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Existence and uniqueness of solution for a class of nonlinear degenerate elliptic equation in weighted Sobolev spaces

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Abstract. In this work we are interested in the existence and uniqueness of solutions for the Navier problem associated to the degenerate nonlinear elliptic equations

$$\Delta(\nu(x)\,|\Delta u|^{r-2}\Delta u) - \sum_{j=1}^{n} D_j\left[\omega_1(x)\mathcal{A}_j(x,u,\nabla u)\right]$$
$$+\, b(x,u,\nabla u)\,\omega_2(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \quad \text{in } \Omega$$

in the setting of the Weighted Sobolev Spaces.

1 Introduction

In this work we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X = W^{2,r}(\Omega,\nu) \cap W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 4 and Definition 5) for the Navier problem

$$(P)\begin{cases} Lu(x) = f_0(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x), & \text{in } \Omega \\ u(x) = \Delta u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

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where L is the partial differential operator

$$Lu(x) = \Delta(\nu(x)\,|\Delta u|^{r-2}\Delta u) - \sum_{j=1}^{n} D_j\left[\omega_1(x)\mathcal{A}_j(x, u(x), \nabla u(x))\right]$$
$$+\ b(x, u, \nabla u)\,\omega_2(x)$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2 and ν are three weight functions, Δ is the Laplacian operator, $1 < p < \infty$ and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1, \ldots, n)$ and $b : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfy the following assumptions:
(H1) The function $x\mapsto\mathcal{A}_j(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R}\times\mathbb{R}^n$. The function $(\eta, \xi)\mapsto\mathcal{A}_j(x, \eta, \xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H2) there exists a constant $\theta_1 > 0$ such that

$$[\mathcal{A}(x, \eta, \xi) - \mathcal{A}(x, \tilde{\eta}, \tilde{\xi})].(\xi - \tilde{\xi})\geq\theta_1\,|\xi - \tilde{\xi}|^p,$$

whenever $\xi, \tilde{\xi}\in\mathbb{R}^n$, $\xi\neq\tilde{\xi}$, $\mathcal{A}(x, \eta, \xi) = (\mathcal{A}_1(x, \eta, \xi), \ldots, \mathcal{A}_n(x, \eta, \xi))$ (where a dot denote here the Euclidian scalar product in $\mathbb{R}^n$).
(H3) $\mathcal{A}(x, \eta, \xi).\,\xi \geq \lambda_1|\xi|^p + \Lambda_1|\eta|^p - g_1(x)|\eta| - g_2(x)|\xi|$, where λ_1 and Λ_1 are nonnegative constants, $g_1/\omega_2 \in L^{p\,'}(\Omega, \omega_2)$ and $g_2/\omega_1 \in L^{p\,'}(\Omega, \omega_1)$.
(H4) $|\mathcal{A}(x, \eta, \xi)| \leq K_1(x) + h_1(x)|\eta|^{p/p\,'} + h_2(x)|\xi|^{p/p\,'}$, where K_1, h_1 and h_2 are nonegative functions, with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p\,'}(\Omega, \omega_1)$ (with $1/p + 1/p\,' = 1$).
(H5) The function $x\mapsto b(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R}\times\mathbb{R}^n$. The function $(\eta, \xi)\mapsto b(x, \eta, \xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.
(H6) there exists a constant $\theta_2 > 0$ such that

$$[b(x, \eta, \xi) - b(x, \tilde{\eta}, \tilde{\xi})](\eta - \tilde{\eta}) \geq \theta_2\,|\eta - \tilde{\eta}|^p,$$

whenever $\eta, \tilde{\eta}\in\mathbb{R}$, $\eta\neq\tilde{\eta}$.
(H7) $b(x, \eta, \xi)\eta \geq \lambda_2|\xi|^p + \Lambda_2|\eta|^p - g_3(x)\,|\eta| - g_4(x)|\xi|$, where $\lambda_2\geq0$ and $\Lambda_2 > 0$ are constants, $g_3/\omega_2\in L^{p\,'}(\Omega, \omega_2)$ and $g_4\omega_2/\omega_1\in L^{p\,'}(\Omega, \omega_1)$.
(H8) $|b(x, \eta, \xi)| \leq K_2(x) + h_3(x)|\eta|^{p/p\,'} + h_4(x)|\xi|^a$, where K_2, h_3 and h_4 are nonnegative functions, with $K_2 \in L^{p\,'}(\Omega, \omega_2)$, h_3 and $h_4 \in L^\infty(\Omega)$, and $a = (p - 1)/q'$, where $1 < q < \infty$ $(1/q + 1/q\,' = 1)$.
(H9) $\lambda_1 + \lambda_2 > 0$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1, 2, 4, 8, 13]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [10]). These classes have found many useful applications in harmonic analysis (see [12]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), & \text{in} \quad \Omega \\ u(x) = 0, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ (see [7]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), & \text{in} \quad \Omega \\ u(x) = 0, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W^{1,p}_0(\Omega)$ (see [3]), where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [11] and the references therein), and the degenerated p-Laplacian has been studied in [4]. The problem with degenerated p-Laplacian and p-Biharmonic operators

$$\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \text{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \text{div}(G(x)), & \text{in} \quad \Omega \\ u(x) = \Delta u(x) = 0, \quad \text{in} \quad \partial\Omega \end{cases}$$

has been studied by the author in [2].

