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Convergence of Multi-step Iterates with Errors for Uniformly Quasi-Lipschitzian Mappings

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Abstract. The aim of this paper is to give some necessary and sufficient conditions for multi-step iterative scheme with errors for a finite family of uniformly quasi-Lipschitzian mappings to converge to common fixed point in a real Banach space. Our results extend and improve the corresponding results of Quan [7], Liu [5,6], Xu and Noor [9] and many others.

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1 Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space and C is a nonempty convex subset of E. Let $F(T)$ and $\mathbb{N}$ denote the set of fixed points and the set of natural numbers, respectively. We recall the following definitions:

Definition 1.1. (see [7]) *Let $T\colon C \to C$ be a mapping:*

(1) T is said to be uniformly quasi-Lipschitzian if there exists $L \in [1, +\infty)$, such that

$$\|T^n x - p\| \quad \leq \quad L\|x - p\|, \tag{1.1}$$

for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

(2) T is said to be uniformly L-Lipschitzian if there exists $L \in [1, +\infty)$, such that

$$\|T^n x - T^n y\| \quad \leq \quad L\|x - y\|, \tag{1.2}$$

for all $x, y \in C$, and all $n \in \mathbb{N}$.

(3) T is said to be asymptotically quasi-nonexpansive if there exists $k_n \in [1, +\infty)$ with $\lim_{n\to+\infty} k_n = 1$, such that

$$\|T^n x - p\| \quad \leq \quad k_n\|x - p\|, \tag{1.3}$$

for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

From the above definitions, it follows that if $F(T)$ is nonempty, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian, and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian. But the converse does not hold.

In 1973, Petryshyn and Williamson in [4] proved a sufficient and necessary condition for Picard iterative sequences and Mann [3] iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [1] extended the result of [4] and gave a sufficient and necessary condition for Ishikawa [2] iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 2001, Liu [5,6] extended the above results and obtained some sufficient and necessary conditions for Ishikawa iterative sequences with errors members for asymptotically quasi-nonexpansive mappings to converge to fixed points.

Recently, in 2006 Quan in [7] gave the sufficient condition for convergence of three-step iterative sequences with errors (TSISE) to converge to fixed point for uniformly quasi-Lipschitzian mappings, he proved the following:

Theorem Q. *Let E be a Banach space and C be a nonempty closed convex subset of E and $T\colon C \to C$ be a uniformly quasi-Lipschitzian mapping with the nonempty fixed point set $F(T)$. For arbitrary $x_1 \in C$, let iterative sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined by:*

$$\begin{aligned} z_n &= (1 - \gamma_n - \nu_n)x_n + \gamma_n T^n x_n + \nu_n u_n, \\ y_n &= (1 - \beta_n - \mu_n)x_n + \beta_n T^n z_n + \mu_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n T^n x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{$TSISE$}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *are bounded sequences in* C *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ *are appropriate sequences in [0,1] with the restrictions that* $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$. *Then* $\{x_n\}$ *converges to a fixed point if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(y, C)$ *denotes the distance of* y *to set* C, *that is,* $d(y, C) = \inf d(y, x), \forall x \in C$.

The aim of this paper is to study convergence of multi-step iterative sequences with error term to converge to common fixed point for uniformly quasi-Lipschitzian mappings and give the sufficient condition for convergence of common fixed point for such maps in Banach spaces. The multi-step iteration scheme with errors defined as follows:

Definition 1.2. *Let* C *be a nonempty convex subset of a normed space* E, *and let* $T_1, T_2 \ldots, T_N \colon C \to C$ *be* N *uniformly quasi-Lipschitzian mappings. For a given* $x_1 \in C$, *and a fixed* $N \in \mathbb{N}$ *(*$\mathbb{N}$ *denote the set of all positive integers), compute the sequence* $\{x_n\}$ *by*

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)},\\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)},\\
x_n^{(3)} &= \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)},\\
&\vdots\\
x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}, \qquad (1.4)
\end{aligned}$$

where$\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ *are appropriate sequences in* $[0, 1]$ *with* $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for each* $i \in \{1, 2, \ldots, N\}$, *and* $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \ldots, \{u_n^{(N)}\}$ *are bounded sequences in* C.

