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Extensional quotient coalgebras

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Abstract. Given an endofunctor F of an arbitrary category, any maximal element of the lattice of congruence relations on an F-coalgebra (A, a) is called a coatomic congruence relation on (A, a). Besides, a coatomic congruence relation K is said to be factor split if the canonical homomorphism $\nu : A_K \to A_{\nabla_A}$ splits, where ∇_A is the largest congruence relation on (A, a). Assuming that F is a covarietor which preserves regular monos, we prove under suitable assumptions on the underlying category that, every quotient coalgebra can be made extensional by taking the regular quotient of an F-coalgebra with respect to a coatomic and not factor split congruence relation or its largest congruence relation.

1 Introduction

The study of coalgebras developed by J. J. M. M. Rutten [15] concerns the particular case of Set-endofunctors. The author develops the theory of universal coalgebras with the assumption that the functors preserve weak pullbacks. This property can see bisimulation equivalences corresponding notions as of congruence relations in universal algebras. In the same context, the largest bisimulation on any coalgebra is again the largest congruence on this coalgebra.

Many theoretical computer science structures, including automata, transition systems, object oriented systems and lazy data types can be modeled with a type functor preserving weak pullbacks. However there are viable examples

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of coalgebras (topological spaces, for instance) whose type functors do not obey such a restriction.

Certainly, the major advantage of coalgebras is that the theory can naturally deal with nondeterminism and undefinedness, concepts which are hard, or even impossible, to treat algebraically.

A universal algebra is called simple if it does not have any nontrivial congruence relation. The notion of simple coalgebra is obtained by applying the same definition. In other words, the largest congruence relation on a simple coalgebra is its diagonal. An extensional coalgebra is a coalgebra on which the largest bisimulation is its diagonal. Assuming the type functor preserves weak pullbacks, every extensional coalgebra is simple (see [4]).

A quotient algebra also called a factor algebra, is obtained by partionning the elements of an algebra into equivalence classes given by a congruence relation, that is an equivalence relation compatible with all the operators of the algebra. This is equivalent to consider the quotient of an algebra with respect to a congruence relation. The quotient algebra A/θ is simple if and only if θ is a maximal congruence on A or θ is the largest congruence relation on A (see [3]).

The purpose of this paper is to give a characterization theorem for extensional quotient coalgebras of an endofunctor, given an arbitrary underlying category. To this end, let F denote an endofunctor of an arbitrary category. Any maximal element of the lattice of congruence relations on an F-coalgebra (A, a) is called a coatomic congruence relation on (A, a). Besides, a coatomic congruence relation K is said to be factor split if the canonical homomorphism $\nu : A_K \to A_{\nabla_A}$ splits, where ∇_A is the largest congruence relation on (A, a). Suppose that the underlying category is regularly well powered, cocomplete, exact and equipped with epi-(regular mono) factorizations. If more, F is a covarietor which preserves regular monos then, every quotient coalgebra can be made extensional by taking the regular quotient of an F-coalgebra with respect to a coatomic and not factor split congruence relation or its largest congruence relation.

2 Basic notions

We recall here some definitions and usual properties for the following sections.

2.1 Factorization systems

They will be often used throughout this paper.

A *factorization system* (F.S) for a category $\mathcal{C}$ consists of a pair $(\mathcal{E},\mathcal{M})$ of classes of morphisms in $\mathcal{C}$ such that:

FS1. $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$ and are closed under composition.

FS2. Every morphism f of $\mathcal{C}$ can be factored as $f = m \circ e$ for some morphisms $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

FS3. For all commutative squares

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, dashed, "w"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a unique arrow w making both triangles commute.

2.2 Subobjects

Unions of regular subobjects are revisited (for more details, see [2]). Their existence allows one to construct pullbacks.

Recall that a regular mono is a morphism in some category which occurs as the equalizer of some parallel pair of morphisms. The dual concept is that of regular epi.

Let A be an object of a category $\mathcal{C}$. Denote by $\mathcal{M}_A$ the class of all regular monos of codomain A. Any member $f : B \to A$ of $\mathcal{M}_A$ is written (B, f). The relation $\leq_A$ defined on $\mathcal{M}_A$ by $(B, f) \leq_A (C, g)$ iff there is $h : B \to C$ such that $f = g \circ h$ is a preorder. This preorder induces an equivalence relation $\sim_A$ in $\mathcal{M}_A$, where $(B, f) \sim_A (C, g)$ iff $(B, f) \leq_A (C, g)$ and $(C, g) \leq_A (B, f)$. Also, the preorder $\leq_A$ in $\mathcal{M}_A$ induces an order, again denoted $\leq_A$, in the quotient class $\overline{\mathcal{M}}_A = \mathcal{M}_A/\sim_A$; more precisely $[(B, f)] \leq_A [(C, g)]$ iff $(B, f) \leq_A (C, g)$. A member of an equivalence class is called a *regular subobject* of A.

Definition 1 *A category $\mathcal{C}$ is said to be regularly well powered, if for each A in $\mathcal{C}$, $\overline{\mathcal{M}}_A$ is a set.*

An equivalence class $[(B, f)]$ will be also denoted by its representative f or simply by the domain B; and in this case one also says that f or B is a regular subobject of A.

Definition 2 *A regular image of a morphism* $f : A \longrightarrow C$ *is a regular mono* $m : B \rightarrowtail C$ *through which* f *factors, which is minimal in the sense that, if* f *factors through any other regular mono* $B' \rightarrowtail C$*, then* B *is a regular subobject of* B'.

Suppose that $\mathcal{C}$ is a regularly well powered category admitting coproducts and epi-(regular mono) factorizations. The regular image of a cospan $(f_\alpha : A_\alpha \longrightarrow A)_\alpha$ in $\mathcal{C}$ is the smallest regular subobject E of A through which each f_α factors; that is, there exists a regular mono $m : E \longrightarrow A$ and an epi sink $(g_\alpha : A_\alpha \longrightarrow E)_\alpha$ such that $(f_\alpha) = m \circ (g_\alpha)$. It is constructed in two steps as follows:

- By the universal property of coproducts, consider the unique morphism $f : \coprod_\alpha A_\alpha \longrightarrow A$ such that $(f_\alpha) = f \circ (\mu_\alpha)$, where $(\mu_\alpha)_\alpha$ is the cospan of structural injections.
- Consider the epi-(regular mono) factorization of f: $\coprod_\alpha A_\alpha \xrightarrow{e} E \overset{m}{\rightarrowtail} A$.

Hence, the collection of morphisms $(e \circ \mu_\alpha : A_\alpha \longrightarrow E)_\alpha$ is an epi sink given that e is an epimorphism and $(\mu_\alpha)_\alpha$ is an epi sink. Particularly, the regular image or *union* of a cospan $(m_\alpha : S_\alpha \rightarrowtail A)_\alpha$ of regular subobjects in $\mathcal{C}$ is their supremum in the ordered set $(\overline{\mathcal{M}}_A, \leq_A)$. It will be denoted $\bigcup_{\alpha\in\lambda} Im(m_\alpha)$.

