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# Solvable and nilpotent right loops

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**Abstract.** In this paper the notions of solvable right transversal and nilpotent right transversal are defined. Further, it is proved that if a core-free subgroup has a generating solvable transversal, then the whole group is solvable.

## 1 Introduction

Let $G$ be a group and $H$ a proper subgroup of $G$. A *normalized right transversal* is a subset of $G$ obtained by selecting one and only one element from each right coset of $H$ in $G$, including the identity from the coset $H$. Now we will call it a *transversal* in place of normalized right transversal. Suppose that $S$ is a transversal of $H$ in $G$. We define an operation $\circ$ on $S$ as follows: for $x, y \in S$, $\{x \circ y\} := S \cap Hxy$. It is easy to check that $(S, \circ)$ is a right loop, that is the equation of the type $X \circ a = b$, $X$ is unknown and $a, b \in S$ has a unique solution in $S$, and $(S, \circ)$ has a two-sided identity. In [5], it has been shown that for each right loop there exists a pair $(G, H)$ such that $H$ is a core-free subgroup of the group $G$ and the given right loop can be identified with a transversal of $H$ in $G$. Not all transversals of a subgroup generate the group. But for finite groups, it is proved by Cameron in [2], that if a subgroup is core-free, then always there exists a transversal which generates the whole group. We call such a transversal a *generating transversal.*

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Let $(S, \circ)$ be a right loop (identity denoted by $1$). Let $x, y, z \in S$. Define a map $R(y, z)$ from $S$ to $S$ as follows: $R(y, z)(x)$ is the unique solution of the equation $X \circ (y \circ z) = (x \circ y) \circ z$, where $X$ is unknown. It is easy to verify that $R(y, z)$ is a bijective map. For a set $X$, let $Sym(X)$ denote the symmetric group on $X$. We denote by $RInn(S)$ the subgroup of $Sym(S)$, generated by the set $\{R(y, z) \mid y, z \in S\}$. This group is called the *right inner mapping group* of the right loop $S$. It measures the deviation of a right loop from being a group. Also note that right multiplication $R_s$ by an element $s \in S$ gives a bijective map from $S$ to $S$. The subgroup generated by $\{R_s \mid s \in S\}$ is the *right multiplication group* $RMlt(S)$ of the right loop $S$. One should note that the right multiplication group $RMlt(S)$ factorizes as $RInn(S)R(S)$ in $Sym(S)$, where $R(S) = \{R_s \mid s \in S\}$. We will follow the right action convention for the map, that is the image of an element $x$ under a map $f$ is denoted by $xf$. Note that if $H$ is a core-free subgroup of a group $G$ and $S$ is a generating transversal of $H$ in $G$, then $G \cong RMlt(S)$ such that $H \cong RInn(S)$ (see [6, Lemma, p. 1343]).

A non-empty subset $T$ of right loop $S$ is called a *right subloop* of $S$, if it is right loop with respect to induced binary operation on $T$ ([7, Definition 2.1, p. 2683]). An equivalence relation $R$ on a right loop $S$ is called a *congruence* in $S$, if it is a right subloop of $S \times S$. Also an *invariant right subloop* of a right loop $S$ is precisely the equivalence class of the identity of a congruence in $S$ ([7, Definition 2.8, p. 2689]). It is observed in the proof of [7, Proposition 2.10, p. 2690] that if $T$ is an invariant right subloop of $S$, then the set $S/T = \{T \circ x | x \in S\}$ becomes right loop called as *quotient of S mod T*. Let $R$ be the congruence associated to an invariant right subloop $T$ of $S$. Then we also denote $S/T$ by $S/R$.

## 2 Some properties of right loops

In this section, we will recall some basic facts about right loops and also prove some of the results which will be used in next sections. Let $(S, \circ, /, 1)$ be a right loop, with right division $/$ and two-sided identity $1$. Then $(S, \circ, /, 1)$ is Mal'tsev algebra with Mal'tsev term $P(x, y, z) = (x/y) \circ z$.

