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Some fixed point results for rational type and subrational type contractive mappings

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Abstract. In this paper, we introduce the concepts of rational type and subrational type contractive mappings. We expand and improve some fixed point theorems obtained by Alsulami *et al.* (Fixed Point Theory Appl., 2015, 2015: 97). Moreover, we give an example to support our results.

1 Introduction and preliminaries

Fixed point theory gains very large impetus due to its wide range of applications in various fields such as engineering, economics, computer science, and many others. It is well known that the contractive condition is indispensable in the study of fixed point theory. Banach fixed point theorem [1] is one of the pivotal results in mathematical analysis. Many authors (see, e.g., [2]-[8]) not only extend this theorem but also consider fixed points in various abstract spaces.

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In 2000, Branciari [2] introduced the notion of generalized metric space. A generalized metric is a semi-metric which does not satisfy the triangle inequality but satisfies a weaker condition called quadrilateral inequality.

Throughout this paper, denote $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}^* = \{1, 2, 3, \ldots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$. For the sake of author, we give some notations and notions as follows.

Definition 1 [2] *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ be a mapping. If for all $x, y \in X$ and all distinct $u, v \in X$, each of which is different from x and y,*

(GMS1) $d(x, y) = 0$ *if and only if* $x = y$;

(GMS2) $d(x, y) = d(y, x)$;

(GMS3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then d is called a generalized metric and is abbreviated as GM. Here, the pair (X, d) is called a generalized metric space and is abbreviated as GMS.

Note that if d satisfies only (GMS1) and (GMS2), then it is called a semi-metric (see *e.g.*, [3]).

Definition 2 *Let (X, d) be a GMS. Then*

1. *a sequence $\{x_n\}$ in (X, d) is said to be GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$;*
2. *a sequence $\{x_n\}$ in (X, d) is said to be GMS Cauchy if and only if for every $\varepsilon > 0$, there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m > N(\varepsilon)$;*
3. *(X, d) is said to be complete if every GMS Cauchy sequence in X is GMS convergent;*
4. *a mapping $T : X \to X$ is said to be continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.*

Definition 3 *Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be two mappings. We say that T is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.*

Definition 4 [4] *Let (X, d) be a GMS and $\alpha : X \times X \to \mathbb{R}^+$. We say that X is called α-regular if, for any sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}^*$.*

Definition 5 *Let* X *be a nonempty set,* $T : X \to X$ *and* $\mu : X \times X \to \mathbb{R}^+$ *be two mappings. We say that* T *is a* μ*-subadmissible mapping if* $\mu(x, y) \leq 1$ *implies* $\mu(Tx, Ty) \leq 1$ *for all* $x, y \in X$.

Definition 6 *Let* (X, d) *be a GMS and* $\mu : X \times X \to \mathbb{R}^+$*. We say that* X *is called* μ*-subregular if, for any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *and* $\mu(x_n, x_{n+1}) \leq 1$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\mu(x_{n_k}, x) \leq 1$ *for all* $k \in \mathbb{N}^*$.

Definition 7 *A mapping* $H : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is called a function of subclass of type I if it is continuous and* $x \geq 1$ *implies* $H(1, y) \leq H(x, y)$ *for all* $y \in \mathbb{R}^+$.

Example 1 *We have the following functions of subclass of type I, for all* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}^+$*:*

(1) $H(x, y) = (y + l)^x, \ l \geq 1$;
(2) $H(x, y) = (x + l)^y, \ l \geq 0$;
(3) $H(x, y) = xy^n, \ n \in \mathbb{N}$;
(4) $H(x, y) = x^n y, \ n \in \mathbb{N}$;
(5) $H(x, y) = \frac{(x+1)y}{2}$;
(6) $H(x, y) = \frac{(2x+1)y}{3}$;
(7) $H(x, y) = \frac{y}{n+1} \sum_{i=0}^{n} x^i, \ n \in \mathbb{N}$;
(8) $H(x, y) = \left(\frac{1}{n+1} \sum_{i=0}^{n} x^i + l \right)^y, \ l \geq 0, \ n \in \mathbb{N}$.

