}\right)\\ &= x^4\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1}\right)''',\end{aligned}$$

but

$$\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1} = x^3\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-4}\right) = \frac{2x^3}{(1-x)^3}$$

thus, we have

$$\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n = \frac{12x^4+72x^5+36x^6}{(1-x)^6},$$

which yields

$$\sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2x^n = \frac{48x^4(1+8x+6x^2)}{(1-x)^7}. \tag{2.6}$$

Therefore, from (2.3)- (2.6) with $|\delta|^2 = x$, we obtain

$$\begin{aligned}&\sum_{n=2}^{\infty} n(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|\delta|^n|b_n|\\ \leq\ & |\beta_1|\frac{|\delta|^2\sqrt{2(1-\alpha-\alpha|b_1|)}}{(1-|\delta|^2)^{3/2}} + |\beta_2|\frac{|\delta|^3\sqrt{(6+18|\delta|^2)(1-\alpha-\alpha|b_1|-2|b_2|^2)}}{(1-|\delta|^2)^{5/2}}\\ & |\beta_3|\frac{|\delta|^4\sqrt{48(1+8|\delta|^2+6|\delta|^4)(1-\alpha-\alpha|b_1|-2|b_2|^2-6|b_3|^2)}}{\left(1-|\delta|^2\right)^{7/2}}\end{aligned}$$

Now, let us consider the complex number δ $(0<|\delta|<1)$ such that

$$\begin{aligned}
&|\beta_1| \frac{|\delta|^2 \sqrt{2(1-\alpha)}}{(1-|\delta|^2)^{3/2}} + |\beta_2| \frac{|\delta|^3 \sqrt{6(1+3|\delta|^2)(1-\alpha-2|b_2|^2)}}{(1-|\delta|^2)^{5/2}} \\
&|\beta_3| \frac{|\delta|^4 \sqrt{48(1+8|\delta|^2+6|\delta|^4)(1-\alpha-2|b_2|^2-6|b_3|^2)}}{\left(1-|\delta|^2\right)^{7/2}} \\
= \ & \lambda.
\end{aligned}$$

If we define the function $h(|\delta|)$ by

$$\begin{aligned}
h(|\delta|) \ = \ & |\beta_1| |\delta|^2 (1-|\delta|^2)^2 \sqrt{2(1-\alpha-\alpha|b_1|)} \\
& + |\beta_2| |\delta|^3 \left(1-|\delta|^2\right) \sqrt{(6+18|\delta|^2)(1-\alpha-\alpha|b_1|-2|b_2|^2)} \\
& + |\beta_3| |\delta|^4 \sqrt{48(1+8|\delta|^2+6|\delta|^4)(1-\alpha-\alpha|b_1|-2|b_2|^2-6|b_3|^2)} \\
& - \lambda(1-|\delta|^2)^{7/2},
\end{aligned}$$

then we have
$h(0) = -\lambda < 0$ and $h(1) = |\beta_3| \sqrt{720(1-\alpha-\alpha|b_1|-2|b_2|^2-6|b_3|^2)} > 0$. This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ $(0 < |\delta_0| < 1)$. This completes the proof of the theorem. ∎

Acknowledgement

The authors wishes to thank the referees for his/her useful comments and suggestions.

References

[1] **H. Kobashi, K. Kuroki, H. Shiraishi, and S. Owa**, Radius problems of certain analytic functions, *Internat. J. Open Problems Complex Anal.*, **1**, (2009), 8-12.

[2] **H. Kobashi, K. Kuroki, and S. Owa**, Notes on radius problems of certain univalent functions, *General Mathematics*, **17, No. 4 (2009)**, 5-12.

[3] **H. Kobashi, H. Shiraishi, and S. Owa**, Radius problems of certain starlike functions, *J. Adv. Math. Studies*, **3, No. 2 (2010)**, 57-64.

Basem Aref Frasin
Al al-Bayt University, Faculty of Sciences,
Department of Mathematics
P.O. Box 130095, Mafraq 25113, Jordan
E-mail: bafrasin@yahoo.com

Jin-Lin Liu
Department of Mathematics, Yangzhou University,
Yangzhou, 225002, P.R.China
E-mail: jlliu@yzu.edu.cn

Received: 16.07.2012
Accepted: 4.10.2012