The proof of the following Fundamental Theorem of homomorphism for right loops is as usual.

**Proposition 1** *Let $\rho : S \to S'$ be a homomorphism of right loops. Then there exists a unique injective homomorphism $\bar{\rho} : S/Ker\rho \to S'$ such that $\bar{\rho} \circ \nu = \rho$, where $\nu : S \to S/Ker\rho$ is the natural homomorphism.*

**Lemma 1** *Let* $G$ *be a group,* $H$ *a subgroup of* $G$ *and* $S$ *a transversal of* $H$ *in* $G$*. Suppose that* $N \trianglelefteq G$ *containing* $H$*. Then*

$$G/N = HS/N \cong S/N \cap S.$$

**Proof.** Suppose that $\circ$ denotes the induced right loop operation on $S$. Consider the map $\psi : S \to HS/N$ defined as $x\psi = xN$. This is a homomorphism, for

$$\begin{aligned}(x \circ y)\psi &= (x \circ y)N \\ &= hxyN \text{ for some } h \in H \\ &= (x)\psi(y)\psi \quad (H \subseteq N).\end{aligned}$$

Also, $\mathrm{Ker}\psi = \{x \in S|\ xN = N\} = S \cap N$. Since for $h \in H$ and $x \in S$, we have $hxN = xN$ and $x\psi = xN$, $\psi$ is onto and so by Proposition 1, $S/N \cap S \cong HS/N$. □

Let $G$ be a group, $H$ a subgroup and $S$ a transversal of $H$ in $G$. Suppose that $\circ$ is the induced right loop structure on $S$. We define a map $f : S \times S \to H$ as: for $x, y \in S$, $f(x, y) := xy(x \circ y)^{-1}$. We further define the action $\theta$ of $H$ on $S$ as $\{x\theta h\} := S \cap Hxh$ where $h \in H$ and $x \in S$. Identifying $S$ with the set $H\backslash G$ of all right cosets of $H$ in $G$, we get a transitive permutation representation $\chi_S : G \to Sym(S)$ defined by $\{(x)(g)\chi_S\} = S \cap Hxg,\ g \in G, x \in S$. The kernel $\ker\chi_S$ of this action is $\mathrm{Core}_G(H)$, the core of $H$ in $G$.
One can check that $(\langle S\rangle \cap H)\chi_S \cong RInn(S)$, where $\langle S\rangle$ denotes the subgroup of $G$ generated by $S$. Since $\chi_S$ is injective on $S$ and if we identify $S$ with $(S)\chi_S$, then $(\langle S\rangle)\chi_S \cong RMlt(S)$. One can also verify that $\ker(\chi_S|_{\langle S\rangle} : \langle S\rangle \to RMlt(S)) = \ker(\chi_S|_{\langle S\rangle \cap H} : \langle S\rangle \cap H \to RInn(S)) = \mathrm{Core}_{\langle S\rangle}(\langle S\rangle \cap H)$ and $\chi_S|_S$=the identity map on $S$.

With these notations it is easy to prove following lemma.

**Lemma 2** *For* $x, y, z \in S$*, we have* $xR(y, z) = x\theta f(y, z)$*.*

**Lemma 3** *Let* $H$ *be a subgroup of a group* $G$ *and* $S$ *a transversal of* $H$ *in* $G$*. Let* $U$ *be a congruence on* $S$ *considered as a right loop such that* $\{(x, x\theta h) \mid h \in H, x \in S\} \subseteq U$*. Let* $T$ *be the equivalence class of* $1$ *under* $U$*. Then* $S/U$ *is a group. Moreover,* $N = HT \trianglelefteq HS = G$ *(and so* $H \le N$ *and* $N \cap S = T$*) and* $G/N \cong S/U$*.*