Definition 8 *Let* $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *and* $H : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be two mappings. We say that the pair* $(\mathcal{F}, H)$ *is an upper class of type* I *if* H *is a function of subclass of type I and satisfies for all* $r, s, t \in \mathbb{R}^+$,

(1) $0 \leq s < 1 \Rightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$;
(2) $H(1, r) \leq \mathcal{F}(s, t) \Rightarrow r \leq st$.

Example 2 *We have the following upper classes of type I, for all* $x \in \mathbb{R}$, $y, t \in \mathbb{R}^+$, $s \in [0, 1)$*:*

(1) $H(x, y) = (y + l)^x$, $\mathcal{F}(s, t) = st + l$, $l \geq 1$;
(2) $H(x, y) = (x + l)^y$, $\mathcal{F}(s, t) = (1 + l)^{st}$, $l \geq 0$;
(3) $H(x, y) = xy^n$, $\mathcal{F}(s, t) = s^n t^n, \ n \in \mathbb{N}$;
(4) $H(x, y) = x^n y$, $\mathcal{F}(s, t) = st, \ n \in \mathbb{N}$;
(5) $H(x, y) = \frac{(2x+1)y}{3}$, $\mathcal{F}(s, t) = st$;
(6) $H(x, y) = \frac{(x+1)y}{2}$, $\mathcal{F}(s, t) = st$;

(7) $H(x,y) = \frac{y}{n+1}\sum_{i=0}^{n} x^i$, $\mathcal{F}(s,t) = st,\ n \in \mathbb{N}$;

(8) $H(x,y) = \left(\frac{1}{n+1}\sum_{i=0}^{n} x^i + l\right)^y$, $\mathcal{F}(s,t) = (1+l)^{st}$, $l \geq 0,\ n \in \mathbb{N}$.

Proposition 1 [6] Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n,u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n,z) = d(u,z)$, for all $z \in X$. In particular, the sequence $\{x_n\}$ does not converge to z if $z \neq u$.

2 Main results

In this section, let $F(T)$ denote the set of fixed points of the mapping T. Let Ψ be a family of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following properties:

(i) ψ is upper semi-continuous and nondecreasing;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

By the above properties, we have $\psi(t) < t$, for each $t > 0$. Indeed, if there exists $t_0 > 0$ such that $\psi(t_0) \geq t_0$, then by the monotonicity of ψ, it establishes that

$$\psi^n(t_0) \geq t_0, \qquad (n = 1,2,\ldots),$$

thus

$$0 = \lim_{n\to\infty} \psi^n(t_0) \geq t_0.$$

This is a contradiction.

Definition 9 *Let* (X,d) *be a GMS and* $\alpha : X\times X \to \mathbb{R}^+$ *a mapping. A mapping* $T : X \to X$ *is said to be an* $(\alpha,\psi,\mathcal{F},H)$*-rational type-*I *contractive mapping if there exists a function* $\psi \in \Psi$*, such that for all* $x,y \in X$*, the following condition holds:*

$$H\left(\alpha\left(x,y\right), d(Tx,Ty)\right) \leq \mathcal{F}\left(1, \psi(M(x,y))\right), \tag{1}$$

where $(\mathcal{F},H)$ *is an upper class of type I and*

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}. \tag{2}$$

Theorem 1 *Let* (X,d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*
(ii) T *is an* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$ satisfy $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct a sequence $\{x_n\}$ in X as $x_n = T^n x_0 = Tx_{n-1}$, for $n \in \mathbb{N}^*$. It is obvious that x_{n_0} is a fixed point of T if $x_{n_0} = x_{n_0+1}$, for some $n_0 \in \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is α-admissible, then

$$\begin{aligned}\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 &\Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \\ &\Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1,\end{aligned}$$

and by induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

By similar argument, we get $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \in \mathbb{N}$ from $\alpha(x_0, T^2x_0) \geq 1$. Consider (1) with $x = x_n$ and $y = x_{n+1}$, it follows that

$$\begin{aligned}H(1, d(x_{n+1}, x_{n+2})) &= H(1, d\,(Tx_n, Tx_{n+1})) \\ &\leq H(\alpha\,(x_n, x_{n+1})\,, d\,(Tx_n, Tx_{n+1})) \\ &\leq \mathcal{F}(1, \psi(M(x_n, x_{n+1}))),\end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})),$$

where

$$\begin{aligned}M(x_n, x_{n+1}) = \max\Bigg\{ & d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ & \frac{d(x_n, Tx_n) d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n) d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})} \Bigg\} \\ = \max\{ & d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},\end{aligned}$$