In the sequel we need the following lemma.

Lemma 1.1.([8]; Lemma 1) *Let* $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \ \ \forall n \in N.$$

If $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty}\beta_n < \infty$. *Then*

(i) $\lim_{n\longrightarrow\infty} a_n$ *exists.*

(ii) If $\liminf_{n\longrightarrow\infty} a_n = 0$, *then* $\lim_{n\longrightarrow\infty} a_n = 0$.

2 Main Results

In this section, we prove strong convergence theorems of multi-step iterative sequences with error term to converge to common fixed point for uniformly quasi-Lipschitzian mappings in the framework of Banach spaces.

Lemma 2.1. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N \colon C \to C$ be N uniformly quasi-Lipschitzian mappings. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) with the restrictions $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$ where $\theta_n = (L^N - 1)$. Then*

(i) $\|x_{n+1} - x^*\| = \left\|x_n^{(N)} - x^*\right\| \leq (1 + \theta_n)\|x_n - x^*\| + d_n^{(N-1)}$, *for all $n \geq 1$, $x^* \in \mathcal{F}$ and nondecreasing sequence $\{d_n^{(i-1)}\}$ for all $i = 1, 2, \dots, N$ of numbers such that $\sum_{n=1}^{\infty} d_n^{(i-1)} < \infty$.*

(ii) There exists a constant $M > 0$ such that $\|x_{n+m} - x^*\| \leq M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{(N-1)}$ for all $n, m \geq 1$ and $x^* \in \mathcal{F}$.

Proof. (i) Let $x^* \in \mathcal{F}$, then from (1.4) we have

$$\begin{aligned}
\left\|x_n^{(1)} - x^*\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - x^*\right\| \\
&\leq \alpha_n^{(1)}\|T_1^n x_n - x^*\| + \beta_n^{(1)}\|x_n - x^*\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x^*\right\| \\
&\leq \alpha_n^{(1)} L\|x_n - x^*\| + \beta_n^{(1)}\|x_n - x^*\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x^*\right\| \\
&\leq (1 - \beta_n^{(1)}) L\|x_n - x^*\| + \beta_n^{(1)} L\|x_n - x^*\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x^*\right\| \\
&\leq L\|x_n - x^*\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x^*\right\| \\
&\leq L\|x_n - x^*\| + d_n^{(0)} \qquad (2.1)
\end{aligned}$$

where $d_n^{(0)} = \gamma_n^{(1)}\left\|u_n^{(1)} - x^*\right\|$. Since $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, then $\sum_{n=1}^{\infty} d_n^{(0)} < \infty$. Next, we note that

$$\begin{aligned}
\left\|x_n^{(2)} - x^*\right\| &= \left\|\alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)} - x^*\right\| \\
&\leq \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - x^*\right\| + \beta_n^{(2)} \|x_n - x^*\| + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq \alpha_n^{(2)} L \left\|x_n^{(1)} - x^*\right\| + \beta_n^{(2)} \|x_n - x^*\| + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq \alpha_n^{(2)} L[L \|x_n - x^*\| + d_n^{(0)}] + \beta_n^{(2)} \|x_n - x^*\| + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq [L^2 \alpha_n^{(2)} + \beta_n^{(2)}] \|x_n - x^*\| + \alpha_n^{(2)} L d_n^{(0)} + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq (\alpha_n^{(2)} + \beta_n^{(2)}) L^2 \|x_n - x^*\| + \alpha_n^{(2)} L d_n^{(0)} + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&= (1 - \gamma_n^{(2)}) L^2 \|x_n - x^*\| + \alpha_n^{(2)} L d_n^{(0)} + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq L^2 \|x_n - x^*\| + \alpha_n^{(2)} L d_n^{(0)} + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\| \\
&\leq L^2 \|x_n - x^*\| + d_n^{(1)} \qquad (2.2)
\end{aligned}$$

