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Tournaments, oriented graphs and football sequences

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Abstract. Consider the result of a soccer league competition where n teams play each other exactly once. A team gets three points for each win and one point for each draw. The total score obtained by each team v_i is called the f-score of v_i and is denoted by f_i. The sequences of all f- scores $[f_i]_{i=1}^{n}$ arranged in non-decreasing order is called the f-score sequence of the competition. We raise the following problem: Which sequences of non-negative integers in non-decreasing order is a football sequence, that is the outcome of a soccer league competition. We model such a competition by an oriented graph with teams represented by vertices in which the teams play each other once, with an arc from team u to team v if and only if u defeats v. We obtain some necessary conditions for football sequences and some characterizations under restrictions.

1 Introduction

Ranking of objects is a typical practical problem. One of the popular ranking methods is the pairwise comparison of the objects. Many authors describe

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different applications: e.g., biological, chemical, network modeling, economical, human relation modeling, and sport applications.

A tournament is an irreflexive, complete, asymmetric digraph, and the score s_v of a vertex v in a tournament is the number of arcs directed away from that vertex. We interpret a tournament as the result of a competition between n teams with teams represented by vertices in which the teams play each other once (ties not allowed), with an arc from team u to team v if and only if u defeats v. A team receives one point for each win. With this scoring system, team v receives a total of s_v points. We call the sequence $S = [s_1, s_2, \cdots, s_n]$ as the score sequence, if s_i is the score of some vertex v_i. Thus a sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers in non-decreasing order is a score sequence if it realizes some tournament. Landau [21] in 1953 characterized the score sequences of a tournament.

Theorem 1 [21] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \tag{1}$$

with equality when $|I| = n$, where $|I|$ is the cardinality of the set $|I|$.

Since $s_1 \leq \cdots \leq s_n$, the inequality (1), called Landau inequalities, are equivalent to $\sum_i^k s_i \geq \binom{k}{2}$, for $k = 1, 2, \cdots, n-1$, and equality for $k = n$.

There are now several proofs of this fundamental result in tournament theory, clever arguments involving gymastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, a constructive argument utilizing network flows, another one involving systems of distinct representatives. Landau's original proof appeared in 1953 [21], Matrix considerations by Fulkerson [15] (1960) led to a proof, discussed by Brauldi and Ryser [10] in (1991). Berge [7] in (1960) gave a network flow proof and Alway [3] in (1962) gave another proof. A constructive proof via matrices by Fulkerson [16] (1965), proof of Ryser (1964) appears in the monograph of Moon (1968). An inductive proof was given by Brauer, Gentry and Shaw [8] (1968). The proof of Mahmoodian [23] given in (1978) appears in the textbook by Behzad, Chartrand and Lesnik-Foster [6](1979). A proof by contradiction was given by Thomassen [33] (1981) and was adopted by Chartrand and Lesniak [13] in subsequent revisions of their 1979 textbook, starting with their 1986 revision. A nice proof was given by Bang and Sharp [5](1979) using systems of distinct representatives. Three years later in 1982, Achutan, Rao and Ramachandra-Rao [1] obtained a proof as result of some slightly more general work. Bryant [12] (1987) gave a proof via a slightly different use of distinct representatives. Partially ordered sets were employed in a proof by Aigner [2] in 1984 and described by Li [22] in 1986 (his version appeared in 1989). Two proofs of sufficiency appeared in a paper by Griggs and Reid [17] (1996) one a direct proof and the second is self contained. Again two proofs appeared in 2009 one by Brauldi and Kiernan [11] using Rado's theorem from Matroid theory, and another inductive proof by Holshouser and Reiter [19] (2009). More recently Santana and Reid [32] (2012) have given a new proof in the vein of the two proofs by Griggs and Reid (1996).

The following is the recursive method to determine whether or not a sequence is the score sequence of some tournament. It also provides an algorithm to construct the corresponding tournament.