If $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$ for some n, then

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) < d(x_{n+1}, x_{n+2}),$$

which is impossible. Hence, $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) = \psi(d\,(x_n,x_{n+1})) < d\,(x_n,x_{n+1}) \\ &\leq \psi(M(x_{n-1}, x_n)) = \psi(d\,(x_{n-1},x_n)) < d\,(x_{n-1},x_n)\,, \end{aligned} \tag{3}$$

and by the monotonicity of ψ, it is easy to see that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \psi(M(x_n, x_{n+1})) = \psi(d(x_n,x_{n+1})) \\ &\leq \psi(\psi(M(x_{n-1}, x_n))) = \psi(\psi(d(x_{n-1}, x_n))) = \psi^2(d(x_{n-1}, x_n)) \\ &\leq \psi^3(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^{n+1}(d(x_0, x_1)) \to 0 \ (n \to \infty), \end{aligned}$$

so

$$\lim_{n\to\infty} d\,(x_{n+1},x_{n+2}) = 0. \tag{4}$$

Putting $x = x_{n-1}$ and $y = x_{n+1}$ in (1), we have

$$\begin{aligned} H(1, d(x_n, x_{n+2})) &= H(1, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq H(\alpha(x_{n-1}, x_{n+1}), d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \mathcal{F}(1, \psi(M(x_{n-1}, x_{n+1}))), \end{aligned}$$

which establishes that

$$d(x_n, x_{n+2}) \leq \psi(M(x_{n-1}, x_{n+1})), \tag{5}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\ &\quad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})} \Big\} \\ &= \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ &\quad \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})} \Big\} \\ &< \max\Big\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \Big\} \end{aligned}$$

based on (3).

If $M(x_{n-1}, x_{n+1}) < d(x_{n-1}, x_{n+1})$, then by (5), it is not hard to verify that

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \psi(M(x_{n-1}, x_{n+1})) \leq \psi(d(x_{n-1}, x_{n+1})) \leq \psi(\psi(M(x_{n-2}, x_n))) \\ &\leq \psi(\psi(d(x_{n-2}, x_n))) = \psi^2(d(x_{n-2}, x_n)) \\ &\leq \cdots \leq \psi^n(d(x_0, x_2)) \to 0 \ (n \to \infty). \end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$.

If

$$M(x_{n-1}, x_{n+1}) < \max\left\{ d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \right\},$$

then by (5), we arrive at

$$d(x_n, x_{n+2}) < \max\left\{ d(x_{n-1}, x_n), \frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}, \frac{d^2(x_{n-1}, x_n)}{1 + d(x_n, x_{n+2})} \right\}. \quad (6)$$

Taking the limit as $n \to \infty$ from (6), we get $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$ because of (4).

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \quad (7)$$

Now, we prove that $\{x_n\}$ is a GMS Cauchy sequence, that is, $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$, for all $k \in \mathbb{N}^*$. We have already proved the cases for $k = 1$ and $k = 2$ in (4) and (7), respectively. Take arbitrary $k \geq 3$. We discuss two cases.

Case 1. Suppose that $k = 2m + 1$, where $m \geq 1$. Using the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n, x_{n+k}) = d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p(d(x_0, x_1)) \leq \sum_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 0 \ (n \to \infty). \end{aligned}$$

Case 2. Suppose that $k = 2m$, where $m \geq 2$. Again, by applying the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \ldots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(d(x_0, x_1)) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 0 \ (n \to \infty). \end{aligned}$$

Uniting Case 1 and Case 2, we have $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$ for all $k \geq 3$. Thus, again by (4) and (7), we have $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$ for all $k \geq 1$. Hence we claim that $\{x_n\}$ is a GMS Cauchy sequence in (X, d). Since (X, d) is complete, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{8}$$

We shall show x^* is a fixed point of T. First, assume that T is continuous, then by (8), it is clear that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Due to Proposition 1, we conclude that $x^* = Tx^*$, that is, x^* is a fixed point of T.