Definition 3 *In a category with binary products, a binary relation from* A *to* B *is a regular subobject of* $A \times B$*. This is represented by a regular mono* $m : R \rightarrowtail A \times B$ *or equivalently, by a pair of arrows*

$$\begin{tikzcd} & A \\ R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & \\ & B \end{tikzcd}$$

with the property that the induced arrow $\langle r_1, r_2\rangle : R \longrightarrow A \times B$ *is a regular mono. Also,* r_1 *and* r_2 *form a mono source because* $r_1 = p_1 \circ \langle r_1, r_2\rangle$ *and* $r_2 = p_2 \circ \langle r_1, r_2\rangle$ *with* p_1 *and* p_2 *which form a mono source as structural morphisms of the product of* A *and* B*. A relation from* A *to* A *is called a relation on* A.

Binary relations are ordered (as regular subobjects of $A \times A$) and can be composed. The relational composition is defined by applying the standard pullback construction as in the category of sets: given a binary relation R

(represented by $r_1 : R \to A$ and $r_2 : R \to B$) in a finitely complete category with epi-(regular mono) factorizations, form the pullback of r_1 and r_2.

$$\begin{tikzcd}
R \times_A R \arrow[r, "t_1"] \arrow[d, "t_2"'] & R \arrow[r, "r_1"] \arrow[d, "r_2"] & A \\
R \arrow[r, "r_1"'] \arrow[d, "r_2"'] & A & \\
A & &
\end{tikzcd}$$

Factorize $\langle r_1 \circ t_1, r_2 \circ t_2 \rangle : R \times_A R \to A \times A$ as an epimorphism followed by a regular mono, then the latter represents the composite $R \circ R$. R is said to be *transitive* if $R \circ R$ is smaller than R. The relation R is called *reflexive* if the diagonal map $\langle 1_A, 1_A \rangle : A \to A \times A$ factors through it and, *symmetric* if there is an arrow $\tau : R \to R$ such that $r_1 \circ \tau = r_2$ and $r_2 \circ \tau = r_1$. We say that R is an *equivalence relation* if it is reflexive, symmetric and transitive.

Pullbacks are constructed in the presence of unions of regular subobjects as follows.

Proposition 1 *Suppose that $\mathcal{C}$ is a regularly well powered category with coproducts, finite products and admitting epi-(regular mono) factorizations. Then it has pullbacks.*

Proof. Consider a cospan $(A \xrightarrow{f_1} C \xleftarrow{f_2} B)$ in $\mathcal{C}$.

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & \\
& C \\
B \arrow[ur, "f_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "f_1"] & \\
R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & & C \\
& B \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Let us denote by $\mathrm{Rel}(A, B)$ the class of all binary relations R from A to B such that $f_1 \circ r_1 = f_2 \circ r_2$. This class is nonempty as we are going to show. Let 0 be the initial objest of $\mathcal{C}$ (the coproduct in $\mathcal{C}$ over the empty index set). The canonical arrow $! : 0 \to A \times B$ factorizes through a regular subobject $0'$ of $A \times B$, which is a member of $\mathrm{Rel}(A, B)$. Since the category $\mathcal{C}$ is regularly well powered, the class $\mathrm{Rel}(A, B)$ is a set. Let R denote again its supremum (the union); this supremum exists since $\mathcal{C}$ has coproducts. Denote by $u \colon R \rightarrowtail A \times B$ the regular mono making R a binary relation.

Consider a span $(Q, (g_i)_{i=1,2})$ such that $f_1 \circ g_1 = f_2 \circ g_2$. By the universal property of products, there is a unique arrow $g : Q \longrightarrow A \times B$ such that

$p_1 \circ g = g_1$ and $p_2 \circ g = g_2$; p_1 and p_2 being the structural morphisms of the product of A and B. Factorize g as an epimorphism followed by a regular mono:

$$\begin{tikzcd} Q \arrow[rr, "g"] \arrow[dr, "e_g"'] & & A \times B \\ & W \arrow[ur, "m_g"'] & \end{tikzcd}$$

Then $(W, (h_i)_{i=1,2})$, with $h_i = p_i \circ m_g$, is a binary relation from A to B such that $f_1 \circ h_1 = f_2 \circ h_2$. Hence, there is an arrow $s \colon W \to R$ such that $m_g = u \circ s$. As a result, $g_i = p_i \circ g = p_i \circ m_g \circ e_g = p_i \circ u \circ s \circ g = r_i \circ s \circ e_g$ with $r_i = p_i \circ u$; $i = 1, 2$. This implies that for any arrow $j : Q \to R$ such that $r_1 \circ j = g_1$ and $r_2 \circ j = g_2$, we have $r_1 \circ j = r_1 \circ (s \circ e_g)$ and $r_2 \circ j = r_2 \circ (s \circ e_g)$. Thereafter $s \circ e_g = j$ since the pair (r_1, r_2) is a mono source. Consequently, $s \circ e_g$ is the unique arrow from Q to R such that $r_1 \circ (s \circ e_g) = g_1$ and $r_2 \circ (s \circ e_g) = g_2$. This proves that R together with arrows $r_1 = p_1 \circ u$ and $r_2 = p_2 \circ u$ is the pullback of the cospan $(A \xrightarrow{f_1} C \xleftarrow{f_2} B)$. □

Under Proposition 1, the category $\mathcal{C}$ is finitely complete; this is because it has finite products and pullbacks (see [14]).

2.3 Exact sequences

Set, the category of sets and mappings has *exact sequences*; this means that every equivalence relation is a kernel pair of its coequalizer. In other words, there is a ono-to-one correspondence between equivalence relations and regular quotients.

Replacing Set by a finitely complete category $\mathcal{C}$ with coequalizers, an exact sequence in $\mathcal{C}$ is a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "e"] & B \end{tikzcd}$$

where R is the kernel pair of e and e is the coequalizer of the parallel pair (r_1, r_2). The category $\mathcal{C}$ is said to have exact sequences if every equivalence relation in $\mathcal{C}$ is the kernel pair of its coequalizer. Every topos has exact sequences (see [7]).

A category $\mathcal{C}$ will be called *regular* if every finite diagram has a limit, if every parallel pair of morphisms has a coequalizer and if regular epis are stable under pullbacks. A regular category with exact sequences is called *exact.*

2.4 Kleisli categories

Only monads on Set will be considered.

A *monad* on Set consists of a Set-endofunctor T together with

• a *unit* natural transformation $\eta : \mathrm{id} \Rightarrow T$; that is, a function $\eta_X : X \to TX$ for each set X satisfying a suitable naturality condition; and

• a *multiplication* natural transformation $\mu : T^2 \Rightarrow T$, consisting of functions $\mu_X : T^2X \to TX$ with X ranging over sets.

The unit and multiplication are required to satisfy the following compatibility conditions.

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[dr, "\mathrm{id}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "T\eta_X"'] \arrow[dl, "\mathrm{id}"] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

The powerset functor $\mathcal{P}$ is a monad with a unit given by singletons and a multiplication given by unions. Every adjunction gives rise to a monad (see [10]).