where $d_n^{(1)} = \alpha_n^{(2)} L d_n^{(0)} + \gamma_n^{(2)} \left\|u_n^{(2)} - x^*\right\|$. Since $\sum_{n=1}^{\infty} d_n^{(0)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{(2)} < \infty$, and so $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. Similarly, we have

$$\begin{aligned}
\left\|x_n^{(3)} - x^*\right\| &= \left\|\alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)} - x^*\right\| \\
&\leq \alpha_n^{(3)} \left\|T_3^n x_n^{(2)} - x^*\right\| + \beta_n^{(3)} \|x_n - x^*\| + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\| \\
&\leq \alpha_n^{(3)} L \left\|x_n^{(2)} - x^*\right\| + \beta_n^{(3)} \|x_n - x^*\| + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\| \\
&\leq \alpha_n^{(3)} L[L^2 \|x_n - x^*\| + d_n^{(1)}] + \beta_n^{(3)} \|x_n - x^*\| + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\| \\
&\leq [\alpha_n^{(3)} L^3 + \beta_n^{(3)}] \|x_n - x^*\| + \alpha_n^{(3)} L d_n^{(1)} + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\| \\
&\leq (\alpha_n^{(3)} + \beta_n^{(3)}) L^3 \|x_n - x^*\| + \alpha_n^{(3)} L d_n^{(1)} + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\| \\
&= (1 - \gamma_n^{(3)}) L^3 \|x_n - x^*\| + d_n^{(2)} \\
&\leq L^3 \|x_n - x^*\| + d_n^{(2)} \qquad (2.3)
\end{aligned}$$

where $d_n^{(2)} = \alpha_n^{(3)} L d_n^{(1)} + \gamma_n^{(3)} \left\|u_n^{(3)} - x^*\right\|$. Since $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{(3)} < \infty$, thus $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$.

By continuing the above process, there exists a nonnegative real sequence $\{d_n^{(l-1)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(l-1)} < \infty$ and

$$\left\|x_n^{(i)} - x^*\right\| \leq L^i \|x_n - x^*\| + d_n^{(i-1)}, \ \forall n \geq 1, \ \forall i = 1, 2, \ldots, N. \quad (2.4)$$

Thus

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|x_n^{(N)} - x^*\right\| \\
&\leq L^N \|x_n - x^*\| + d_n^{(N-1)} \\
&= (1 + L^N - 1) \|x_n - x^*\| + d_n^{(N-1)} \\
&= (1 + \theta_n) \|x_n - x^*\| + d_n^{(N-1)} \qquad (2.5)
\end{aligned}$$

for all $n \in N$, where $\theta_n = (L^N - 1)$ with $\sum_{n=1}^{\infty} \theta_n < \infty$. This completes the proof of part (i).

(ii) Since $1 + x \leq e^x$ for all $x > 0$. Then from (i) it can be obtained that

$$\begin{aligned}
\|x_{n+m} - x^*\| &\leq (1+\theta_{n+m-1})\|x_{n+m-1} - x^*\| + d_{n+m-1}^{(N-1)} \\
&\leq e^{\theta_{n+m-1}}\|x_{n+m-1} - x^*\| + d_{n+m-1}^{(N-1)} \\
&\leq e^{\theta_{n+m-1}}[e^{\theta_{n+m-2}}\|x_{n+m-2} - x^*\| + d_{n+m-2}^{(N-1)}] + d_{n+m-1}^{(N-1)} \\
&\leq e^{(\theta_{n+m-1}+\theta_{n+m-2})}\|x_{n+m-2} - x^*\| + e^{\theta_{n+m-1}}[d_{n+m-1}^{(N-1)} + d_{n+m-2}^{(N-1)}] \\
&\leq \dots \\
&\leq \dots \\
&\leq \left(e^{\sum_{k=n}^{n+m-1}\theta_k}\right)\|x_n - x^*\| + \left(e^{\sum_{k=n}^{n+m-1}\theta_k}\right)\sum_{k=n}^{n+m-1} d_k^{(N-1)} \qquad (2.6)
\end{aligned}$$

for all $x^* \in \mathcal{F}$ and $n, m \geq 1$. Setting $M = e^{\sum_{k=n}^{n+m-1}\theta_k}$, then $\|x_{n+m} - x^*\| \leq M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{(N-1)}$. This completes the proof of part (ii).