Second, assume that X is α-regular. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}^*$. Choose $x = x_{n_k}$ and $y = x^*$ in (1), we get

$$\begin{aligned} H(1, d(x_{n_k+1}, Tx^*)) &= H(1, d(Tx_{n_k}, Tx^*)) \\ &\leq H(\alpha(x_{n_k}, x^*), d(Tx_{n_k} Tx^*)) \\ &\leq \mathcal{F}(1, \psi(M(x_{n_k}, x^*))), \end{aligned}$$

which follows that

$$d\left(x_{n_k+1}, Tx^*\right) \leq \psi(M(x_{n_k}, x^*)), \tag{9}$$

where

$$M(x_{n_k}, x^*) = \max\left\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*),\right.$$
$$\left.\frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1+d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1+d(Tx_{n_k}, Tx^*)}\right\}$$
$$= \max\left\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*),\right.$$
$$\left.\frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1+d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1+d(x_{n_k+1}, Tx^*)}\right\}.$$

Consider the upper semi-continuity of ψ, it derives from (9) that

$$d(x^*, Tx^*) = \limsup_{k\to\infty} d(x_{n_k+1}, Tx^*) \le \limsup_{k\to\infty} \psi(M(x_{n_k}, x^*))$$
$$\le \psi(\lim_{k\to\infty} M(x_{n_k}, x^*)) = \psi(d(x^*, Tx^*)),$$

which implies $x^* = Tx^*$ (otherwise, if $d(x^*, Tx^*) > 0$, then

$$d(x^*, Tx^*) \le \psi(d(x^*, Tx^*)) < d(x^*, Tx^*)$$

is a contradiction).

Finally, assume that x^* and y^* are two different fixed points of T. Then by the hypothesis, $\alpha(x^*, y^*) \ge 1$. Hence, from (1) with $x = x^*$ and $y = y^*$ we conclude that

$$H(1, d(x^*, y^*)) = H(1, d(Tx^*, Ty^*)) \le H(\alpha(x^*, y^*), d(Tx^*, Ty^*))$$
$$\le \mathcal{F}(1, \psi(M(x^*, y^*))),$$

which establishes that

$$d(x^*, y^*) \le \psi(M(x^*, y^*)),$$

where

$$M(x^*, y^*) = \max\left\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*),\right.$$
$$\left.\frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1+d(x^*, y^*)}, \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1+d(Tx^*, Ty^*)}\right\}$$
$$= d(x^*, y^*).$$

Thus $d(x^*, y^*) \le \psi(d(x^*, y^*)) < d(x^*, y^*)$. This is a contradiction. Hence T has a unique fixed point. $\square$

Corollary 1 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*

(ii) *for all* $x, y \in X$,

$$(l+d(Tx, Ty))^{\alpha(x,y)} \leq \psi(M(x, y)) + l,$$

where $l \geq 1$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*

(iv) *either* T *is continuous, or* X *is* α*-regular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Choose $H(x, y) = (y + l)^x$ and $\mathcal{F}(s, t) = st + l$, then by Theorem 1, the desired proof is obtained. □

Corollary 2 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*

(ii) *for all* $x, y \in X$,

$$(l + \alpha(x, y))^{d(Tx,Ty)} \leq (1 + l)^{\psi(M(x,y))},$$

where $l \geq 0$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*

(iv) *either* T *is continuous, or* X *is* α*-regular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Choose $H(x, y) = (x + l)^y$ and $\mathcal{F}(s, t) = (1 + l)^{st}$, then by Theorem 1, the proof is valid. □

Corollary 3 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is an* α*-admissible mapping;*