Given any monad T, its *Kleisli category* $\mathcal{K}l(T)$ is defined as follows. Its objects are the objects of the base category, hence sets in our consideration. An arrow $X \to Y$ in $\mathcal{K}l(T)$ is the same thing as an arrow $X \to TX$. Identities and composition of arrows are defined using the unit and the multiplication of T. Moreover, there is a canonical adjunction $J \dashv H$, where the functor $J : \mathsf{Set} \to \mathcal{K}l(T)$ carries a mapping $f : X \to Y$ to $\eta_Y \circ f : X \to TY$ in $\mathcal{K}l(T)$ (see [10]). For instance, the Kleisli category $\mathcal{K}l(\mathcal{P})$ of the powerset monad is up to isomorphism the category **Rel** of sets and binary relations (see [5]).

A functor $\bar{F} : \mathcal{K}l(T) \to \mathcal{K}l(T)$ is said to be a *lifting* of a Set-endofunctor F if the following diagram commutes.

$$\begin{tikzcd} \mathcal{K}l(T) \arrow[r, "\bar{F}"] & \mathcal{K}l(T) \\ \mathsf{Set} \arrow[u, "J"] \arrow[r, "F"'] & \mathsf{Set} \arrow[u, "J"'] \end{tikzcd}$$

A lifting $\bar{F}$ of a Set-endofunctor F is in bijective with a *distributive law* $\lambda : FT \Rightarrow TF$ (see [12]).

3 Coalgebras of an endofunctor

Let F be an endofunctor of a category $\mathcal{C}$. An F-*coalgebra* or a *coalgebra of type* F is a pair (A, α) consisting of an object A in $\mathcal{C}$ together with a $\mathcal{C}$-morphism

$\mathsf{a} : \mathsf{A} \to \mathsf{FA}$. A is called the *carrier* or the *underlying object* and the arrow a the *coalgebra structure* of (A, a).

Given F-coalgebras (A, a) and (B, b), the arrow $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ in $\mathcal{C}$ is called an F-*morphism*, if the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "b"'] & FB \end{tikzcd}$$

It is straightforward to check that the F-morphisms are stable under composition. We write $\mathcal{C}_{\mathsf{F}}$ the category of F-coalgebras and their homomorphisms.

Throughout all that follows, unless otherwise stated,

- $\mathcal{C}$ is a regularly well powered category equipped with epi-(regular mono) factorizations and admitting products;
- F denotes an endofunctor of $\mathcal{C}$.

3.1 Congruences

Definition 4 *Let* (A, a) *and* (B, b) *be* F-*coalgebras. A binary relation* K *from* A *to* B *is a precongruence if for every cospan* $(\mathsf{A} \xrightarrow{i} \mathsf{Z} \xleftarrow{j} \mathsf{B})$,

if $\begin{tikzcd}[row sep=small] & A \arrow[dr, "i"] & \\ K \arrow[ur] \arrow[dr] & & Z \\ & B \arrow[ur, "j"'] & \end{tikzcd}$ *commutes then so does* $\begin{tikzcd}[row sep=small] & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\ K \arrow[ur] \arrow[dr] & & & FZ \\ & B \arrow[r, "b"'] & FB \arrow[ur, "Fj"'] & \end{tikzcd}$

A congruence relation is a precongruence which is an equivalence relation.

Consider a Set-endofunctor F that preserves weak pullbacks. There exists a distributive law $\lambda : \mathsf{F}\mathcal{P} \Rightarrow \mathcal{P}\mathsf{F}$ given by

$$\lambda_{\mathsf{X}}(u) = \{v \in \mathsf{FX} : (v, u) \in \mathsf{Rel}_{\mathsf{F}}(\epsilon_{\mathsf{X}})\}$$

where $u \in \mathsf{F}\mathcal{P}\mathsf{X}$ and $\mathsf{Rel}_{\mathsf{F}}(\epsilon_{\mathsf{X}}) \subseteq \mathsf{FX} \times \mathsf{F}\mathcal{P}\mathsf{X}$ is the F-relation lifting of the membership relation ϵ_{X} (see [5]). The functor $\bar{\mathsf{F}} : \mathbf{Rel} \to \mathbf{Rel}$ induced by this distributive law carries and arrow $\mathsf{R} : \mathsf{X} \to \mathsf{Y}$ in $\mathcal{K}l(\mathcal{P})$ which is a binary relation from X to Y to its F-relation lifting $\mathsf{Rel}_{\mathsf{F}}(\mathsf{R})$. That is, $\bar{\mathsf{F}}\mathsf{R} = \mathsf{Rel}_{\mathsf{F}}(\mathsf{R}) : \mathsf{FX} \to \mathsf{FY}$ in $\mathcal{K}l(\mathcal{P}) \cong \mathbf{Rel}$.

Given $\bar{F}$-coalgebras (A, a) and (B, a). Let $K : A \to B$ be an $\bar{F}$-morphism. The following diagram commutes as $\bar{F}$ and F coincide on objects.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "K"'] & FA \arrow[d, "\bar{F}K"] \\ B \arrow[r, "b"'] & FB \end{tikzcd}$$

Also, for every cospan $(A \xrightarrow{i} Z \xleftarrow{j} B)$, if $j \circ K = i$ then $\bar{F}(j) \circ \bar{F}K = \bar{F}(i)$; hence $\bar{F}(j) \circ b \circ K = \bar{F}(j) \circ \bar{F}K \circ a = \bar{F}(i) \circ a$. This results the commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dd, "K"'] & FA \arrow[dr, "\bar{F}i"] & \\ & & FZ \\ B \arrow[r, "b"'] & FB \arrow[ur, "\bar{F}j"'] & \end{tikzcd}$$

Then K is a precongruence. Consequently, any $\bar{F}$-morphism is a precongruence.

Proposition 2 *Assume the category $\mathcal{C}$ has colimits. Congruence relations on an F-coalgebra (A, a) form a sup-complete lattice denoted $\boldsymbol{Con}(A, a)$. The supremum is given by*

$$\bigvee_{\alpha \in \Lambda} K_\alpha = [\cup\{\mathrm{Im}(m_\alpha : K_\alpha \to A \times A); \alpha \in \Lambda\}]^*$$

the smallest congruence relation greater than the union of all m_α.