**Proof.** Let $R$ be a congruence on $S$ generated by $\{(x, xR(y, z))|x, y, z \in S\}$. Then, clearly $R \subseteq U$ and $S/U$ is a group. Let $\phi : G \to S/U$ be the map defined by $(hx)\phi = T \circ x,\ h \in H,\ x \in S$. This is a homomorphism, because for all $h_1,\ h_2 \in H$ and $x_1,\ x_2 \in S$,

$$\begin{aligned}(h_1x_1h_2x_2)\phi &= (h_1h(x_1\theta h_2 \circ x_2))\phi \text{ for some } h \in H\\ &= T \circ (x_1\theta h_2 \circ x_2)\\ &= (T \circ x_1)\ \circ\ (T \circ x_2) \qquad (\text{for } (x_1, x_1\theta h_2) \in U)\\ &= (h_1x_1)\phi(h_2x_2)\phi.\end{aligned}$$

Let $h \in H$ and $x \in S$. Then $hx \in \mathrm{Ker}\phi$ if and only if $x \in T$. Hence $\mathrm{Ker}\phi = HT = N(say)$. This proves the lemma. □

## 3 Solvable right loops

In this section, we will define a solvable right loop and obtain some of its properties.

**Definition 1** *A right loop $S$ is said to be a solvable right loop if it has a finite composition series with abelian group factors.*

**Definition 2** *Let $S$ be a transversal of a subgroup $H$ of $G$. We call $S$ a solvable transversal if it is solvable with respect to the induced right loop structure.*

We define $S^{(1)}$ to be the smallest invariant right subloop of $S$ such that $S/S^{(1)}$ is an abelian group. We define $S^{(n)}$ by induction. Suppose $S^{(n-1)}$ is defined. Then $S^{(n)}$ is an invariant right subloop of $S$ such that $S^{(n)} = (S^{(n-1)})^{(1)}$.

**Theorem 1** *If a group has a solvable generating transversal with respect to a core-free subgroup, then the group is solvable.*

**Proof.** Let $G$ be a group and $H$ a core-free subgroup of it. Suppose that $S$ is a generating transversal of $H$ in $G$. Then the group $G$ can be written as $HS$. By Lemma 1, $G/HG^{(1)} \cong S/S \cap HG^{(1)}$. So,

$$S^{(1)} \subseteq S \cap HG^{(1)}. \tag{1}$$

By Lemma 3, $HS^{(1)}$ is a normal subgroup of $G$. Thus $G/HS^{(1)} = S/S^{(1)}$ (Lemma 1). Since $S/S^{(1)}$ is abelian, $G^{(1)} \subseteq HS^{(1)}$. Thus

$$S \cap HG^{(1)} \subseteq S^{(1)}. \tag{2}$$

From (2) and (1), it is clear that

$$S \cap HG^{(1)} = S^{(1)} \tag{3}$$

and

$$HG^{(1)} = HS^{(1)}. \tag{4}$$

We will use induction to prove that $HS^{(n)} = H(HS^{(n-1)})^{(1)}$ for $n \geq 1$. Define $S^{(0)} = S$. For $n = 1$, $HS^{(1)} = HG^{(1)} = H(HS^{(0)})^{(1)}$ (by (4)). By induction, suppose that $HS^{(n-1)} = H(HS^{(n-2)})^{(1)}$.

Since $S^{(n-1)}/S^{(n)} \cong HS^{(n-1)}/HS^{(n)}$ is an abelian group, $(HS^{(n-1)})^{(1)} \subseteq HS^{(n)}$. Thus $H(HS^{(n-1)})^{(1)} \subseteq HS^{(n)}$.