(ii) *for all* $x, y \in X$,

$$\alpha(x, y)(d(Tx, Ty))^n \leq (\psi(M(x, y)))^n, \ n \in \mathbb{N},$$

where $M(x, y)$ *is defined by* (2)*;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Take $H(x, y) = xy^n$ and $\mathcal{F}(s, t) = s^n t^n$, then by Theorem 1, the proof is completed. □

Corollary 4 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) *for all* $x, y \in X$*,*

$$\frac{k\alpha(x, y) + 1}{k + 1}(d(Tx, Ty))^n \leq (\psi(M(x, y)))^n, \ n \in \mathbb{N},$$

where $k \geq 0$ *is a constant and* $M(x, y)$ *is defined by* (2)*;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Take $H(x, y) = \frac{kx+1}{k+1} y^n$ and $\mathcal{F}(s, t) = s^n t^n$, then by Theorem 1, the claim holds. □

Remark 1 *Assume that* $H(x, y) = xy$ *and* $\mathcal{F}(s, t) = t$ *in Definition 9 and let (2) from Definition 8 be replaced by* $H(1, r) \leq \mathcal{F}(1, t) \Rightarrow r \leq t$*, then* $T : X \to X$ *is said to be an* (α, ψ)*-rational type-*I *contractive mapping. This is a definition from* [4]*. In this case, we easily get the following Corollary 5 from Theorem 1.*

Corollary 5 [4] *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\alpha : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) T *is an* (α, ψ)*-rational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Definition 10 *Let* (X, d) *be a GMS and* $\alpha : X \times X \to \mathbb{R}^+$ *a mapping. A mapping* $T : X \to X$ *is said to be* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-II contractive mapping if there exists a function* $\psi \in \Psi$*, such that for all* $x, y \in X$*, the following condition holds:*

$$H(\alpha(x, y), d(Tx, Ty)) \leq \mathcal{F}(1, \psi(M(x, y))),$$

where $(\mathcal{F}, H)$ *is an upper class of type* I *and*

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \right.$$
$$\left. \frac{d(x,Ty)d(x,y)}{1+d(x,Tx)+d(y,Tx)+d(y,Ty)} \right\}.$$

For this class of mappings we state a similar existence and uniqueness theorem.

Theorem 2 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *and* $\alpha : X \times X \to \mathbb{R}^+$ *be two mappings. Suppose that the following conditions are satisfied:*
(i) T *is an* α*-admissible mapping;*
(ii) T *is an* $(\alpha, \psi, \mathcal{F}, H)$*-rational type-II contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*
(iv) *either* T *is continuous, or* X *is* α*-regular.*
Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^* *. Further, if for all* $x, y \in F(T)$*, we have* $\alpha(x, y) \geq 1$*, then* T *has a unique fixed point in* X*.*

Proof. The proof can be done by the similar proof as the following Theorem 3. □

Definition 11 *Let* (X, d) *be a GMS and* $\mu : X \times X \to \mathbb{R}^+$ *a mapping. A mapping* $T : X \to X$ *is said to be a* $(\mu, \psi, \mathcal{F}, H)$*-subrational type-I contractive mapping if there exists a function* $\psi \in \Psi$*, such that for all* $x, y \in X$*, the following condition holds:*

$$H(1, d(Tx, Ty)) \leq \mathcal{F}(\mu(x, y), \psi(M(x, y))), \tag{10}$$

where $(\mathcal{F}, H)$ *is an upper class of type* I *and* $M(x, y)$ *is defined by* (2)*.*

Theorem 3 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a mapping and* $\mu : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*
(i) T *is a* μ*-subadmissible mapping;*