The following theorem will be proved in section 3.

Theorem 1 *Assume* (H1)-(H9). *If*
(i) $\nu \in A_r$ *and* $\omega_1, \omega_2 \in A_p$ $(1 < p, r, \infty)$, $\omega_1 \le \omega_2$ *a.e.*, $\omega_2/\omega_1 \in L^q(\Omega, \omega_1)$ $(1 < q < \infty)$,
(ii) $f_0/\omega_2 \in L^{p'}(\Omega, \omega_2)$ *and* $f_j/\omega_1 \in L^{p'}(\Omega, \omega_1)$ $(j = 1, \dots, n)$.
Then the problem (P) has a unique solution

$$u \in X = W^{2,r}(\Omega, \nu) \cap W^{1,p}_0(\Omega, \omega_1, \omega_2).$$

2 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [6, 8, 12] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2r)) \leq C\,\mu(B(x;r))$ for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [8]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [12]).

If $\omega \in A_p$, then $\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)}$ whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [8]). Therefore, if $\mu(E) = 0$ then $|E| = 0$.

Definition 1 *Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that*

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [13]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2 *Let $\Omega \subset \mathbb{R}^n$ be open, k be a nonnegative integer and $\omega \in A_p$ $(1 < p < \infty)$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$ for $1 \leq |\alpha| \leq k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by*

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx + \sum_{1\leq|\alpha|\leq k}\int_\Omega |D^\alpha u(x)|^p\,\omega(x)\,dx\right)^{1/p}. \quad (1)$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{k,p}(\Omega,\omega)}$.

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (1) (see Theorem 2.1.4 in [13]). The spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces.

It is evident that the weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), gives nothing new (the space $W_0^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W_0^{k,p}(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish in somewhere $\Omega \cup \partial\Omega$ or increase to infinity (or both).

Definition 3 *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, and let ω_1 and ω_2 be A_p-weights. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ as the set of functions $u \in L^p(\Omega,\omega_2)$ with weak derivatives $D_j u \in L^p(\Omega,\omega_1)$, $j = 1,\dots,n$. The norm of u in $W^{1,p}(\Omega,\omega_1,\omega_2)$ is given by*

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p \omega_2(x)\,dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega_1(x)\,dx\right)^{1/p}. \quad (2)$$

The space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2). The dual space of $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the space

$$\begin{aligned}&[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = W^{-1,p'}(\Omega,\omega_1,\omega_2)\\ &= \{T = f_0 - \operatorname{div} F : F = (f_1,\dots,f_n),\ \frac{f_0}{\omega_2} \in L^{p'}(\Omega,\omega_2),\ \frac{f_j}{\omega_1} \in L^{p'}(\Omega,\omega_1)\}.\end{aligned}$$

In this article we use the following results.

Theorem 2 *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, *μ-a.e. on Ω;*
(ii) $|u_{m_k}(x)| \le \Phi(x)$, *μ-a.e. on Ω;*
(*where* $\mu(E) = \int_E \omega(x)\,dx$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [5]. □

Lemma 1 *Let $1 < p < \infty$.*

(a) *There exists a constant* α_p *such that*

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq \alpha_p \, |x-y|(|x|+|y|)^{p-2}, \ \forall \, x, y \in \mathbb{R}^n;$$

(b) *There exist two positive constants* β_p, γ_p *such that for every* $x, y \in \mathbb{R}^n$

$$\beta_p \, (|x|+|y|)^{p-2}|x-y|^2 \leq (|x|^{p-2}x - |y|^{p-2}y).(x-y) \leq \gamma_p \, (|x|+|y|)^{p-2}|x-y|^2.$$

Proof. See [3], Proposition 17.2 and Proposition 17.3. □

Lemma 2 *If* $\omega \in A_p$, *then* $\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega} \frac{\mu(E)}{\mu(B)}$, *whenever* B *is a ball in* $\mathbb{R}^n$ *and* E *is a measurable subset of B (where* $\mu(E) = \int_E \omega(x)\, dx$*).*

Proof. See Theorem 15.5 *Strong doubling of* A_p*-weights* in [8]. □

By Lemma 2, if $\mu(E) = 0$ then $|E| = 0$.