Proof. Let $(m_\alpha : K_\alpha \to A \times A)_{\alpha \in \Lambda}$ be a nonempty family of congruences on an F-coalgebra (A, a) with projections k_1^α and k_2^α given $\alpha \in \Lambda$. Since the category $\mathcal{C}$ is regularly well powered, this family of regular subobjects of $A \times A$ is a set. Its supremum K exists therefore in $\mathcal{C}$. This is equivalent to consider a regular mono $m : K \to A \times A$ and an epi sink $(e_\alpha)_\alpha$ such that $(m_\alpha) = m \circ (e_\alpha)$. Furthermore, the category $\mathcal{C}$ has pullbacks under Proposition 1. Given $(A \xrightarrow{u} B \xleftarrow{v} A)$ the pushout of the projections k_1 and k_2 of K. Denote by $\mathrm{Pb}(u, v)$ the pullback of u and v. There is a unique arrow $s : K \to \mathrm{Pb}(u, v)$ such that $w_1 \circ s = k_1$ and $w_2 \circ s = k_2$; w_1 and w_2 being the structural morphisms of $\mathrm{Pb}(u, v)$. Consider a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dr, "i"] & \\ \mathrm{Pb}(u,v) \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & Z \\ & A \arrow[ur, "j"'] & \end{tikzcd}$$

Then $i \circ k_1 = i \circ w_1 \circ s = j \circ w_2 \circ s = j \circ k_2$. By the universal property of pushouts, there is a unique arrow $w : B \to Z$ such that $w \circ u = i$ and $w \circ v = j$. For each $\alpha \in \Lambda$, $i \circ k_1^\alpha = j \circ k_2^\alpha$; this follows from the fact that $k_1^\alpha = k_1 \circ e_\alpha$ and $k_2^\alpha = k_2 \circ e_\alpha$. Hence $F(i) \circ a \circ k_1 \circ e_\alpha = F(j) \circ a \circ k_2 \circ e_\alpha$, because K_α is a precongruence. The equality $F(i) \circ a \circ k_1 = F(j) \circ a \circ k_2$ due to the collection $(e_\alpha)_\alpha$ is an epi sink. Particularly, the equality $F(u) \circ a \circ k_1 = F(v) \circ a \circ k_2$ holds. There is therefore a unique arrow $b : B \to FB$ turning u and v into F-morphisms. So, we have $F(i) \circ a \circ w_1 = F(w \circ u) \circ a \circ w_1 = F(w) \circ F(u) \circ a \circ w_1 = F(w) \circ b \circ u \circ w_1 = F(w) \circ b \circ v \circ w_2 = F(w) \circ F(v) \circ a \circ w_2 = F(w \circ v) \circ a \circ w_2 = F(j) \circ a \circ w_2$. This proves that the following diagram commutes.

$$\begin{tikzcd}
 & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\
Pb(u,v) \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & & FZ \\
 & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] &
\end{tikzcd}$$

Thus $Pb(u,v)$ is a precongruence. Besides $u = v$ given that K is reflexive. That is, u is the coequalizer of the two projections k_1 and k_2. Consequently, $Pb(u,v)$ is an equivalence relation as the kernel pair of a regular mono. Hence $Pb(u,v)$ is a congruence relation on (A,a). It is easy to check that this is in fact the supremum of the family $(m_\alpha : K_\alpha \to A \times A)_{\alpha \in \Lambda}$.

The supremum of a family of congruences on an F-coalgebra (A,a) indexed over the empty set is $\Delta_A = \ker(1_A)$. It is the smallest congruence on (A,a). □

Write ∇_A to denote the largest congruence relation on (A,a).

Proposition 3 *Suppose that the category $\mathcal{C}$ is exact with colimits. For every* F*-coalgebra* (B,b)*, there is at most one* F*-morphism* $\varphi : (B,b) \to A_{\nabla_A}$.

Proof. By Proposition 1, the category $\mathcal{C}$ has pullbacks. Let us prove that there is at most one F-morphism with codomain A_{∇_A}. Assume there are two different F-morphisms $\varphi_1, \varphi_2 : (B,b) \to A_{\nabla_A}$. Let $\psi : A_{\nabla_A} \to C$ be their coequalizer. Then $\ker(\psi \circ \pi_{\nabla_A})$ is an equivalence relation on A, where $\pi_{\nabla_A} : A \to A_{\nabla_A}$ is the coequalizer of ∇_A. In addition, $\ker(\psi \circ \pi_{\nabla_A})$ is a precongruence. Indeed, consider a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes; t_1 and t_2 being the projections of $\ker(\psi \circ \pi_{\nabla_A})$.

$$\begin{tikzcd}
 & A \arrow[dr, "i"] & \\
\ker(\psi \circ \pi_{\nabla_A}) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & Z \\
 & A \arrow[ur, "j"'] &
\end{tikzcd}$$

Then $i = j$ given that $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ is a reflexive relation on A. Also, $\psi \circ \pi_{\nabla_A}$ is the coequalizer of $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ as the category $\mathcal{C}$ is regular. By the universal property of coequalizers, there is a unique arrow $u : C \to Z$ such that $u \circ (\psi \circ \pi_{\nabla_A}) = i$. Furthermore, π_{∇_A} is an F-morphism; this follows from the fact that ∇_A is a congruence relation. Hence $\psi \circ \pi_{\nabla_A}$ is an F-morphism. This implies that $F(i) \circ a \circ t_1 = F(j) \circ a \circ t_2$; that is, the following diagram commutes.

$$\begin{tikzcd}
 & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\
\mathsf{ker}(\psi \circ \pi_{\nabla_A}) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & & FZ \\
 & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] &
\end{tikzcd}$$

So, $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ is a congruence relation on (A, a). Under condition that the category $\mathcal{C}$ has exact sequences, ∇_A is the kernel pair of π_{∇_A}. Consequently, ∇_A is properly smaller than $\mathsf{ker}(\psi \circ \pi_{\nabla_A})$ because φ_1 and φ_2 are different. This contradicts the fact that ∇_A is the largest congruence relation on (A, a). $\square$

Any maximal element of the lattice of congruence relations on (A, a) is called a *coatomic* congruence relation on (A, a).

3.2 Bisimulations

In the coalgebraic context, there are four notions of bisimulation that generalize the standard notion of bisimulation for labelled transition systems (i.e., coalgebras of the Set-endofunctor $\mathcal{P}(L \times (-))$), due to Milner [11] and Park [13]. Further, the four notions are related under certain conditions (see [16]). The definition we adopt here is a simplification of the bisimulation of Hermida and Jacobs [6].

Definition 5 *For any relation R from A to B in $\mathcal{C}$, we define the relation $\bar{F}R$ from FA to FB to be the regular image of the composite morphism $FR \to F(A \times B) \to FA \times FB$.*

A bisimulation between F-coalgebras (A, a) and (B, b) is a binary relation R from A to B such that there is a morphism $R \to \bar{F}R$ making the following diagram commute.

$$\begin{tikzcd}
A \arrow[d, "a"'] & R \arrow[l] \arrow[r] \arrow[d, dashed] & B \arrow[d, "b"] \\
FA & \bar{F}R \arrow[l] \arrow[r] & FB
\end{tikzcd}$$

A bisimulation on (A, a) *is a bisimulation between* (A, a) *and* (A, a)*. Any bisimulation on* (A, a) *which is an equivalence relation is called a bisimulation equivalence.*

Proposition 4 *Suppose that the category* $\mathcal{C}$ *has coproducts and the endofunctor* F *preserves regular monos. Then the union of any collection of bisimulations is a bisimulation.*