Further, $HS^{(n-1)}/H(HS^{(n-1)})^{(1)} \cong S^{(n-1)}/(S^{(n-1)} \cap H(HS^{(n-1)})^{(1)})$ by Lemma 1. So $S^{(n)} \subseteq S^{(n-1)} \cap H(HS^{(n-1)})^{(1)} = S \cap H(HS^{(n-1)})^{(1)}$. That is,

$$HS^{(n)} = H(HS^{(n-1)})^{(1)} \text{ for all } n \geq 1. \tag{5}$$

Now (5) implies that

$$HS^{(n)} = H(HS^{(n-1)})^{(1)} \supseteq H(H(HS^{(n-2)})^{(1)})^{(1)} \supseteq H(HS^{(n-2)})^{(2)}. \tag{6}$$

Proceeding inductively, we have $HS^{(n)} \supseteq H(HS)^{(n)} = HG^{(n)}$. Suppose that $S$ is a solvable right loop, that is there exists $n \in \mathbb{N}$ such that $S^{(n)} = \{1\}$. Then $G^{(n)} \subseteq H$. Since $G^{(n)}$ is a normal subgroup of $G$ contained in $H$, so $G^{(n)} = \{1\}$. This proves the theorem. □

The converse of the above theorem is not true. For example take $G$ to be the symmetric group on three symbols and $H$ to be any two order subgroup of it. Then $H$ has no solvable generating transversal but we know that $G$ is solvable. Following is an easy consequence of the above theorem.

**Corollary 1** *The right multiplication group of a solvable right loop is a solvable group.*

# 4 Nilpotent right loops

In this section, we define nilpotent right loops as a special case of the nilpotent Mal'tsev algebras defined in [8]. We will obtain some properties of nilpotent right loops. This will generalize a result of [1].

**Definition 3** [8, Definition 211, p. 24] *Let $\beta$ and $\gamma$ be congruences on a right loop $S$. Let $(\gamma|\beta)$ be a congruence on $\beta$. Then $\gamma$ is said to centralize $\beta$ by means of the centering congruence $(\gamma|\beta)$ such that following conditions are satisfied:*

(i) $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow x\ \gamma\ u$, *for all* $(x, y), (u, v) \in \beta$.

(ii) *For all* $(x, y) \in \beta$, *the map* $\pi : (\gamma|\beta)_{(x,y)} \to \gamma_x$ *defined by* $(u, v) \mapsto u$ *is a bijection, where for a set* $X$ *and an equivalence relation* $\delta$ *on* $X$, $\delta_w$ *denotes the equivalence class of* $w \in X$ *under* $\delta$.

(iii) *For all* $(x, y) \in \gamma$, $(x, x)\ (\gamma|\beta)\ (y, y)$.

(iv) $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow (y, x)\ (\gamma|\beta)\ (v, u)$, *for all* $(x, y), (u, v) \in \beta$.

(v) $(x, y)\ (\gamma|\beta)\ (u, v)$ *and* $(y, z)\ (\gamma|\beta)\ (v, w) \Rightarrow (x, z)\ (\gamma|\beta)\ (u, w)$, *for all* $(x, y), (u, v), (y, z)$ *and* $(v, w)$ *in* $\beta$.

By (i) and (iv), we observe that $(x, y)\ (\gamma|\beta)\ (u, v) \Rightarrow y\ \gamma\ v$.

Let $S$ be a right loop. If a congruence $\alpha$ on $S$ is centralized by $S \times S$, then it is called a *central congruence* (see [8, p. 42]). By [8, Proposition 221, p. 34] and [8, Proposition 226, p. 38], there exists a unique maximal central congruence $\zeta(S)$ on $S$, called as the *center congruence* of $S$. For a right loop, it is product of all centralizing congruences. The *center* $\mathcal{Z}(S)$ of $S$ is defined as $\zeta_1$, the equivalence class of the identity $1$. In [4, Proposition 3.3, p. 6], it is observed that if $x \in \mathcal{Z}(S)$, then $x \circ (y \circ z) = (x \circ y) \circ z$ for all $y, z \in S$. In [4, Proposition 3.4, p. 6], it is observed that if $x \in \mathcal{Z}(S)$, then $x \circ y = y \circ x$ for all $y \in S$. This means that the center $\mathcal{Z}(S)$ is an abelian group.