Theorem 2 [21] *Let S be a sequence of n non-negative integers not exceeding $n-1$, and let S' be obtained from S by deleting one entry s_k and reducing $n-1-s_k$ largest entries by one. Then S is the score sequence of some tournament if and only if S' is the score sequence.*

Brauldi and Shen [9] obtained stronger inequalities for scores in tournaments. These inequalities are individually stronger than Landau's inequalities, although collectively the two sets of inequalities are equivalent.

Theorem 3 [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} s_i \geq \frac{1}{2}\sum_{i\in I}(i-1) + \frac{1}{2}\binom{|I|}{2} \tag{2}$$

with equality when $|I| = n$

It can be seen that equality can often occur in (2), for example, equality hold for regular tournaments of odd order n whenever $|I| = k$ and $I = \{n-k+1, \cdots, n\}$. Further Theorem 2 is best possible in the sense that, for any real $\epsilon > 0$, the inequality

$$\sum_{i\in I} s_i \geq (\frac{1}{2}+\epsilon)\sum_{i\in I}(i-1) + (\frac{1}{2}-\epsilon)\binom{|I|}{2}$$

fails for some I and some tournaments, for example, regular tournaments. Brauldi and Shen [9] further observed that while an equality appears in (2), there are implications concerning the strong connectedness and regularity of every tournament with the score sequence S. Brauldi and Shen also obtained the upper bounds for scores in tournaments.

Theorem 4 [9] *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \leq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{4}|I|(2n - |I| - 1),$$

with equality when $|I| = n$

An oriented graph is a digraph with no symmetric pairs of directed arcs and without self loops. If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and if $d^+(v)$ and $d^-(v)$ are respectively, the outdegree and indegree of a vertex v, then $a_v = n - 1 + d^+(v) - d^-(v)$ is called the score of v. Clearly, $0 \leq a_v \leq 2n - 2$. The score sequence $A(D)$ of D is formed by listing the scores in non-decreasing order. One of the interpretations of an oriented graph is a competition between n teams in which each team competes with every other exactly once, with ties allowed. A team receives two points for each win and one point for each tie. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.
(i). An arc directed from u to v, denoted by $u(1-0)v$, (ii). An arc directed from v to u, denoted by $u(0-1)v$, (iii). There is no arc from u to v and there is no arc from v to u, and is denoted by $u(0-0)v$.

If $d^*(v)$ is the number of those vertices u in D which have $v(0-0)u$, then $d^+(v) + d^-(v) + d^*(v) = n - 1$. Therefore, $a_v = 2d^+(v) + d^*(v)$. This implies that each vertex u with $v(1-0)u$ contributes two to the score of v. Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0-0)u$ contributes two(one each at u and v) to scores, therefore the sum total of all the scores is $2\binom{n}{2}$. With this scoring system, player v receives a total of a_v points.

Avery [4] obtained the following characterization of score sequences in oriented graphs.

Theorem 5 [4] *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each*

$I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i \in I} a_i \geq 2\binom{|I|}{2}, \qquad (3)$$

with equality when $|I| = n$.

Since $a_1 \leq a_2 \leq \cdots \leq a_n$, the inequality (3) are equivalent to

$$\sum_{i}^{k} a_i \geq 2\binom{k}{2}, \text{ for } k = 1, 2, \cdots, n-1$$

with equality for $k = n$.

A constructive proof of Avery's theorem can be seen in Pirzada, Merajuddin and Samee [29] and another proof in Pirzada et. al [28]. A recursive characterization of score sequences in oriented graphs also appears in Avery [4].

Theorem 6 [4] *Let* A *be a sequence of integers between 0 and* $2n-2$ *inclusive and let* A' *be obtained from* A *by deleting the greatest entry* $2n-2-r$ *say, and reducing each of the greatest* r *remaining entries in* A *by one. Then* A *is a score sequence if and only if* A' *is a score sequence.*

Theorem 6 provides an algorithm for determining whether a given non-decreasing sequence A of non-negative integers is a score sequence of an oriented graph and for constructing a corresponding oriented graph. Pirzada, Merajuddin and Samee (2008) obtained the stronger inequalities for oriented graph scores.

An r-digraph is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i)$= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, so that $0 \leq p_{v_i} \leq 2r(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D.