Proof. Given F-coalgebras (A, a) and (B, b). The class $(R_t)_{t\in T}$ of bisimulations between (A, a) and (B, b) is nonempty, since the category $\mathcal{C}$ has coproducts. Also, this class is a set because $\mathcal{C}$ is regularly well powered. Denote by $(\coprod_{t\in T} R_t, (\sigma_t)_{t\in T})$ the coproduct of R_t's. Each R_t is a regular subobject of $A \times B$ represented by a regular mono $m_t : R_t \to A \times B$. Let $u : \coprod_{t\in T} R_t \to A \times B$ be the unique arrow such that $u \circ \sigma_t = m_t$, for all $t \in T$. Denote by $\bar{F} \coprod_{t\in T} R_t$ the regular image of the composite morphism $F \coprod_{t\in T} R_t \to F(A \times B) \to FA \times FB$. The following diagram commutes.

$$\begin{tikzcd}
FR_t \arrow[r, "\bar{e}_t"] \arrow[d, "F(\sigma_t)"'] & \bar{F}R_t \arrow[ddl, dashed, "d"] \arrow[dd, "\bar{m}_t"] \\
F\coprod_{t\in T} R_t \arrow[d, "\bar{v}"'] & \\
\bar{F}\coprod_{t\in T} R_t \arrow[r, "\bar{u}"'] & FA \times FB
\end{tikzcd}$$

Under condition that the category $\mathcal{C}$ has epi-(regular mono) factorizations, there is a unique arrow $d : \bar{F}R_t \to \bar{F} \coprod_{t\in T} R_t$ making both triangles commute. By the universal property of coproducts, there is a unique arrow $\rho : \coprod_{t\in T} R_t \to \bar{F} \coprod_{t\in T} R_t$ such that $\rho \circ \sigma_t = d \circ r_t$, for all $t \in T$.

Factorize u as an epimorphism e followed by a regular mono $m : R \rightarrowtail A \times B$. Consider $m_1 : R_1 \rightarrowtail A$ and $m_2 : R_2 \rightarrowtail B$ the respective regular images of the morphisms $p_1 \circ u$ and $p_2 \circ u$; p_1 and p_2 being structural morphisms of the product of A and B. Then $(p_i \circ m) \circ e = m_i \circ e_i$; $i = 1, 2$. Hence, there is a unique arrow $w_i : R \to R_i$ such that $m_i \circ w_i = p_i \circ m$ and $w_i \circ e = e_i$. The morphisms w_1 and w_2 induce a unique arrow $\langle w_1, w_2\rangle : R \to R_1 \times R_2$ such that $v_1 \circ \langle w_1, w_2\rangle = w_1$ and $v_2 \circ \langle w_1, w_2\rangle = w_2$; v_1 and v_2 being the structural morphisms of the product of R_1 and R_2. Let $s : R_1 \times R_2 \to FA \times FB$ be the unique arrow such that $h_1 \circ s = a \circ m_1 \circ v_1$ and $h_2 \circ s = b \circ m_2 \circ v_2$; h_1 and h_2 being the structural morphisms of the product of FA and FB. In addition, consider the unique arrow $k : F(A \times B) \to FA \times FB$ such that $h_1 \circ k = Fp_1$

and $h_2 \circ k = Fp_2$. So, $\bar{u} \circ \bar{v} = k \circ F(u) = k \circ F(m) \circ F(e) = \bar{m} \circ \bar{e} \circ F(e)$. Because $\bar{v}$ is an epimorphism and $\bar{m}$ is a regular mono, there is a unique arrow $z : \bar{F} \coprod_{t \in T} R_t \to \bar{F}R$ such that $\bar{m} \circ z = \bar{u}$ and $z \circ \bar{v} = \bar{e} \circ F(e)$. Likewise, $F(m_i) \circ F(e_i) = F(m_i \circ e_i) = F(p_i \circ u) = F(p_i) \circ F(u) = h_i \circ k \circ F(u) = h_i \circ \bar{u} \circ \bar{v}$; $i = 1, 2$. Since the endofunctor F preserves regular monos, there is a unique arrow $c_i : \bar{F} \coprod_{t \in T} R_t \to FR_i$ such that $F(m_i) \circ c_i = h_i \circ \bar{u}$ and $c_i \circ \bar{v} = F(e_i)$; $i = 1, 2$. It follows that for all $t \in T$, $F(m_1) \circ c_1 \circ \rho \circ \sigma_t = h_1 \circ \bar{u} \circ \rho \circ \sigma_t = h_1 \circ \bar{u} \circ d \circ r_t = h_1 \circ \bar{m}_t \circ r_t = a \circ p_1 \circ m_t = a \circ p_1 \circ u \circ \sigma_t$ and $F(m_2) \circ c_2 \circ \rho \circ \sigma_t = h_2 \circ \bar{u} \circ \rho \circ \sigma_t = h_2 \circ \bar{u} \circ d \circ r_t = h_2 \circ \bar{m}_t \circ r_t = b \circ p_2 \circ m_t = b \circ p_2 \circ u \circ \sigma_t$. Whence $F(m_1) \circ c_1 \circ \rho = a \circ m_1 \circ e_1$ and $F(m_2) \circ c_2 \circ \rho = b \circ m_2 \circ e_2$ as the cospan $(\sigma_t)_{t \in T}$ is an epi sink.

These equalities are used to establish the following commutative diagram.

$$
\begin{tikzcd}
\coprod_{t \in T} R_t \arrow[r, "e"] \arrow[d, "\rho"'] & R \arrow[d, "\langle w_1, w_2 \rangle"] \arrow[ddl, dashed, "r"] \\
\bar{F} \coprod_{t \in T} R_t \arrow[d, "z"'] & R_1 \times R_2 \arrow[d, "s"] \\
\bar{F}R \arrow[r, "\bar{m}"'] & FA \times FB
\end{tikzcd}
$$

Indeed, we have $h_1 \circ s \circ \langle w_1, w_2 \rangle \circ e = a \circ m_1 \circ v_1 \circ \langle w_1, w_2 \rangle \circ e = a \circ m_1 \circ w_1 \circ e = a \circ m_1 \circ e_1 = F(m_1) \circ c_1 \circ \rho = h_1 \circ \bar{u} \circ \rho = h_1 \circ \bar{m} \circ z \circ \rho$ and $h_2 \circ s \circ \langle w_1, w_2 \rangle \circ e = b \circ m_2 \circ v_2 \circ \langle w_1, w_2 \rangle \circ e = b \circ m_2 \circ w_2 \circ e = b \circ m_2 \circ e_2 = F(m_2) \circ c_2 \circ \rho = h_2 \circ \bar{u} \circ \rho = h_2 \circ \bar{m} \circ z \circ \rho$. Since the pair (h_1, h_2) is a mono source, the equality $s \circ \langle w_1, w_2 \rangle \circ e = \bar{m} \circ z \circ \rho$ holds. Consequently, there is a unique arrow $r : R \to \bar{F}R$ making both triangles commute. Furthermore, we have that $(h_1 \circ \bar{m}) \circ r = h_1 \circ s \circ \langle w_1, w_2 \rangle = a \circ m_1 \circ v_1 \circ \langle w_1, w_2 \rangle = a \circ m_1 \circ w_1 = a \circ (p_1 \circ m)$ and $(h_2 \circ \bar{m}) \circ r = h_2 \circ s \circ \langle w_1, w_2 \rangle = b \circ m_2 \circ v_2 \circ \langle w_1, w_2 \rangle = b \circ m_2 \circ w_2 = b \circ (p_2 \circ m)$. Subsequently, R is a bisimulation as union of a collection of bisimulations. $\square$

Any bisimulation equivalence is a congruence relation (see [16]). But the converse is not true (see [1]). Now, we are going to investigate the relationship between bisimulations and congruences.