Definition 4 *We denote by* $X = W^{2,r}(\Omega, \nu) \cap W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *with the norm*

$$\|u\|_X = \|u\|_{L^p(\Omega,\omega_2)} + \|\nabla u\|_{L^p(\Omega,\omega_1)} + \|\Delta u\|_{L^r(\Omega,\nu)}.$$

Definition 5 *We say that an element* $u \in X$ *is a (weak) solution of problem (P) if, for all* $\varphi \in X$,

$$\begin{aligned} &\int_\Omega |\Delta u|^{r-2} \, \Delta u \, \Delta \varphi \, \nu \, dx + \sum_{j=1}^n \int_\Omega \omega_1 \, \mathcal{A}_j(x, u(x), \nabla u(x)) D_j \varphi(x) dx \\ &+ \int_\Omega b(x, u, \nabla u) \varphi \, \omega_2 \, dx \\ &= \int_\Omega f_0(x) \varphi(x) dx + \sum_{j=1}^n \int_\Omega f_j(x) D_j \varphi(x) dx. \end{aligned}$$

3 Proof of Theorem 1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3 *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X. *Then for each* $T \in X^*$ *the equation* $Au = T$ *has a solution* $u \in X$.

Proof. See Theorem 26.A in [15]. □

To prove the existence of solutions, we define $B, B_1, B_2, B_3 : X \times X \to \mathbb{R}$ and $T : X \to \mathbb{R}$ by

$$
\begin{aligned}
B(u,\varphi) &= B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi),\\
B_1(u,\varphi) &= \sum_{j=1}^{n} \int_\Omega \omega_1\, \mathcal{A}_j(x,u,\nabla u) D_j\varphi\, dx = \int_\Omega \omega_1\, \mathcal{A}(x,u,\nabla u).\nabla\varphi\, dx,\\
B_2(u,\varphi) &= \int_\Omega b(x,u,\nabla u)\, \varphi\, \omega_2\, dx,\\
B_3(u,\varphi) &= \int_\Omega |\Delta u|^{r-2}\Delta u\, \Delta\varphi\, \nu\, dx,\\
T(\varphi) &= \int_\Omega f_0(x)\, \varphi(x)\, dx + \sum_{j=1}^{n} \int_\Omega f_j(x)\, D_j\varphi(x)\, dx.
\end{aligned}
$$

Then $u \in X$ is a (weak) solution to problem (P) if for all $\varphi \in X$

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi) = T(\varphi).$$

Step 1. For $j = 1, \dots, n$ we define the operator $F_j : X \to L^{p'}(\Omega,\omega_1)$ by

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that operator F_j is bounded and continuous.

(i) Using (H4) and $\omega_1 \le \omega_2$ we obtain

$$
\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u(x)|^{p'}\omega_1\, dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega_1\, dx\\
&\le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'} \omega_1\, dx\\
&\le C_p \int_\Omega \left[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega_1\right] dx \qquad (3)\\
&\le C_p \left[\int_\Omega K_1^{p'}\, \omega_1\, dx + \int_\Omega h_1^{p'}|u|^p\, \omega_2\, dx\right.\\
&\qquad \left. + \int_\Omega h_2^{p'}|\nabla u|^p \omega_1\, dx\right],
\end{aligned}
$$

where the constant C_p depends only on p. We have,

$$\int_\Omega h_1^{p'}|u|^p\, \omega_2\, dx \le \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^p\, \omega_2\, dx \le \|h_1\|^{p'}_{L^\infty(\Omega)}\, \|u\|^p_X,$$

and

$$\int_\Omega h_2^{p'}|\nabla u|^p\omega_1\,dx \le \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\,\omega_1\,dx \le \|h_2\|^{p'}_{L^\infty(\Omega)}\|u\|^p_X.$$

Therefore, in (3) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega_1)} \le C_p\left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\,\|u\|_X^{p/p'}\right),$$

and hence the boundedness.

(ii) Let $u_m \to u$ in X as $m\to\infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue Dominated Theorem. If $u_m\to u$ in X, then $u_m\to u$ in $L^p(\Omega,\omega_2)$ and $|\nabla u_m|\to|\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and two functions $\Phi_1\in L^p(\Omega,\omega_1)$ and $\Phi_2\in L^p(\Omega,\omega_2)$ such that

$$\begin{aligned}
&u_{m_k}(x)\to u(x),\ \mu_2-\text{a.e. in }\Omega,\\
&|u_{m_k}(x)|\le\Phi_2(x),\ \mu_2-\text{a.e. in }\Omega,\\
&|\nabla u_{m_k}(x)|\to|\nabla u(x)|,\ \mu_1-\text{a.e. in }\Omega,\\
&|\nabla u_{m_k}(x)|\le\Phi_1(x),\ \mu_1-\text{a.e. in }\Omega.
\end{aligned}$$

where $\mu_i=\int_E\omega_i(x)\,dx$ $(i=1,2)$. Hence, using (H4) and $\omega_1\le\omega_2$, we obtain