(ii) T *is a* $(\mu, \psi, \mathcal{F}, H)$*-subrational type-I contractive mapping;*
(iii) *there exists* $x_0 \in X$ *such that* $\mu(x_0, Tx_0) \leq 1$ *and* $\mu(x_0, T^2x_0) \leq 1$*;*
(iv) *either* T *is continuous, or* X *is* μ*-subregular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**.Further, if for all* $x, y \in F(T)$*, we have* $\mu(x, y) \leq 1$*, then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$ satisfy $\mu(x_0, Tx_0) \leq 1$ and $\mu(x_0, T^2x_0) \leq 1$. We construct a sequence $\{x_n\}$ in X as $x_n = T^n x_0 = Tx_{n-1}$, for $n \in \mathbb{N}^*$. It is clear that x_{n_0} is a fixed point of T if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. As a result, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is μ-subadmissible, then

$$\begin{aligned}\mu(x_0, Tx_0) = \mu(x_0, x_1) \leq 1 &\Rightarrow \mu(Tx_0, Tx_1) = \mu(x_1, x_2) \leq 1 \\ &\Rightarrow \mu(Tx_1, Tx_2) = \mu(x_2, x_3) \leq 1,\end{aligned}$$

and by induction, we get $\mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$.

By similar argument, on account of $\mu(x_0, T^2x_0) \leq 1$, we have $\mu(x_n, x_{n+2}) \leq 1$ for all $n \in \mathbb{N}$. Considering (10) with $x = x_n$ and $y = x_{n+1}$, we acquire that

$$\begin{aligned}H(1, d(x_{n+1}, x_{n+2})) &= H(1, d\left(Tx_n, Tx_{n+1}\right)) \\ &\leq \mathcal{F}(\mu\left(x_n, x_{n+1}\right), \psi(M(x_n, x_{n+1}))) \\ &\leq \mathcal{F}(1, \psi(M(x_n, x_{n+1}))),\end{aligned}$$

which follows that

$$d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})).$$

Similar to the process of proof of Theorem 1, we get (4). Next consider (10) with $x = x_{n-1}$ and $y = x_{n+1}$, it is easy to get (7). By similar proof of Theorem 1, we prove that there exists $x^* \in X$ such that (8) holds.

We shall show that the limit x^* of $\{x_n\}$ is a fixed point of T. First, we assume that T is continuous, then by (8), it is clear that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Due to Proposition 1, we obtain $x^* = Tx^*$, that is, x^* is a fixed point of T.

Now, we suppose that X is μ-subregular. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\mu(x_{n_k}, x^*) \leq 1$ for all $k \in \mathbb{N}^*$. Choosing $x = x_{n_k}$ and $y = x^*$ in (10) , we arrive at

$$\begin{aligned}H(1, d\left(x_{n_k+1}, Tx^*\right)) &= H(1, d\left(Tx_{n_k}, Tx^*\right)) \\ &\leq \mathcal{F}(\mu\left(x_{n_k}, x^*\right), \psi(M(x_{n_k}, x^*))) \\ &\leq \mathcal{F}(1, \psi(M(x_{n_k}, x^*))),\end{aligned}$$

which implies that (9). Similar argument of Theorem 1, we claim that x^* is a fixed point of T.

Finally, assume that x^* and y^* are two different fixed points of T. Then by the hypothesis, $\mu(x^*, y^*) \leq 1$. Hence, from (10) with $x = x^*$ and $y = y^*$ we conclude that

$$\begin{aligned} H(1, d(x^*, y^*)) &= H(1, d(Tx^*, Ty^*)) \\ &\leq \mathcal{F}(\mu(x^*, y^*), \psi(M(x^*, y^*))) \\ &\leq \mathcal{F}(1, \psi(M(x^*, y^*))), \end{aligned}$$

which establish that

$$d(x^*, y^*) \leq \psi(M(x^*, y^*)).$$

Hence by the same proof of as one in Theorem 1, T has a unique fixed point. □

Corollary 6 *Let* (X, d) *be a complete GMS,* $T : X \to X$ *a self mapping and* $\mu : X \times X \to \mathbb{R}^+$ *a given function. Suppose that the following conditions are satisfied:*

(i) T *is a* μ*-subadmissible mapping;*

(ii) *for all* $x, y \in X$,

$$d(Tx, Ty) \leq \mu(x, y)\psi(M(x, y)),$$

where $M(x, y)$ *is defined by* (2)*;*

(iii) *there exists* $x_0 \in X$ *such that* $\mu(x_0, Tx_0) \leq 1$ *and* $\mu(x_0, T^2x_0) \leq 1$*;*

(iv) *either* T *is continuous, or* X *is* μ*-subregular.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^**. Further, if for all* $x, y \in F(T)$*, we have* $\mu(x, y) \leq 1$*, then* T *has a unique fixed point in* X.