**Definition 4** *A right loop* $S$ *is said to be nilpotent if it has a central series*

$$\{1\} = \mathcal{Z}_0 \leq \mathcal{Z}_1 \leq \cdots \leq \mathcal{Z}_n = S$$

*for some* $n \in \mathbb{N}$, *where*

$$\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(S/\mathcal{Z}_i) \text{ and } \mathcal{Z}_1 = \mathcal{Z}(S).$$

On can observe that $\mathcal{Z}_i$ $(0 \leq i \leq n)$ is an invariant right subloop of $S$. We call a transversal $S$ of a subgroup $H$ of a group $G$ to be *nilpotent*, if it is nilpotent with respect to the induced right loop structure.

**Lemma 4** *Every nilpotent right loop is a solvable right loop.*

**Proof.** It follows from the fact that the central series of a nilpotent right loop is a composition series with abelian group factors. □

Since a nilpotent right loop $S$ is solvable, by Corollary 1, $\mathrm{RMlt}(S)$ is solvable. But in this proof we do not know much about the structure of $\mathrm{RInn}(S)$. We will obtain that if $S$ is a nilpotent right loop of prime power order, then the order of $\mathrm{RMlt}(S)$ will be a prime power.

**Proposition 2** *Let $S$ be a right loop. Let $\theta : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}(S))$ be the onto homomorphism induced by the natural projection $\nu : S \to S/\mathcal{Z}(S)$. Then $\mathrm{Ker}\theta$ is isomorphic to a subgroup of an abelian group $\prod_{\mathcal{A}} \mathcal{Z}(S)$ for some indexing set $\mathcal{A}$.*

**Proof.** Let $\mathcal{A} = \{x_1, \cdots, x_i, \cdots\}$ be a set obtained by choosing one element from each right coset of $\mathcal{Z}(S)$ in $S$, with $x_1 = 1 \in \mathcal{Z}(S)$. Then $S = \sqcup_{x_i \in \mathcal{A}}(\mathcal{Z}(S) \circ x_i)$. Let $h \in \mathrm{Ker}\theta$. Then $x_i h = z \circ x_i$ for some $z \in \mathcal{Z}(S)$. If $y = u_i \circ x_i$, where $u_i \in \mathcal{Z}(S)$, then
$(y)h = (u_i \circ x_i)h = u_i \circ (x_i)h$ (by condition (C7) of [5, Definition 2.1, p. 71] and [4, Proposition 3.3, p. 6])

$$
\begin{aligned}
&= u_i \circ (z \circ x_i)\\
&= (u_i \circ z) \circ x_i \quad (\text{for } u_i \in \mathcal{Z}(S))\\
&= (z \circ u_i) \circ x_i\\
&= z \circ (u_i \circ x_i) \quad (\text{for } z \in \mathcal{Z}(S))\\
&= z \circ y.
\end{aligned}
$$

Thus $h \in \mathrm{Ker}\theta$ is completely determined by $x_i h$ ($x_i \in \mathcal{A}$). Therefore, it defines a map $\eta : \mathrm{Ker}\theta \to \prod_{\mathcal{A}} \mathcal{Z}(S)$ by $(h)\eta = (z_i)_{\mathcal{A}}$, where $(x_i)h = z_i \circ x_i$. One can check that $\eta$ is injective homomorphism. □

Following is the finite version of above proposition.

**Corollary 2** *Let $S$ be a finite right loop with $|S/\mathcal{Z}(S)| = k$. Let $\theta : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}(S))$ be the onto homomorphism induced by natural projection $\nu : S \to S/\mathcal{Z}(S)$. Then $\mathrm{Ker}\theta$ is isomorphic to a subgroup of abelian group $\mathcal{Z}(S) \times \cdots \times \mathcal{Z}(S)$ ($k-1$ times).*

Let $S$ be a nilpotent right loop with central series

$$\{1\} = \mathcal{Z}_0 \leq \mathcal{Z}_1 \leq \cdots \leq \mathcal{Z}_n = S. \tag{7}$$