$$\begin{aligned}
&\|F_ju_{m_k}-F_ju\|^{p'}_{L^{p'}(\Omega,\omega_1)} = \int_\Omega |F_ju_{m_k}(x)-F_ju(x)|^{p'}\omega_1\,dx\\
&=\int_\Omega |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})-\mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega_1\,dx\\
&\le C_p\int_\Omega\left(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'}+|\mathcal{A}_j(x,u,\nabla u)|^{p'}\right)\omega_1\,dx\\
&\le C_p\left[\int_\Omega\left(K_1+h_1|u_{m_k}|^{p/p'}+h_2|\nabla u_{m_k}|^{p/p'}\right)^{p'}\omega_1\,dx\right.\\
&\quad\left.+\int_\Omega\left(K_1+h_1|u|^{p/p'}+h_2|\nabla u|^{p/p'}\right)^{p'}\omega_1\,dx\right]\\
&\le 2\,C_p\int_\Omega\left(K_1+h_1\Phi_2^{p/p'}+h_2\Phi_1^{p/p'}\right)^{p'}\omega_1\,dx\\
&\le 2\,C_p\left[\int_\Omega K_1^{p'}\,\omega_1\,dx+\int_\Omega h_1^{p'}\Phi_2^p\,\omega_1\,dx+\int_\Omega h_2^{p'}\Phi_1^p\,\omega_1\,dx\right]
\end{aligned}$$

$$\leq 2\,C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_2^p\,\omega_2\,dx + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_1^p\,\omega_1\,dx\right]$$
$$\leq 2\,C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\,\|\Phi_2\|^p_{L^p(\Omega,\omega_2)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\|\Phi_1\|^p_{L^p(\Omega,\omega_1)}\right].$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \to 0$, that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega,\omega_1)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [14]), we have

$$F_j u_m \to F_j u \ \text{ in } \ L^{p'}(\Omega,\omega_1). \tag{4}$$

Step 2. Define the operator $G : X \to L^{r'}(\Omega,\nu)$, $(Gu)(x) = |\Delta u(x)|^{r-2}\Delta u(x)$. We also have that the operator G is continuous and bounded. In fact:
(i) We have

$$\begin{aligned}\|Gu\|^{r'}_{L^{r'}(\Omega,\nu)} &= \int_\Omega \big|\,|\Delta u|^{r-2}\,\Delta u\big|^{r'}\,\nu\,dx\\ &= \int_\Omega |\Delta u|^{(r-2)\,r'}|\Delta u|^{r'}\,\nu\,dx = \int_\Omega |\Delta u|^r\,\nu\,dx\\ &\leq \|u\|^r_X.\end{aligned}$$

Hence, $\|Gu\|_{L^{r'}(\Omega,\nu)} \leq \|u\|_X^{r/r'}$.
(ii) If $u_m \to u$ in X then $\Delta u_m \to \Delta u$ in $L^r(\Omega,\nu)$. By Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_3 \in L^r(\Omega,\nu)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x),\ \ \mu_3 - \text{a.e. in } \Omega$$
$$|\Delta u_{m_k}(x)| \leq \Phi_3(x),\ \ \mu_3 - \text{a.e. in } \Omega,$$

where $\mu_3(E) = \int_E \nu(x)\,dx$. Hence, using Lemma 1(a), we obtain, if $r \neq 2$

$$\begin{aligned}\|Gu_{m_k} - Gu\|^{r'}_{L^{r'}(\Omega,\nu)} &= \int_\Omega |\,Gu_{m_k} - Gu|^{r'}\,\nu\,dx\\ &= \int_\Omega \Big|\,|\Delta u_{m_k}|^{r-2}\,\Delta u_{m_k} - |\,\Delta u|^{r-2}\,\Delta u\Big|^{r'}\,\nu\,dx\\ &\leq \int_\Omega \Big[\alpha_r\,|\Delta u_{m_k} - \Delta u|\,(\,|\Delta u_{m_k}| + |\Delta u|)^{(r-2)}\Big]^{r'}\,\nu\,dx\end{aligned}$$

$$\le \alpha_r^{r'} \int_\Omega |\Delta u_{m_k} - \Delta u|^{r'} (2\,\Phi_3)^{(r-2)\,r'}\,\nu\,dx$$
$$\le \alpha_r^{r'}\, 2^{(r-2)r'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^r\,\nu\,dx\right)^{r'/r} \times \left(\int_\Omega \Phi_3^{(r-2)\,r\,r'/(r-r')}\,\nu\,dx\right)^{(r-r')/r}$$
$$\le \alpha_r^{r'}\, 2^{(r-2)\,r'}\, \|u_{m_k} - u\|_X^{r'}\, \|\Phi\|_{L^r(\Omega,\nu)}^{r-r'},$$

since $(r-2)\,r\,r'/(r-r') = r$ if $r \neq 2$. If $r = 2$, we have

$$\|Gu_{m_k} - Gu\|_{L^2(\Omega,\nu)}^2 = \int_\Omega |\Delta u_{m_k} - \Delta u|^2\,\nu\,dx \le \|u_{m_k} - u\|_X^2.$$