Theorem 2.1. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N \colon C \to C$ be N uniformly quasi-Lipschitzian mappings. Assume that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) and some $a, b \in (0,1)$ with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$;

(iii) $\sum_{n=1}^{\infty} \theta_n < \infty$ where $\theta_n = (L^N - 1)$.

Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point x^* of the mappings $\{T_1, T_2, \dots, T_N\}$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. The necessity is obvious, we only prove the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Then from Lemma 2.1(i), we have $\|x_{n+1} - x^*\| \leq (1+\theta_n)\|x_n - x^*\| + d_n^{(N-1)}$, for all $n \geq 1$. Therefore

$$d(x_{n+1}, \mathcal{F}) \leq (1+\theta_n)d(x_n, \mathcal{F}) + d_n^{(N-1)}. \qquad (2.7)$$

Since $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\sum_{n=1}^{\infty} d_n^{(N-1)} < \infty$, so by Lemma 1.1 and $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we get that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 2.1(ii), we have

$$\|x_{n+m} - x^*\| \leq M \cdot \|x_n - x^*\| + M \cdot \sum_{k=n}^{n+m-1} d_k^{(N-1)} \tag{2.8}$$

for all $x^* \in \mathcal{F}$ and $n, m \geq 1$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, for each $\varepsilon > 0$, there exists a natural number n_1 such that $d(x_n, \mathcal{F}) < \frac{\varepsilon}{6M}$, for all $n \geq n_1$. Hence, there exists $q \in \mathcal{F}$ such that

$$\|x_{n_1} - q\| < \frac{\varepsilon}{3M}, \qquad \sum_{k=n_1}^{n+m-1} d_k^{(N-1)} < \frac{\varepsilon}{3M}. \tag{2.9}$$

From (2.8) and (2.9), for all $n \geq n_1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq M \cdot \|x_{n_1} - q\| + M \cdot \sum_{k=n_1}^{n+m-1} d_k^{(N-1)} + M \cdot \|x_{n_1} - q\| \\ &< M \cdot \frac{\varepsilon}{3M} + M.\frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{3M} \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{2.10}$$

Thus $\{x_n\}$ is a Cauchy sequence in E. By the completeness of E, we also have that $\{x_n\}$ is a convergent sequence. Assume that $\{x_n\}$ converges to a point q^*, that is, $\lim_{n\to\infty} x_n = q^*$. It will be prove that q^* is a common fixed point, that is, $q^* \in \mathcal{F}$.

Since $\lim_{n\to\infty} x_n = q^*$, for each $\hat{\varepsilon} > 0$, there exists a natural number n_2 such that when $n \geq n_2$,

$$\|x_n - q^*\| < \frac{\hat{\varepsilon}}{2(1+L)}. \tag{2.11}$$

Moreover, $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ implies that there exists a natural number $n_3 \geq n_2$, such that when $n \geq n_3$,

$$d(x_n, \mathcal{F}) < \frac{\hat{\varepsilon}}{2(1+L)}, \quad d(x_{n_3}, \mathcal{F}) < \frac{\hat{\varepsilon}}{2(1+L)}. \tag{2.12}$$

Thus there exists a $w^* \in \mathcal{F}$, such that

$$\|x_{n_3} - w^*\| = d(x_{n_3}, w^*) < \frac{\hat{\varepsilon}}{2(1+L)}. \tag{2.13}$$