Let $\theta_j : \mathrm{RInn}(S) \to \mathrm{RInn}(S/\mathcal{Z}_j)$ ($0 \leq j \leq n-1$) be onto homomorphism induced by the natural projection $\nu_j : S \to S/\mathcal{Z}_j$. Then this will give a series

$$\{1\} = \mathrm{Ker}\theta_0 \leq \cdots \leq \mathrm{Ker}\theta_{n-1} = \mathrm{RInn}(S).$$

Let $\theta : \mathrm{RInn}(S/\mathcal{Z}_j) \to \mathrm{RInn}((S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j))$ be onto homomorphism induced by the natural projection $\nu : S/\mathcal{Z}_j \to (S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. By Proposition 2, $\mathrm{Ker}\theta$ is isomorphic to a subgroup of $\prod_{\mathcal{B}} \mathcal{Z}_{j+1}/\mathcal{Z}_j$ for some indexing set $\mathcal{B}$.

We now observe that each member of $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ induces a member of $\mathrm{Ker}\theta$. For this, we will see the action of an element of $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ on the elements of $(S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. Let $h_{j+1}\mathrm{Ker}\theta_j \in \mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$, where $h_{j+1} \in \mathrm{Ker}\theta_{j+1}$ and $(\mathcal{Z}_{j+1}/\mathcal{Z}_j) \circ (\mathcal{Z}_j \circ x) \in (S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j)$. By the definition of $\theta_j$, each element of $\mathrm{Ker}\theta_j$ acts trivially on the cosets of $\mathcal{Z}_j$. Since $\mathrm{RInn}(S/\mathcal{Z}_{j+1}) \cong \mathrm{RInn}((S/\mathcal{Z}_j)/(\mathcal{Z}_{j+1}/\mathcal{Z}_j))$, by definition of $\theta_{j+1}$, $h_{j+1}$ also acts trivially on $(\mathcal{Z}_{j+1}/\mathcal{Z}_j) \circ (\mathcal{Z}_j \circ x)$. Thus, we have proved the following:

**Proposition 3** *Let $S$ be a nilpotent right loop with central series 7. Then there exists a series*

$$\{1\} = \mathrm{Ker}\theta_0 \leq \cdots \leq \mathrm{Ker}\theta_{n-1} = \mathrm{RInn}(S)$$

*such that $\mathrm{Ker}\theta_{j+1}/\mathrm{Ker}\theta_j$ is isomorphic to a subgroup of $\prod_{\mathcal{B}} \mathcal{Z}_{j+1}/\mathcal{Z}_j$ for some indexing set $\mathcal{B}$.*

**Corollary 3** *Let $S$ be a nilpotent right loop. Then the right inner mapping group $\mathrm{RInn}(S)$ is a solvable group.*

**Proof.** By Proposition 3, central series of $S$ gives a series of $\mathrm{RInn}(S)$ with abelian quotients. □

**Corollary 4** *If a group $G$ has a nilpotent generating transversal with respect to a core-free subgroup $H$, then $H$ is solvable.*

**Corollary 5** *Let $S$ be a nilpotent generating transversal with respect to a core-free subgroup $H$ of a finite group $G$ such that $|S| = p^n$ for some prime $p$ and $n \in \mathbb{N}$. Then both $H$ and $G$ are $p$-groups.*

## 5 Some examples

In this section, we will observe some examples and counterexamples. We have seen that the concepts of solvability and nilpotency of a right loop can be transferred in term of a generating transversal of a core-free subgroup of a group. There are examples of groups where no non-trivial subgroup is core-free. Following is an example of such a group:

**Example 1** *Consider the group* $G = \langle x_1, x_2, x_3, x_4 | x_1^{p^n} = x_2^{p^3} = x_3^{p^2} = x_4^{p^2} = 1, [x_1, x_2] = x_2^{p^2}, [x_1, x_3] = x_3^{p}, [x_1, x_4] = x_4^{p}, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^{p^2}, [x_3, x_4] = x_4^{p}\rangle$ *where $p$ is an odd prime and $n$ is the natural number greater than 2. The above example has been taken from [3]. This is a nilpotent group of class 2 having no nontrivial core-free subgroup.*

We now observe that a solvable right loop which is not a group need not be a nilpotent right loop.