Therefore (for $1 < r < \infty$), by the Lebesgue Dominated Convergence Theorem, we obtain $\|Gu_{m_k} - Gu\|_{L^r(\Omega,\nu)} \to 0$, that is, $Gu_{m_k} \to Gu$ in $L^{r'}(\Omega,\nu)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [14]), we have

$$Gu_m \to Gu \text{ in } L^{r'}(\Omega,\nu). \tag{5}$$

Step 3. We define $H : X \to L^{p'}(\Omega,\omega_2)$ by $(Hu)(x) = b(x,u(x),\nabla u(x))$. We also have that the operator H is continuous and bounded. In fact,
(i) Using (H8) and $a = (p-1)/q'$, we obtain

$$\begin{aligned}
\|Hu\|_{L^{p'}(\Omega,\omega_2)}^{p'} &= \int_\Omega |Hu|^{p'}\,\omega_2\,dx = \int_\Omega |b(x,u,\nabla u)|^{p'}\,\omega_2\,dx \\
&\le \int_\Omega \left(K_2 + h_3|u|^{p/p'} + h_4|\nabla u|^a\right)^{p'} \omega_2\,dx \\
&\le C_p \int_\Omega \left[(K_2^{p'} + h_3^{p'}|u|^p + h_4^{p'}|\nabla u|^{a\,p'})\omega_2\right] dx \\
&= C_p \left[\int_\Omega K_2^{p'}\,\omega_2\,dx + \int_\Omega h_3^{p'}|u|^p\,\omega_2\,dx + \int_\Omega h_4^{p'}|\nabla u|^{a\,p'}\omega_2\,dx.\right.
\end{aligned}$$

We have

$$\int_\Omega h_3^{p'}|u|^p\omega_2 \le \|h_3\|_{L^\infty(\Omega)}^{p'} \int_\Omega |u|^p\,\omega_2\,dx \le \|h_3\|_{L^\infty(\Omega)}^{p'}\|u\|_X^p,$$

and

$$\int_\Omega h_4^{p'} |\nabla u|^{a p'}\,\omega_2\,dx \le \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{p/q'}\,\omega_2\,dx$$
$$= \|h_4\|^{p'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^{p/q'}\,\frac{\omega_2}{\omega_1}\,\omega_1\,dx$$
$$\le \|h_4\|^{p'}_{L^\infty(\Omega)} \left(\int_\Omega |\nabla u|^p\,\omega_1\,dx\right)^{1/q'} \left(\int_\Omega \left(\frac{\omega_1}{\omega_2}\right)^q \omega_1\,dx\right)^{1/q}$$
$$\le \|h_4\|^{p'}_{L^\infty(\Omega)} \|u\|_X^{p/q'}\,\|\omega_2/\omega_1\|_{L^q(\Omega,\omega_1)}.$$

Hence,

$$\|Hu\|_{L^{p'}(\Omega,\omega_2)} \le C_p\Big[\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}$$
$$+ \|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1}\|u\|_X^{(p-1)/q'}\Big].$$

(ii) By the same argument used in Step 1 (ii)(and condition (H5)), we obtain analogously, if $u_m \to u$ in X then

$$Hu_m \to Hu, \quad \text{in} \quad L^{p'}(\Omega,\omega_2). \tag{6}$$

Step 4. We also have

$$|T(\varphi)| \le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx$$
$$= \int_\Omega \frac{|f_0|}{\omega_2}|\varphi|\omega_2\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1}|D_j\varphi|\,\omega_1\,dx$$
$$\le \|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)}\|\varphi\|_{L^p(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\|D_j\varphi\|_{L^p(\Omega,\omega_1)}$$
$$\le \left(\|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\varphi\|_X.$$

Moreover, using (H4), (H8) and the Hölder inequality, we also have

$$|B(u,\varphi)| \le |B_1(u,\varphi)| + |B_2(u,\varphi)| + |B_3(u,\varphi)|$$
$$\le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega_1\,dx + \int_\Omega |\Delta u|^{r-2}\,|\Delta u|\,|\Delta\varphi|\,\nu\,dx \tag{7}$$
$$+ \int_\Omega |b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx.$$