An analogous result to Landau$'$s theorem on tournament scores [21] is the following characterization of marks in r-digraphs and is due to Pirzada [27].

Theorem 7 [27] *A sequence* $P = [p_i]_1^n$ *of non-negative integers in non-decreasing order is the mark sequence of an* r*-digraph if and only*

$$\sum_{i=1}^{t} p_i \geq rt(t-1),$$

for $1 \leq t \leq n$, *with equality when* $t = n$.

Various results on mark sequences in digraphs are given in [25, 27] and we can find certain stronger inequalities of marks for digraphs in [26] and for multidigraphs in [30].

2 Football sequences

If D is an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and if $d^+(v_i)$ and $d^-(v_i)$ are respectively the outdegree and indegree of a vertex v_i, define f_{v_i} (or briefly f_i) as

$$f_i = n - 1 + 2d^+(v_i) - d^-(v_i)$$

and call f_i as the football score(or briefly f-score) of v_i. Clearly

$$0 \leq f_{v_i} \leq 3(n-1).$$

The f-score sequence $F(D)$ (or briefly F) of D is formed by listing the f-scores in non-decreasing or non-increasing order. For any two vertices u and v in an oriented graph D, we have one of the following possibilities.

(i). An arc directed from u to v, denoted by $u \rightarrow v$ and we write this as $u(1--0)v$.

(ii). An arc directed from v to u, denoted by $u \leftarrow v$ and we write this as $u(0--1)v$.

(iii). There is no arc directed from u to v and there is no arc directed from v to u, denoted by $u \sim v$ and we write this as $u(0--0)v$.

If $d^*(v)$ is the number of those vertices u in D for which we have $v(0--0)u$, then

$$d^+(v) - d^-(v) + d^*(v) = n - 1.$$

Therefore,

$$f_v = d^+(v) - d^-(v) + d^*(v) + 2d^+(v) - d^-(v) = 3d^+(v) + d^*(v).$$

This implies that each vertex u with $v(1--0)u$ contributes three to the f-score of v, and each vertex u with $v(0--0)u$ contributes one to the f-score of v.

Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$, and each $v(0--0)u$ contributes two (one each at u and v) to f-scores, therefore

$$2\binom{n}{2} \leq \sum_{i=1}^{n} f_i \leq 3\binom{n}{2}.$$

We interpret an oriented graph as the result of a football tournament with teams represented by vertices in which the teams play each other once, with an arc from team u to team v if and only if u defeats v. A team receives three points for each win and one point for each draw (tie). With this f-scoring system, team v receives a total of f_v points.

We call the sequence $F = [f_1, f_2, \cdots, f_n]$ as the football sequence, if f_i is the f-score of some vertex v_i. Thus a sequence $F = [f_1, f_2, \cdots, f_n]$ of non-negative integers in non-decreasing order is a football sequence if it realizes some oriented graph. Several results on football sequences can be found in Ivanyi [20].

In an oriented graph the vertex of indegree zero is called a transmitter. This means that the transmitter represents that team in the game which does not lose any match.

Theorem 8 *If the sequence $F = [f_1, f_2, \cdots, f_n]$ of non-negative integers in non-decreasing order is a football sequence then for $1 \leq k \leq n-1$ and $2\binom{k}{2} \leq x_k \leq 3\binom{k}{2}$,*

$$\sum_{i=1}^{k} f_i \geq x_k,$$

and for $2\binom{n}{2} \leq x_n \leq 3\binom{n}{2}$

$$\sum_{i=1}^{n} f_i = x_n.$$

Lemma 1 *There is no oriented graph with n vertices whose f-score of some vertex is $3n-4$.*

Proof. Let D be an oriented graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Let v_i be the vertex with f-score f_i. In case $v_i(1-0)v$ to all $v \in V - \{v_i\}$, then f-score of v_i is $3(n-1)$. If $v_i(1-0)v$ for all $v \in V - \{v_i, v_j\}$, for some $v_j \in V$ and $i \neq j$, then f-score of v_i is $3(n-2)+1 = 3n-5$. We note that the possible f score can be $3(n-1)$ or $3(n-2)+1$. Thus the f-score f_i is either $3(n-1)$ or $f_i \leq 3(n-2)+1 = 3n-5$. These imply that the f-score cannot be $3n-4$. $\square$