The following fact is a generalization of the H. P. Gumm's result presented in [4].

Proposition 5 *Assume the category $\mathcal{C}$ has colimits and exact sequences. For every bisimulation R on an F-coalgebra (A, a) there is a smallest congruence relation $\langle R \rangle$ greater than R provided that R is reflexive.*

Proof. According to Proposition 1, the category $\mathcal{C}$ has pullbacks. Consider a bisimulation $(R, (r_i)_{i=1,2})$ on an F-coalgebra (A, a). Let $(A \xrightarrow{u} B \xleftarrow{v} A)$ be the pushout of r_1 and r_2. Denote by $Pb(u, v)$ the pullback of u and v. Then $Pb(u, v)$ is a precongruence. Indeed, given a cospan $(A \xrightarrow{i} Z \xleftarrow{j} A)$ such that the following diagram commutes; t_1 and t_2 being the projections of $Pb(u, v)$.

$$\begin{tikzcd} & A \arrow[dr, "i"] & \\ Pb(u,v) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & Z \\ & A \arrow[ur, "j"'] & \end{tikzcd}$$

By the universal property of pullbacks, there is a unique arrow $s : R \to Pb(u, v)$ such that $t_1 \circ s = r_1$ and $t_2 \circ s = r_2$. This implies that $i \circ r_1 = j \circ r_2$. Hence there is a unique arrow $w : B \to Z$ such that $w \circ u = i$ and $w \circ v = j$. In addition, $F(u) \circ a \circ r_1 = F(v) \circ a \circ r_2$ due to R is a precongruence as bisimulation (see [16]). Thus B is equipped with a coalgebra structure turning u and v into F-morphisms. For this reason, the equality $F(i) \circ a \circ t_1 = F(j) \circ a \circ t_2$ holds; that is, the following diagram commutes.

$$\begin{tikzcd} & A \arrow[r, "a"] & FA \arrow[dr, "Fi"] & \\ Pb(u,v) \arrow[ur, "t_1"] \arrow[dr, "t_2"'] & & & FZ \\ & A \arrow[r, "a"'] & FA \arrow[ur, "Fj"'] & \end{tikzcd}$$

Also, $Pb(u, v)$ is an equivalence relation on A because R is a reflexive bisimulation. Finally, $Pb(u, v)$ is a congruence relation on (A, a) which is greater than R. Since the category $\mathcal{C}$ satisfies the exactness property, it is not hard to see that $Pb(u, v)$ is the smallest congruence relation with this property. □

Denote by $\mathbf{R\text{-}Bis}(A, a)$ the ordered set of reflexive bisimulations on (A, a). The Proposition 5 yields a functorial correspondence

$$\begin{array}{rccc} \Diamond_{(A,a)} : & \mathbf{R\text{-}Bis}(A, a) & \longrightarrow & \mathbf{Con}(A, a) \\ & R & \longmapsto & \langle R \rangle \end{array}$$

Otherwise, every congruence relation K on (A, a) is a reflexive relation on A as equivalence relation. Then the diagonal map $\langle 1_A, 1_A \rangle : A \to A \times A$ factors through K. But the diagonal map is a split mono and therefore a regular mono. Also, A is equipped with a bisimulation structure that comes from its coalgebra structure by epi-(regular mono) factorization of the composite morphism $FA \to F(A \times A) \to FA \times FA$. Hence A is a bisimulation on (A, a) smaller than K. If more, the endofunctor F preserves regular monos, there exists under Proposition 4, a largest bisimulation on (A, a) smaller than K, that we denote $\Box_{(A,a)}K$. Since A is a bisimulation on (A, a) smaller than K, the diagonal map factors through $\Box_{(A,a)}K$. So $\Box_{(A,a)}K$ is a reflexive bisimulation on (A, a). This defines a correspondence

$$\begin{array}{rccc} \Box_{(A,a)}: & \mathbf{Con}(A, a) & \longrightarrow & \mathbf{R\text{-}Bis}(A, a) \\ & K & \longmapsto & \Box_{(A,a)}K \end{array}$$

which extends to a functor.
Given a reflexive bisimulation R and a congruence relation K on (A, a), the following are equivalent:

(i) $\langle R \rangle$ is a regular subobject of K.

(ii) R is a regular subobject of $\Box_{(A,a)}K$.

Hence, assuming that F preserves regular monos, the functor $\Diamond_{(A,a)}$ is the left adjoint of the functor $\Box_{(A,a)}$.

Definition 6 *An endofunctor* $F : \mathcal{C} \to \mathcal{C}$ *is called a covarietor, provided that the forgetful functor* $U_F : \mathcal{C}_F \to \mathcal{C}$ *has a right adjoint.*

Given a topos $\mathcal{E}$ with a *natural number object* (see [7]). The endofunctor $M : \mathcal{E} \to \mathcal{E}$ that assigns to each object A in $\mathcal{E}$, the *free monoid* generated by A is a covarietor (see [8]).

The largest bisimulation on (A, a) which is denoted $\sim_A$ is a reflexive bisimulation.

Proposition 6 *Assume the category* $\mathcal{C}$ *has colimits with exact sequences and the endofunctor* F *is a covarietor which preserves regular monos. A nontrivial congruence relation* K *on* (A, a) *is coatomic or* $K = \nabla_A$*, provided that* $\Box_{(A,a)}K = \sim_A$.

Proof. Let K be a nontrivial congruence relation on (A, a), different from ∇_A and satisfying the condition $\Box_{(A,a)}K = \sim_A$. Suppose that there is a congruence relation L on (A, a), greater than K and different from ∇_A. By the universal property of coequalizers, there is a unique factorization $r : A_K \to A_L$ such that $\pi_L = r \circ \pi_K$, where π_K and π_L are respectively the coequalizers of K and L. Under

Proposition 1, the category $\mathcal{C}$ has pullbacks. Then the category $\mathcal{C}_F$ has also pullbacks, since the endofunctor F is a covarietor which preserves regular monos (see [9]). Since the category $\mathcal{C}$ has exact sequences, every equivalence relation in $\mathcal{C}$ is the kernel pair of its coequalizer. Furthermore, the coequalizer of any congruence relation is an F-morphism. Thereafter, the canonical arrow from the kernel pair of π_K in $\mathcal{C}_F$ to $A \times A$ factored through the largest bisimulation on (A, a) smaller than K. Likewise, the canonical arrow from the kernel pair of π_L in $\mathcal{C}_F$ to $A \times A$ factored through the largest bisimulation on (A, a) smaller than L. Besides, $\Box_{(A,a)}L$ is a regular subobject of $\Box_{(A,a)}K$, given that $\Box_{(A,a)}K =\sim_A$. As a consequence, there is a unique arrow $s : A_L \to A_K$ such that $\pi_K = s \circ \pi_L$. Then we get $\pi_K = (s \circ r) \circ \pi_K$; whence $s \circ r = 1_{A_K}$ since π_K is an epi. Thus r is an epi from the fact of the equality $\pi_L = r \circ \pi_K$, and a section; that is an iso. Hence K is coatomic.