In (7) we have

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_1\,dx\\
&\le \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)|\nabla\varphi|\,\omega_1\,dx\\
&\le \|K_1\|_{L^{p'}(\Omega,\omega_1)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{L^p(\Omega,\omega_2)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)}\\
&\quad + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)}\|\nabla\varphi\|_{L^p(\Omega,\omega_1)}\\
&\le \Big(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\Big)\|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |\,\Delta u|^{r-2}\,|\,\Delta u|\,|\,\Delta\,\varphi|\,\nu\,dx = \int_\Omega |\,\Delta u|^{r-1}\,|\,\Delta\,\varphi|\,\nu\,dx\\
&\le \left(\int_\Omega |\,\Delta u|^r\,\nu\,dx\right)^{1/r'}\left(\int_\Omega |\,\Delta\varphi|^r\,\nu\,dx\right)^{1/r}\\
&\le \|u\|_X^{r/r'}\|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx \le \int_\Omega \Big(K_2 + h_3|u|^{p/p'} + h_4|\nabla u|^a\Big)|\varphi|\,\omega_2\,dx\\
&\le \int_\Omega K_2\,|\varphi|\,\omega_2\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |u|^{p/p'}|\varphi|\,\omega_2\,dx\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^a|\varphi|\,\omega_2\,dx\\
&\le \Big(\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}\Big)\|\varphi\|_X\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{a\,p'}\omega_2\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\,\omega_2\,dx\right)^{1/p}\\
&\le \Big(\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}\Big)\|\varphi\|_X\\
&\quad + \|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{p/q'}\frac{\omega_2}{\omega_1}\,\omega_1\,dx\right)^{1/p'}\|\varphi\|_X\\
&\le \Big(\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}\Big)\|\varphi\|_X
\end{aligned}$$

$$+\|h_4\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^p\,\omega_1\,dx\right)^{1/(p'q')}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|\varphi\|_X$$
$$\leq\Bigg(\|K_2\|_{L^{p'}(\Omega,\omega_2)}+\|h_3\|_{L^\infty(\Omega)}\|u\|_X^{p/p'}$$
$$+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(q'p')}\Bigg)\|\varphi\|_X.$$

Therefore, in (7) we obtain, for all $u, \varphi \in X$

$$|B(u,\varphi)|\leq\Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)}+\|K_2\|_{L^{p'}(\Omega,\omega_2)}$$
$$+\ (\|h_1\|_{L^\infty(\Omega)}+\|h_2\|_{L^\infty(\Omega)}+\|h_3\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}$$
$$+\|u\|_X^{r/r'}+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(p'q')}\Big]\|\varphi\|_X.$$

Since $B(u,.)$ is linear, for each $u \in X$, there exists a linear and continuous operator $A : X \to X^*$ such that $\langle Au, \varphi\rangle = B(u, \varphi)$, for all $u,\ \varphi \in X$ (where $\langle f, x\rangle$ denotes the value of the linear functional f at the point x) and

$$\|Au\|_*\leq\|K_1\|_{L^{p'}(\Omega,\omega_1)}+\|K_2\|_{L^{p'}(\Omega,\omega_2)}$$
$$+(\|h_1\|_{L^\infty(\Omega)}+\|h_2\|_{L^\infty(\Omega)}+\|h_3\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}$$
$$+\|u\|^{r/r'}+\|h_4\|_{L^\infty(\Omega)}\|\omega_2/\omega_1\|^{1/p'}_{L^q(\Omega,\omega_1)}\|u\|_X^{p/(p'q')}.$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T,\ u \in X.$$

Step 5. Using condition (H2), (H6) and Lemma 1(b), we have

$$\langle Au_1-Au_2,u_1-u_2\rangle=B(u_1,u_1-u_2)-B(u_2,u_1-u_2)$$
$$=\int_\Omega\omega_1\,\mathcal{A}(x,u_1,\nabla u_1).\nabla(u_1-u_2)\,dx+\int_\Omega|\Delta u_1|^{r-2}\Delta u_1\,\Delta(u_1-u_2)\,\nu\,dx$$
$$+\int_\Omega b(x,u_1,\nabla u_1)(u_1-u_2)\,\omega_2\,dx-\int_\Omega b(x,u_2,\nabla u_2)(u_1-u_2)\,\omega_2\,dx$$
$$-\int_\Omega\omega_1\,\mathcal{A}(x,u_2,\nabla u_2).\nabla(u_1-u_2)\,dx-\int_\Omega|\Delta u_2|^{r-2}\Delta u_2\,\Delta(u_1-u_2)\,\nu\,dx$$
$$=\int_\Omega\omega_1\Big(\mathcal{A}(x,u_1,\nabla u_1)-\mathcal{A}(x,u_2,\nabla u_2)\Big).\nabla(u_1-u_2)\,dx$$

$$
\begin{aligned}
&+\int_\Omega (|\,\Delta u_1|^{r-2}\,\Delta u_1 - |\,\Delta u_2|^{r-2}\,\Delta u_2)\,\Delta(u_1-u_2)\,\nu\,dx \\
&+\int_\Omega (b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2))(u_1-u_2)\,\omega_2\,dx \\
\geq{}& \theta_1 \int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r \int_\Omega (|\,\Delta u_1| + |\,\Delta u_2|)^{r-2}\,|\Delta u_1 - \Delta u_2|^2\,\nu\,dx \\
&+\theta_2 \int_\Omega |u_1-u_2|^p \omega_2\,dx \\
\geq{}& \theta_1 \int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r \int_\Omega (|\,\Delta u_1 - \Delta u_2|)^{r-2}\,|\Delta u_1 - \Delta u_2|^2\,\nu\,dx \\
&+\theta_2 \int_\Omega |u_1-u_2|^p \omega_2\,dx \\
={}& \theta_1 \int_\Omega \omega_1\,|\nabla(u_1-u_2)|^p\,dx + \beta_r \int_\Omega |\Delta u_1 - \Delta u_2|^r\,\nu\,dx \\
&+\theta_2 \int_\Omega |u_1-u_2|^p \omega_2\,dx \;\geq 0.
\end{aligned}
$$