Lemma 2 *In an oriented graph with n vertices if the f-score f_i and n are of the same parity, then the vertex v_i with f-score f_i is not the transmitter.*

Proof. Let $D(V, A)$ be an oriented graph with $V = \{v_1, v_2, \cdots, v_n\}$ so that $f_{v_i} = f_i$. Let n and f_i be of same parity, that is either (a) n and f_i both are even or (b) n and f_i both are odd.

In D, let $v_i(1-0)u$, $v_i(0-0)w$ and $v_i(0-1)z$ with $u \in U$, $w \in W$, $z \in Z$ and $V = U \cup W \cup Z \cup \{v_i\}$. Further let $|U| = x$, $|W| = y$ and $|Z| = t$. Clearly

$$x + y + t = n - 1. \tag{4}$$

Case (a) $n - 1$ is odd and f_i is even. We have $f_i = 3x + y$. Since f_i is even, $3x + y$ is even. Thus either (i) x is odd and y is odd, or (ii) x is even and y is even. In both cases, it follows from (4) that t is odd.

Case (b) $n - 1$ is even and f_i is odd. So $3x + y$ is odd. This is possible if (iii) x is even and y is odd, or (ii) x is odd and y is even. In both cases, again it follows from (4) that t is odd.

Thus in all cases we have $|Z| = t = \text{ odd}$, which implies that $|Z| \neq \phi$ so that there is at least one vertex z such that $z(1-0)v_n$. Hence v_i is not a transmitter. □

Lemma 2 shows that if the number of teams n and the f-score f_i are both odd or both even, then the team represented by v_i with f-score is not the transmitter, meaning it loses at least once in the competition.

Theorem 9 *In an oriented graph with n vertices the vertex with f-score f_i is a transmitter if (1) n and f_i are of different parity and (2) $f_i \equiv (n-1)(\mathrm{mod}\ 2)$ and $f_i \equiv 3(n-1)(\mathrm{mod}\ 2)$.*

Proof. Let $D(V, A)$ be the oriented graph with n vertices whose vertex set is $V = \{v_1, v_2, \cdots, v_n\}$. Let f-score of v_i be f_i and let v_i be the transmitter. Then in D, we have either $v_i(1-0)v_j$ or $v_i(0-0)v_j$ for all all $j \neq i$. Let U be the set of vertices for which $v_i(1-0)u$ and W be the set of vertices for which $v_i(1-0)w$ and let $|U| = x$ and $|W| = y$. Clearly

$$x + y = n - 1 \tag{5}$$

and

$$f_i = 3x + y. \tag{6}$$

Two cases can arise, (a) n is odd or (b) n is even.

Case (a) n is odd. Then $n-1$ is even so that $x+y$ is even. This is possible if either (i) x odd and y odd or (ii) x even and y even. In case of (i) $f_i = 3x+y = \mathrm{odd}+\mathrm{odd} = \mathit{even}$ and in case of (ii) $f_i = 3x+y = \mathit{even}+\mathit{even} = \mathit{even}$. Thus we see that n and f_i are of different parity.

Case (a) n is even, so that $n-1$ is odd and $x+y$ is odd. This is possible if either (iii) x odd and y even or (ii) x even and y odd. In both cases we observe that f_i is odd. Therefore again we obtain that n and f_i are of different parity.

Solving (5) and (6) together for x and y, we get

$$x = \frac{1}{2}[f_i - (n-1)] \tag{7}$$

$$y = \frac{1}{2}[3(n-1) - f_i]. \tag{8}$$

Clearly x and y are positive integers, thus the right hand sides of (7) and (8) are positive integers. This implies that $f_i - (n-1)$ and $3(n-1) - f_n$ are both divisible by 2. Hence $f_n \equiv (n-1)(\mathrm{mod}\ 2)$ and $f_n \equiv 3(n-1)(\mathrm{mod}\ 2)$. □

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