From (2.12) and (2.13), for any $i \in I$ and $n \geq n_3$, we have

$$\begin{aligned}
\|T_i q^* - q^*\| &= \|T_i q^* - w^* + w^* - x_{n_3} + x_{n_3} - q^*\| \\
&\leq \|T_i q^* - w^*\| + \|w^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\leq L\,\|q^* - w^*\| + \|w^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\leq L[\|q^* - x_{n_3}\| + \|x_{n_3} - w^*\|] + \|w^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\leq (1+L)\,\|x_{n_3} - q^*\| + (1+L)\,\|x_{n_3} - w^*\| \\
&< (1+L) \cdot \frac{\hat{\varepsilon}}{2(1+L)} + (1+L) \cdot \frac{\hat{\varepsilon}}{2(1+L)} \\
&< \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon}.
\end{aligned} \tag{2.14}$$

This implies that $T_i q^* = q^*$. Hence $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. This completes the proof.

Corollary 2.1. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N \colon C \to C$ be N uniformly L-Lipschitzian mappings. Assume that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) and some $a, b \in (0,1)$ with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$;

(iii) $\sum_{n=1}^{\infty} \theta_n < \infty$ where $\theta_n = (L^N - 1)$.

Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point x^* of the mappings $\{T_1, T_2, \dots, T_N\}$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. Since $F(T_i)$ for all $i = 1, 2, \dots, N$ is nonempty, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian. Thus, Corollary 2.1 can be proved by using Theorem 2.1.

Corollary 2.2. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : C \to C$ be N asymptotically quasi-nonexpansive mappings. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) and some $a, b \in (0,1)$ with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$;

(iii) $\sum_{n=1}^{\infty} \theta_n < \infty$ *where* $\theta_n = (L^N - 1)$.

Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point x^* of the mappings $\{T_1, T_2, \ldots, T_N\}$ if and only if

$$\liminf_{n \to \infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. Since $F(T_i)$ for all $i = 1, 2, \ldots, N$ is nonempty, an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian. Thus, Corollary 2.2 can be proved by using Theorem 2.1.

Theorem 2.2. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : C \to C$ be N uniformly quasi-Lipschitzian mappings. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) and some $a, b \in (0,1)$ with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$;

(iii) $\sum_{n=1}^{\infty} \theta_n < \infty$ *where* $\theta_n = (L^N - 1)$.

Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point p of the family of mappings $\{T_1, T_2, \ldots, T_N\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.

Proof. The proof of Theorem 2.2 follows from Lemma 1.1 and Theorem 2.1.

Theorem 2.3. *Let E be a Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N : C \to C$ be N uniformly quasi-Lipschitzian mappings. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ iteratively by (1.4) and some $a, b \in (0,1)$ with the following restrictions:*

(i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$*;*

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$*;*

(iii) $\sum_{n=1}^{\infty} \theta_n < \infty$ *where* $\theta_n = (L^N - 1)$.

Suppose that there exists a map T_j *which satisfies the following conditions:*

(a) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$*;*

(b) there exists a constant $M > 0$ *such that* $\|x_n - T_j x_n\| \geq M d(x_n, \mathcal{F})$, $\forall n \geq 1$.

Then the iterative sequence $\{x_n\}$ *converges strongly to a common fixed point of the family of mappings* $\{T_1, T_2, \ldots, T_N\}$*.*

Proof. From (a) and (b), it follows that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Theorem 2.1, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings $\{T_1, T_2, \ldots, T_N\}$.

Remark 2.1. Our results extend and improve the corresponding results of Petryshyn and Williamson [4], Ghosh and Debnath [1] and Xu and Noor [9] to the case of quasi-nonexpansive and asymptotically nonexpansive mappings to more general class of mappings, multi-step iteration with errors and finite family of mappings.

Remark 2.2. Our results extend and generalize the corresponding results of Liu [5, 6] to the case of multi-step iteration with errors and finite family of mappings.

Remark 2.3. Our results also extend and generalize the corresponding results of Quan [7] to the case of multi-step iteration with errors and finite family of mappings.

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