Therefore, the operator A is monotone. Moreover, using (H3), (H7), (H9) and $\omega_1 \leq \omega_2$, we obtain

$$
\begin{aligned}
\langle Au, u\rangle ={}& B(u,u) = B_1(u,u) + B_2(u,u) + B_3(u,u) \\
={}& \int_\Omega \omega_1\,\mathcal{A}(x,u,\nabla u).\nabla u\,dx + \int_\Omega |\,\Delta u|^{r-2}\,\Delta u\,\Delta u\,\nu\,dx \\
&+\int_\Omega b(x,u,\nabla u)\,u\,\omega_2\,dx \\
\geq{}& \int_\Omega (\lambda_1|\nabla u|^p + \Lambda_1|u|^p - g_1|u| - g_2|\nabla u|)\,\omega_1\,dx + \int_\Omega |\,\Delta u|^r\,\nu\,dx \\
&+\int_\Omega (\lambda_2|\nabla u|^p + \Lambda_2|u|^p - g_3|u| - g_4|\nabla u|)\,\omega_2\,dx \\
\geq{}& (\lambda_1+\lambda_2)\int_\Omega |\nabla u|^p\,\omega_1\,dx + \int_\Omega |\,\Delta u|^r\,\nu\,dx + \Lambda_2\int_\Omega |u|^p\,\omega_2\,dx \\
&-\int_\Omega g_1|u|^p\,\omega_1\,dx - \int_\Omega g_2|\nabla u|^p\,\omega_1\,dx - \int_\Omega g_3|u|\omega_2\,dx - \int_\Omega g_4|\nabla u|\,\omega_2\,dx \\
\geq{}& \gamma\left(\|u\|^p_{L^p(\Omega,\omega_2)} + \|\nabla u\|^p_{L^p(\Omega,\omega_1)} + \|\Delta u\|^r_{L^r(\Omega,\nu)}\right) - \gamma_1\,\|u\|_X,
\end{aligned}
$$

where $\gamma = \min\{\lambda_1 + \lambda_2, \Lambda_2, 1\}$ and

$$\gamma_1 = \|g_1/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \|g_2/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \|g_3/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \|g_4\omega_2/\omega_1\|_{L^{p'}(\Omega,\omega_1)}.$$

Hence, since $1 < p, r < \infty$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty, \text{ as } \|u\|_X \to +\infty,$$

that is, A is coercive (using that $\lim\limits_{t+s+a\to\infty} \dfrac{t^p + s^p + a^r}{t+s+a} = \infty$, with $t > 0, s > 0$ and $a > 0$).

Step 6. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
&|B_1(u_m, \varphi) - B_1(u, \varphi)| \\
&\leq \sum_{j=1}^{n} \int_\Omega |\mathcal{A}_j(x, u_m, \nabla u_m) - \mathcal{A}_j(x, u, \nabla u)||D_j\varphi|\,\omega_1\,dx \\
&= \sum_{j=1}^{n} \int_\Omega |F_j u_m - F_j u||D_j\varphi|\,\omega_1\,dx \\
&\leq \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} \|D_j\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&|B_3(u_m, \varphi) - B_3(u, \varphi)| \\
&= \left| \int_\Omega |\Delta u_m|^{r-2}\Delta u_m\,\Delta\varphi\,\nu\,dx - \int_\Omega |\Delta u|^{r-2}\Delta u\,\Delta\varphi\,\nu\,dx \right| \\
&\leq \int_\Omega \left| |\Delta u_m|^{r-2}\,\Delta u_m - |\Delta u|^{r-2}\Delta u \right| |\Delta\varphi|\,\nu\,dx \\
&= \int_\Omega |Gu_m - Gu|\,|\Delta\varphi|\,\nu\,dx \\
&\leq \|Gu_m - Gu\|_{L^{r'}(\Omega,\nu)}\,\|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}|B_2(u_m,\varphi)-B_2(u,\varphi)| &\le \int_\Omega |b(x,u_m,\nabla u_m)-b(x,u,\nabla u)|\,|\varphi|\,\omega_2\,dx\\ &= \int_\Omega |Hu_m-Hu||\varphi|\,\omega_2\,dx\\ &\le \|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}\|\varphi\|_X,\end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned}&|B(u_m,\varphi)-B(u,\varphi)|\\ &\le |B_1(u_m,\varphi)-B_1(u,\varphi)|+|B_2(u_m,\varphi)-B_2(u,\varphi)|+|B_3(u_m,\varphi)-B_3(u,\varphi)|\\ &\le \Bigg[\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+\|Gu_m-Gu\|_{L^{r'}(\Omega,\nu)}\\ &\quad+\|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}\Bigg]\|\varphi\|_X.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|Au_m-Au\|_* \le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+\|Gu_m-Gu\|_{L^{r'}(\Omega,\nu)}\\ +\|Hu_m-Hu\|_{L^{p'}(\Omega,\omega_2)}.\end{aligned}$$

Therefore, using (4), (5) and (6) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous (and this implies that A is hemicontinuous).