Proof. Choose $H(x, y) = xy^n$ and $\mathcal{F}(s, t) = s^n t^n$, $n \in \mathbb{N}$, then by Theorem 3, the desired proof is completed. □

The following example illustrates that Theorem 3 is inspired by [7].

Example 3 *Let* $X = \{1, 2, 3, 4\}$ *and define a mapping* $d : X \times X \to \mathbb{R}^+$ *by*

$$\begin{aligned} &d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0, \\ &d(1,2) = d(2,1) = 3, \\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1, \\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4. \end{aligned}$$

Clearly, d *is not a metric on* X *in view of*

$$3 = d(1,2) \geq d(1,3) + d(3,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, d *is a GM on* X*, further,* (X, d) *is a complete GMS. Define* $T : X \to X$ *by*

$$T1 = T2 = 1, \quad T3 = 2, \quad T4 = 3,$$

$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = \frac{9t}{10}$ *and* $\mu : X \times X \to \mathbb{R}^+$ *by*

$$\mu(x,y) = \begin{cases} 1, & \textit{if } x, y \in \{1,2\}, \\ 10, & \textit{others.} \end{cases}$$

Then, for $x, y \in \{1, 2\}$*, we have*

$$d(Tx, Ty) = 0 \leq \mu(x,y)\psi(M(x,y)).$$

On the one hand, for $x \in \{1, 2\}$ *and* $y = 3$ *we obtain that*

$$d(Tx, T3) = d(1,2) = 3$$

and

$$\begin{aligned} M(1,3) &= \max\left\{d(1,3), d(1,T1), d(3,T3), \frac{d(1,T1)d(3,T3)}{1+d(1,3)}, \frac{d(1,T1)d(3,T3)}{1+d(T1,T3)}\right\} \\ &= 1, \\ M(2,3) &= \max\left\{d(2,3), d(2,T2), d(3,T3), \frac{d(2,T2)d(3,T3)}{1+d(2,3)}, \frac{d(2,T2)d(3,T3)}{1+d(T2,T3)}\right\} \\ &= 3. \end{aligned}$$

Thus

$$\begin{aligned} d(T1,T3) &= 3 \leq 10 \cdot \frac{9}{10} \cdot 1 = 9 = \mu(1,3)\psi(M(1,3)), \\ d(T2,T3) &= 3 \leq 10 \cdot \frac{9}{10} \cdot 3 = 27 = \mu(2,3)\psi(M(2,3)). \end{aligned}$$

On the other hand, for $x \in \{1, 2\}$ *and* $y = 4$ *we obtain*

$$d(Tx, T4) = d(1,3) = 1$$

and

$$\begin{aligned} M(1,4) &= \max\left\{d(1,4), d(1,T1), d(4,T4), \frac{d(1,T1)d(4,T4)}{1+d(1,4)}, \frac{d(1,T1)d(4,T4)}{1+d(T1,T4)}\right\} \\ &= 4, \\ M(2,4) &= \max\left\{d(2,4), d(2,T2), d(4,T4), \frac{d(2,T2)d(4,T4)}{1+d(2,4)}, \frac{d(2,T2)d(4,T4)}{1+d(T2,T4)}\right\} \\ &= 6. \end{aligned}$$

Accordingly,

$$\begin{aligned} d(T1,T4) = 1 \leq 10 \cdot \frac{9}{10} \cdot 4 = 36 = \mu(1,4)\psi(M(1,4)), \\ d(T2,T4) = 1 \leq 10 \cdot \frac{9}{10} \cdot 6 = 54 = \mu(2,4)\psi(M(2,4)). \end{aligned}$$

For $x, y \in \{3,4\}$*, the contraction condition is obvious. Clearly,* T *satisfies the conditions of Theorem 3 (or Corollary 6) and hence* T *has a unique fixed point* $x = 1$.

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