**Example 2** *Let* $\mathsf{G} = \mathsf{Alt}(4)$*, the alternating group of degree* $4$ *and* $\mathsf{H} = \{\mathrm{I}, (1,2)(3,4)\}$*, where* $\mathrm{I}$ *denotes the identity permutation. Consider a right transversal* $\mathsf{S} = \{\mathrm{I}, (1,3)(2,4), (1,2,3), (1,3,2), (2,3,4), (1,3,4)\}$ *of* $\mathsf{H}$ *in* $\mathsf{G}$*. Note that* $\langle \mathsf{S} \rangle = \mathsf{G}$ *and* $\mathsf{H}$ *is core-free. Then* $\mathsf{RMlt}(\mathsf{S}) \cong \mathsf{G}$ *and* $\mathsf{RInn}(\mathsf{S}) \cong \mathsf{H}$*. Also note that* $\mathsf{S} \cap \mathsf{N_G}(\mathsf{H}) = \{\mathrm{I}, (1,3)(2,4)\}$*, where* $\mathsf{N_G}(\mathsf{H})$ *denotes the normalizer of* $\mathsf{H}$ *in* $\mathsf{G}$*. By* [4, Proposition 3.3, p. 6]*,* $\mathcal{Z}(\mathsf{S}) \subseteq \mathsf{S} \cap \mathsf{N_G}(\mathsf{H})$*. Let* $\circ$ *be the induced binary operation on* $\mathsf{S}$ *as defined in the Section* $1$*. Observe that* $(1,3)(2,4) \circ (1,3,4) \neq (1,3,4) \circ (1,3)(2,4)$*. This implies that* $\mathcal{Z}(\mathsf{S}) = \{\mathrm{I}\}$*. Hence* $\mathsf{S}$ *can not be nilpotent.*

*Now by Lemma 1,* $\mathsf{S}/(\mathsf{S} \cap \mathsf{N_G}(\mathsf{H}))$ *is isomorphic to the cyclic group of order* $3$*. This implies that* $\mathsf{S}$ *is solvable.*

Now, we observe that, unlike for the case of groups, a right loop of prime power order need not be nilpotent.

**Example 3** *Let* $\mathsf{G} =$

$$\langle (1,3)(2,4)(5,7,6,8), (1,4)(2,3)(5,8,6,7), (1,5)(2,6)(3,7)(4,8) \rangle \leq \mathit{Sym}(8),$$

*where* $\mathit{Sym}(n)$ *denotes the symmetric group of degree* $n$*. Let* $\mathsf{H}$ *be the stabilizer of* $1$ *in* $\mathsf{G}$*. Consider* $\mathsf{S} = \{\mathrm{I}, (1,2)(3,4), (1,3)(2,4)(5,7,6,8), (1,4)(2,3)(5,8,6,7), (1,5)(2,6)(3,7)(4,8), (1,6)(2,5)(3,8)(4,7), (1,7)(2,8)(3,6,4,5), (1,8)(2,7)(3,5,4,6)\}$*. Clearly* $\mathsf{S}$ *is right transversal of* $\mathsf{H}$ *in* $\mathsf{G}$*. Note that the center* $\mathsf{Z}(\mathsf{G}) = \{\mathrm{I}, (1,2)(3,4)(5,6)(7,8)\}$ *and* $\mathsf{N_G}(\mathsf{H}) = \mathsf{HZ}(\mathsf{G})$*. Since* $\mathsf{H}$ *is core-free and* $\langle \mathsf{S} \rangle = \mathsf{G}$*,* $\mathsf{G} \cong \mathsf{RMlt}(\mathsf{S})$ *and* $\mathsf{H} \cong \mathsf{RInn}(\mathsf{S})$*. Observe that* $\mathsf{S} \cap \mathsf{N_G}(\mathsf{H}) = \{\mathrm{I}, (1,2)(3,4)\}$*. By* [4, Proposition 3.3, p. 6]*,* $\mathcal{Z}(\mathsf{S}) \subseteq \mathsf{S} \cap \mathsf{N_G}(\mathsf{H})$*. Let* $\circ$ *be the induced binary operation on* $\mathsf{S}$ *as defined in the section* $1$*. Observe that* $(1,2)(3,4) \circ (1,5)(2,6)(3,7)(4,8) \neq (1,5)(2,6)(3,7)(4,8) \circ (1,2)(3,4)$*. This implies that* $\mathcal{Z}(\mathsf{S}) = \{\mathrm{I}\}$*. Hence* $\mathsf{S}$ *cannot be nilpotent.*