Therefore, by Theorem 3, the operator equation $Au = T$ has a solution $u \in X$ and it is a solution for problem (P).

Step 7. Let us now prove the uniqueness of the solution.

Suppose that $u_1, u_2 \in X$ are two solutions of problem (P). Then

$$\begin{aligned}&\int_\Omega |\Delta u_i|^{r-2}\Delta u_i\,\Delta\varphi\,\nu\,dx+\int_\Omega \omega_1\,\mathcal{A}(x,u_i,\nabla u_i).\nabla\varphi\,dx\\ &\quad+\int_\Omega b(x,u_i,\nabla u_i)\,\varphi\,\omega_2\,dx\\ &=\int_\Omega f_0\,\varphi\,dx+\sum_{j=1}^{n}\int_\Omega f_j\,D_j\varphi\,dx,\end{aligned}$$

for all $\varphi \in X$, and $i = 1, 2$. Hence, we obtain

$$\int_\Omega \Big(|\Delta u_1|^{r-2}\Delta u_1 - |\Delta u_2|^{r-2}\Delta u_2\Big)\Delta\varphi\,\nu\,dx + \int_\Omega \omega_1\Big(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\Big).\nabla\varphi\,dx + \int_\Omega \Big(b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2)\Big)\varphi\,\omega_2\,dx = 0.$$

In particular, for $\varphi = u_1 - u_2 \in X$ we have, by (H2), (H7) and Lemma 1(b),

$$\begin{aligned} 0 &= \int_\Omega \Big(|\Delta u_1|^{r-2}\Delta u_1 - |\Delta u_2|^{r-2}\Delta u_2\Big)(\Delta u_1 - \Delta u_2)\,\nu\,dx \\ &\quad + \int_\Omega \omega_1\Big(\mathcal{A}(x,u_1,\nabla u_1) - \mathcal{A}(x,u_2,\nabla u_2)\Big).(\nabla u_1 - \nabla u_2)\,dx \\ &\quad + \int_\Omega \Big(b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2)\Big)(u - 1 - u_2)\,\omega_2\,dx \\ &\geq \beta_r \int_\Omega |\Delta u_1 - \Delta u_2|^r\,\nu\,dx + \theta_1 \int_\Omega |\nabla u_1 - \nabla u_2|^p\,\omega_1\,dx \\ &\quad + \theta_2 \int_\Omega |u_1 - u_2|^p\,\omega_2\,dx. \end{aligned}$$

Hence $\|u_1 - u_2\|_{L^p(\Omega,\omega_2)} = \|\nabla u_1 - \nabla u_2\|_{L^p(\Omega,\omega_1)} = \|\Delta u_1 - \Delta u_2\|_{L^r(\Omega,\nu)} = 0$. Since $u_1, u_2 \in X$, then $u_1 = u_2$ μ_2 a.e. Therefore, by Lemma 2, $u_1 = u_2$ a.e.

Example 1 *Let* $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. *Consider the weight functions* ω_1, ω_2 *and* ν, $\omega_1(x,y) = (x^2+y^2)^{-1/4}$, $\omega_2(x,y) = (x^2+y^2)^{-1/2}$ *and* $\nu(x,y) = (x^2+y^2)^{-1/6}$ *(we have* $\omega_1, \omega_2 \in A_2$ $(p = 2)$ *and* $\nu \in A_3$ $(r = 3)$*), and the functions* $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *and* $b : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \mathcal{A}((x,y),\eta,\xi) &= h_2(x,y)\ \xi, \\ b((x,y),\eta,\xi) &= \eta\,(\cos^2(xy) + 1), \end{aligned}$$

where $h(x,y) = 2\,e^{(x^2+y^2)}$. *Let us consider the partial differential operator*

$$\begin{aligned} Lu(x,y) &= \Delta((x^2+y^2)^{-1/6}|\Delta u|\,\Delta u) - \operatorname{div}((x^2+y^2)^{-1/4}\,\mathcal{A}((x,y),u,\nabla u)) \\ &\quad + (x^2+y^2)^{-1/2}\,b(x,u,\nabla u). \end{aligned}$$

Therefore, by Theorem 1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{\sqrt{x^2+y^2}} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right), & \text{in } \Omega \\ u(x) = \Delta u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,3}(\Omega, \nu) \cap W_0^{1,2}(\Omega, \omega_1, \omega_2)$.

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