On the other hand, $\sim_A$ is the largest bisimulation on (A, a) smaller than ∇_A. $\Box$

In general though $\langle\sim_A\rangle$ does not need to be the largest congruence on (A, a). For illustration, denote by $()^3_2 : \mathsf{Set} \to \mathsf{Set}$ the functor defined on objects as follows: for a set,

$$A^3_2 = \{(a_1, a_2, a_3) \in A^3 / \mid \{a_1, a_2, a_3\} \mid \leq 2\}$$

and for each mapping $f : A \longrightarrow B$,

$$f^3_2(a_1, a_2, a_3) = (f(a_1), f(a_2), f(a_3))$$

Consider the $()^3_2$-coalgebra (A, a) with $A = \{0, 1, 2\}$, $a(0) = (0, 0, 2)$, $a(1) = (1, 1, 2)$ and $a(2) = (1, 2, 2)$. Since the singleton $\{0\}$ can be provided with a $()^3_2$-coalgebra structure, the unique mapping $!_A : A \to \{0\}$ is a $()^3_2$-morphism. Its kernel pair is $A \times A$ and it is not a bisimulation on (A, a). This implies that $A \times A$ is the largest congruence on (A, a). However the largest bisimulation on (A, a) is the diagonal Δ_A. It is easy to check that $\langle\sim_A\rangle = \Delta_A$. Remark that $K = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$ is a coatomic congruence relation on (A, a).

4 Simple and extensional coalgebras

The largest bisimulation on a final coalgebra is its diagonal. Coalgebras which are not final but satisfy this condition are called extensional. They are said to satisfy the weaker condition of simplicity.

Definition 7 *An* F*-coalgebra* (S, s) *is extensional, if* Δ_S *is the largest bisimulation on* (S, s)*.*

The definition of extensionality reformulates the coinduction proof principle:

$$\frac{x \sim x'}{x = x'}$$

This means, in order to prove that two elements x and y are equal it is enough to prove that there exists a bisimulation R under which x is related to y; i.e., $(x, y) \in R$.

A coalgebra is called *simple* if it does not have any nontrivial congruence relation. Obviously every simple coalgebra is extensional, but the converse holds whenever the endofunctor preserves weak pullbacks.

Proposition 7 *For any* F*-coalgebra* (S, s) *the following are equivalent:*

(i) (S, s) *is extensional.*

(ii) *For every* F*-coalgebra* (A, a)*, there is at most one* F*-morphism* $\psi : (A, a) \to (S, s)$*.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that there are two different F-morphisms $\varphi_1, \varphi_2 : (A, a) \to (S, s)$. There is a unique arrow $\varphi : A \to S \times S$ such that $p_1 \circ \varphi = \varphi_1$ and $p_2 \circ \varphi = \varphi_2$, with p_1 and p_2 the structural morphisms of the product of S with itself. The arrow φ factorizes through a regular subobject R of $S \times S$ which is a nontrivial bisimulation on (S, s).

(ii) $\Longrightarrow$ (i). Suppose that Δ_S is not the largest bisimulation on (S, s). There is a bisimulation $(R, (r_i)_{i=1,2})$ on (S, s) with $r_1 \neq r_2$. $\square$

Recall the set $A = \{0, 1, 2\}$ together with the coalgebra structure $a : A \to A^3_2$ such that $a(0) = (0, 0, 2)$, $a(1) = (1, 1, 2)$ and $a(2) = (1, 2, 2)$, where $K = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$ is a coatomic congruence relation on (A, a). In particular, $K \neq \nabla_A = A \times A$, but the $()^3_2$-coalgebra on the quotient set $A_K = \{\bar{0}, \bar{2}\}$ is extensional; this is because the largest bisimulation on A_K is the diagonal Δ_{A_K} (see [4]).

Definition 8 *A coatomic congruence relation* K *on* (S, s) *is called factor split if the canonical homomorphism* $\nu : S_K \to S_{\nabla_S}$ *splits.*

$H = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$ is an equivalence relation on the set $A = \{0, 1, 2\}$. Let π_H denote the canonical projection of A onto $A_H = \{\bar{0}, \bar{1}\}$, the quotient set with respect to H. Provide A with the $()^3_2$-coalgebra structure a such that $a(0) = (0, 0, 2)$, $a(1) = (1, 1, 2)$ and $a(2) = (1, 2, 2)$. Then π_H is a $()^3_2$-morphism given that A_H is equipped with the coalgebra structure $a_H : A_H \to (A_H)^3_2$ defined as $a_H(\bar{0}) = (\bar{0}, \bar{0}, \bar{1})$ and $a_H(\bar{1}) = (\bar{1}, \bar{1}, \bar{1})$. Consequently, H is a congruence relation on (A, a). Also, H is coatomic as a maximal element of the lattice of congruence relations on (A, a). Since $a_H(\bar{1}) = (\bar{1}, \bar{1}, \bar{1})$, the canonical homomorphism $\nu : A_H \to A_{\nabla_A} = \{0\}$ has a right-sided inverse. Hence, H is factor split.

For any coatomic and factor split congruence relation K on (S, s), denote by $\tau : S_{\nabla_S} \to S_K$ the right-sided inverse of the canonical homomorphism $\nu : S_K \to S_{\nabla_S}$. Then $\tau \circ \nu : S_K \to S_K$ and $1_{S_K} : S_K \to S_K$ are two different F-morphisms with codomain S_K. As a result, S_K is not extensional due to Proposition 7.

Lemma 1 *Suppose that the category $\mathcal{C}$ is exact with colimits. For any coatomic congruence relation K on an F-coalgebra (S, s), the quotient coalgebra S_K is extensional provided that K is not factor split.*

Proof. Given K a coatomic and not factor split congruence relation on (S, s). Suppose that the quotient coalgebra S_K is not extensional. Then the largest bisimulation on S_K is nontrivial. By Proposition 3, the canonical homomorphism ν from S_K to S_{∇_S} coequalizes its projections. Let $\varphi : S_K \to C$ denote the coequalizer of the projections of $\sim_{S_K}$, the largest bisimulation on (S, s). There is a unique arrow $t : C \to S_{\nabla_S}$ such that $t \circ \varphi = \nu$. Hence, $\ker(\varphi)$ is properly smaller than $\ker(\nu)$. Also, K is properly smaller than $\ker(\varphi \circ \pi_K)$ and $\ker(\pi_{\nabla_{S_K}} \circ \pi_K)$; π_K and $\pi_{\nabla_{S_K}}$ being respectively the coequalizer of the projections of K and the coequalizer of the projections of ∇_{S_K}, the largest congruence relation on S_K. But, $\ker(\varphi \circ \pi_K)$ and $\ker(\pi_{\nabla_{S_K}} \circ \pi_K)$ are congruence relations on (S, s). Consequently, $\ker(\varphi \circ \pi_K) = \nabla_A = \ker(\pi_{\nabla_{S_K}} \circ \pi_K)$ as K is coatomic. Besides, $\varphi \circ \pi_K$ and $\pi_{\nabla_{S_K}} \circ \pi_K$ are regular epis given that the category $\mathcal{C}$ is regular. This implies that $\varphi \circ \pi_K = \pi_{\nabla_{S_K}} \circ \pi_K$; that is, $\varphi = \pi_{\nabla_{S_K}}$ due to π_K is an epi. Then $\ker(\varphi) = \nabla_{S_K}$ because the category $\mathcal{C}$ has exact sequences. It follows that ∇_{S_K} is properly smaller than $\ker(\nu)$ which is a congruence relation on S_K. This is a contradiction. So, S_K is extensional. □