Next, we will show that there are core-free subgroups of a nilpotent group which has none of its generating transversals nilpotent. But before proceeding to further examples, we need to prove the following results.

**Proposition 4** *Suppose that* $\mathsf{G}$ *is a nilpotent group of class* $2$*,* $\mathsf{H}$ *is a core-free subgroup of* $\mathsf{G}$ *and* $\mathsf{S}$ *is generating transversal of* $\mathsf{H}$ *in* $\mathsf{G}$*. Then* $\mathcal{Z}(\mathsf{G}) \cap \mathsf{S} = \mathcal{Z}(\mathsf{S})$*.*

**Proof.** Take $x \in \mathcal{Z}(S)$. Then $x \circ y = y \circ x$ for all $y \in S$. This implies $xyx^{-1}y^{-1} \in H$ for all $y \in S$. Since group is nilpotent of class 2, so all commutators are central. For $H$ is core-free, so $H$ will not contain any commutator element. This implies $xyx^{-1}y^{-1} = 1$ or $xy = yx$ for all $y \in S$. This proves that $\mathcal{Z}(S) \subseteq \mathcal{Z}(G) \cap S$ (for $S$ generates $G$). Converse is obvious. This proves the lemma. □

**Proposition 5** *For some prime* $p$*, suppose that* $G$ *is a* $p$*-group of nilpotent class* $2$*,* $H$ *is a core-free subgroup of* $G$ *and* $S$ *is generating transversal of* $H$ *in* $G$*. Then* $\mathcal{Z}(G) \cap \Phi(G) \cap S = \{1\}$ *where* $\Phi(G)$ *is the Frattini subgroup of* $G$*.*

**Proof.** Suppose that $1 \neq x \in \mathcal{Z}(G) \cap \Phi(G) \cap S$. Then by Proposition 4, $x \in \mathcal{Z}(S)$. Also $\mathcal{Z}(S)$ is an invariant right subloop, so $|\mathcal{Z}(S)|$ divides $|S|$. Consider $S' = S \setminus \{x\} \cup \{hx\}$ for some $1 \neq h \in H$. Note that $S'$ also generates $G$. Then by Proposition 4, order of center of $S'$ is one less than the order of center of $S$ and also $|\mathcal{Z}(S')|$ divides $|S|$. This is not possible for order of $S$ is $p$ power. This proves the lemma. □

**Example 4** *Consider the group* $G = \langle x_1, x_2, x_3, x_4 | x_1^{p^n} = x_2^{p^2} = x_3^{p^2} = x_4^{p^4} = 1, [x_1, x_2] = x_2^p, [x_1, x_3] = x_3^p, [x_1, x_4] = x_3^p, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^p, [x_3, x_4] = 1\rangle$ *where* $p$ *is an odd prime. The above example has been taken from* [3]*. By the Lemma 2.1 of* [3]*, this group is a nilpotent group of of class* $2$ *and its center and Frattini subgroup are equal. By Propositions 4 and 5, it follows that center of each generating transversal is trivial. So none of the generating transversal is nilpotent.*

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