A quotient coalgebra can be made extensional by taking a regular quotient with respect to a coatomic and not factor split congruence relation or its largest congruence relation as the following states.

Proposition 8 *Assume the category $\mathcal{C}$ is exact with colimits and the endofunctor F is a covarietor which preserves regular monos. For every F-coalgebra*

(S, s) *and a congruence relation* K *on* (S, s)*, the quotient coalgebra* S_K *is extensional if and only if* K *is coatomic and not factor split or* $K = \nabla_S$*.*

Proof. Suppose that S_K is extensional. Let $(Q, (q_i)_{i=1,2})$ be a bisimulation on (S, s). There is a unique arrow $m : Q \to S \times S$ such that $p_1 \circ m = q_1$ and $p_2 \circ m = q_2$; p_1 and p_2 being the structural morphisms of the product of S with itself. Let π_K denote the coequalizer of K. The universal property of products yields a unique arrow $m_K : Q \to S_K \times S_K$ such that $\overline{p}_1 \circ m_K = \pi_K \circ p_1 \circ m$ and $\overline{p}_2 \circ m_K = \pi_K \circ p_2 \circ m$, with $\overline{p}_1$ and $\overline{p}_2$ the structural morphisms of the product of S_K with itself. The arrow m_K admits the epi-(regular mono) factorization

$$\begin{tikzcd}
Q \arrow[rr, "m_K"] \arrow[dr, "u"'] & & S_K \times S_K \\
& R \arrow[ur, "\nu_K"'] &
\end{tikzcd}$$

Let $r_K : S \times S \to S_K \times S_K$ be the unique arrow such that $\overline{p}_1 \circ r_K = \pi_K \circ p_1$ and $\overline{p}_2 \circ r_K = \pi_K \circ p_2$. Then $F(\nu_K) \circ F(u) = F(r_K) \circ F(m)$ due to $\nu_K \circ u = r_K \circ m$. Hence $h \circ F(\nu_K) \circ F(u) = h \circ F(r_K) \circ F(m)$, where $h : F(S_K \times S_K) \to FS_K \times FS_K$ is the unique arrow such that $\bar{t}_1 \circ h = F(\overline{p}_1)$ and $\bar{t}_2 \circ h = F(\overline{p}_2)$; $\bar{t}_1$ and $\bar{t}_2$ being the structural morphisms of the product of FS_K with itself. Furthermore, there is a unique arrow $\pi : FS \times FS \to FS_K \times FS_K$ such that $\bar{t}_1 \circ \pi = F(\pi_K) \circ t_1$ and $\bar{t}_2 \circ \pi = F(\pi_K) \circ t_2$, with t_1 and t_2 the structural morphisms of the product of FS with itself. Given $k : F(S \times S) \to FS \times FS$ the unique arrow such that $t_1 \circ k = F(p_1)$ and $t_2 \circ k = F(p_2)$, we have that $\bar{t}_i \circ \pi \circ k = F(\pi_K) \circ t_i \circ k = F(\pi_K) \circ F(p_i) = F(\overline{p}_i) \circ F(r_K) = \bar{t}_i \circ h \circ F(r_K)$; $i = 1, 2$. The equality $\pi \circ k = h \circ F(r_K)$ arises from the fact that the pair $(\bar{t}_1, \bar{t}_2)$ is a mono source. One deduces the following commutative diagram.

$$\begin{tikzcd}
FQ \arrow[r, "\bar{e}"] \arrow[d, "F(u)"'] & \bar{F}Q \arrow[d, "\bar{m}"] \arrow[ddl, dashed, "w"] \\
FR \arrow[d, "\bar{u}"'] & FS \times FS \arrow[d, "\pi"] \\
\bar{F}R \arrow[r, "\bar{\nu}_K"'] & FS_K \times FS_K
\end{tikzcd}$$

By the axiom FS3, there is a unique arrow $w : \bar{F}Q \to \bar{F}R$ making both triangles commute. In addition, there is a unique arrow $z : S_K \times S_K \to FS_K \times FS_K$ such that $\bar{t}_1 \circ z = s_K \circ \overline{p}_1$ and $\bar{t}_2 \circ z = s_K \circ \overline{p}_2$, where $s_K : S_K \to FS_K$ is the unique arrow turning π_K into an F-morphism. Denote by $q : Q \to \bar{F}Q$ the arrow such that $s \circ p_1 \circ m = t_1 \circ \bar{m} \circ q$ and $s \circ p_2 \circ m = t_2 \circ \bar{m} \circ q$. For $i = 1, 2$; the following holds:

$$\begin{aligned}
\bar{t}_i \circ z \circ \nu_K \circ u &= s_K \circ \overline{p}_i \circ \nu_K \circ u \\
&= s_K \circ \overline{p}_i \circ r_K \circ m \\
&= s_K \circ \pi_K \circ p_i \circ m \\
&= F(\pi_K) \circ s \circ p_i \circ m \\
&= F(\pi_K) \circ t_i \circ \bar{m} \circ q \\
&= \bar{t}_i \circ \pi \circ \bar{m} \circ q \\
&= \bar{t}_i \circ \bar{\nu}_K \circ w \circ q
\end{aligned}$$

Hence, $z \circ \nu_K \circ u = \bar{\nu}_K \circ w \circ q$ because the pair $(\bar{t}_1, \bar{t}_2)$ is a mono source. Since u is an epimorphism and $\bar{\nu}_K$ a regular mono, there is a unique arrow $r : R \longrightarrow \bar{F}R$ such that $\bar{\nu}_K \circ r = z \circ \nu_K$ and $r \circ u = w \circ q$. In fact, R is a bisimulation on S_K. Thus R is a regular subobject of Δ_{S_K} which is the largest bisimulation on S_K. This implies that $\pi_K \circ p_1 \circ m = \pi_K \circ p_2 \circ m$. Consequently Q is a regular subobject of K. Since Q is a bisimulation on (S, s), it is smaller than $\Box_{(S,s)}K$. Whence $\Box_{(S,s)}K$ is the largest bisimulation on (S, s). The Proposition 6 allows to conclude.

Conversely S_K is extensional arising from Propositions 3, 7 and Lemma 1.

□

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