Acta Univ. Sapientiae, Mathematica, **9**, 1 (2017) 13–25

DOI: 10.1515/ausm-2017-0002

Properties of nearly ω-continuous multifunctions

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Abstract. Erdal Ekici has introduced and studied nearly continuous multifunctions in [5]. The purpose of the present paper is to introduce and study upper and lower nearly ω-continuous multifunctions as a weaker form of upper and lower nearly continuous multifunctions. Basic characterizations, several properties of upper and lower nearly ω-continuous multifunctions are investigated.

1 Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with

2010 Mathematics Subject Classification: 54C10, 54C08, 54C05
Key words and phrases: nearly ω-continuous multifunctions, N-closed complement

such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Several characterizations and properties of ω-closed sets were provided in [1], [2], [7] and [8]. Recently, Zorlutuna [14] introduced and studied the concept of ω-continuous multifunctions in topological spaces. In this paper, we introduce and study a new class of multifunction called near ω-continuous multifunctions in topological spaces.

2 Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [8]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [8] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ, denoted by τ_ω. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \tau_\omega$ and $x \in A\}$ the neighborhood system at x in τ_ω. A point x of X is called a θ-cluster [12] point of $S \subset X$ if $\mathrm{Cl}(U) \cap S \neq \varnothing$ for every open subset of X containing x. The set of all θ-cluster points of S is called the θ-closure of S and is denoted by $\mathrm{Cl}_\theta(S)$. A subset S is said to be θ-closed if and only if $S = \mathrm{Cl}_\theta(S)$. The complement of a θ-closed set is said to be a θ-open set. The θ-interior [12] of A is defined as $\mathrm{Int}_\theta(A) = \{x \in X : \mathrm{Cl}(U) \subset A$ for some open set U containing $x\}$. By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 1 [14] *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be*

1. *upper* ω*-continuous if* $F^+(V) \in \omega O(X)$ *for each open set* V *of* Y*,*
2. *lower* ω*-continuous if* $F^-(V) \in \omega O(X)$ *for each open set* V *of* Y*.*

Definition 2 [4] *A subset* A *of a topological space* (X, τ) *is said to be* N*-closed if every cover of* A *by regular open sets of* X *has a finite subcover.*

Definition 3 [5] *A function* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

1. *upper nearly continuous at a point* $x \in X$ *if for each open set* V *containing* $F(x)$ *and having* N*-closed complement, there exists an open set* U *containing* x *such that* $F(U) \subset V$*.*
2. *lower nearly continuous at a point* $x \in X$ *if for each open set* V *of* Y *meeting* $F(x)$ *and having* N*-closed complement, there exists an open set* U *of* X *containing* x *such that* $F(u) \cap V \neq \emptyset$ *for each* $u \in U$*.*
3. *upper (resp. lower) nearly continuous on* X *if it has this property at every point of* X*.*

3 Upper (Lower) nearly ω-continuous multifunctions

Definition 4 *A function* $F : (X, \tau) \to (Y, \sigma)$ *is said to be:*

1. *upper nearly* ω*-continuous at a point* $x \in X$ *if for each open set* V *containing* $F(x)$ *and having* N*-closed complement, there exists an* ω*-open set* U *containing* x *such that* $F(U) \subset V$*.*
2. *lower nearly* ω*-continuous at a point* $x \in X$ *if for each open set* V *of* Y *meeting* $F(x)$ *and having* N*-closed complement, there exists an* ω*-open set* U *of* X *containing* x *such that* $F(u) \cap V \neq \emptyset$ *for each* $u \in U$*.*
3. *upper (resp. lower) nearly* ω*-continuous on* X *if it has this property at every point of* X*.*

Example 1 *Let* $X = Y = \{a, b, c, d\}$*,* $\tau_X = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $\sigma_Y = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, c\}\}$*. Consider the multifunction* $F : (X, \tau_X) \to (Y, \sigma_Y)$ *defined as follows:* $F(a) = \{c\}, F(b) = \{a, b\}, F(c) = \{d\}, F(d) = \{a, b\}$*. It is easy to see that:* F *is upper (resp. lower) nearly* ω*-continuous on* X*.*

Example 2 *Let $\mathfrak{R}$ be the set of real numbers with the discrete topology τ_d. Consider the multifunction $F : (\mathfrak{R}, \tau_d) \to (\mathfrak{R}, \sigma_d)$ defined as follows: $F(x) = \{x\}$ for all $x \in \mathfrak{R}$. It is easy to see that: F is upper (resp. lower) nearly ω-continuous on X.*

It is clear that every upper (resp. lower) nearly continuous multifunction is upper (resp. lower) nearly ω-continuous multifunction, but the converse is not true in general as shown in the following example.

Example 3 *In the Example 1, F is upper (resp. lower) nearly ω-continuous on X but is not upper (resp. lower) nearly continuous on X*

Theorem 1 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper nearly ω-continuous.*
2. *$F^+(V)$ is ω-open for each open set V of Y having N-closed complement.*
3. *$F^-(K)$ is ω-closed for every N-closed and closed subset K of Y.*
4. *$\omega\,\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure.*
5. *$F^+(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(B))$ for every subset B of Y such that $Y \setminus \mathrm{Int}(B)$ is N-closed.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be any open set of Y having N-closed complement. From (1), there exists an ω-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since any union of ω-open sets is ω-open, $F^+(V)$ is ω-open in (X, τ).
(2)⇒(3): Let K be any N-closed and closed set of Y. Then by (2), $F^+(Y \setminus K) = X \setminus F^-(K)$ is an ω-open set. Then it is obtained that $F^-(K)$ is an ω-closed set.
(3)⇒(4): Let B be any subset of Y having N-closed closure. By (3), we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = \omega\,\mathrm{Cl}(F^-(\mathrm{Cl}(B)))$. Hence $\omega\,\mathrm{Cl}(F^-(B)) \subset \omega\,\mathrm{Cl}(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.
(4)⇒(5): Let B be a subset of Y such that $Y \setminus \mathrm{Int}(B)$ is N-closed. Then by (4), we have $X \setminus \omega\,\mathrm{Int}(F^+(B)) = \omega\,\mathrm{Cl}(X \setminus F^+(B)) = \omega\,\mathrm{Cl}(F^-(Y \setminus B)) \subset F^-(\mathrm{Cl}(Y \setminus B)) = F^-(Y \setminus \mathrm{Int}(B)) = X \setminus F^+(\mathrm{Int}(B))$. Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(B))$.
(5)⇒(1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having N-closed complement. Then by (5), $x \in F^+(V) = F^+(\mathrm{Int}(V)) \subset \omega\,\mathrm{Int}(F^+(V))$.

There exists an ω-open set U containing x such that $U \subset F^{+}(V)$; hence $F(U) \subset V$. This shows that F is upper nearly ω-continuous. $\square$

Theorem 2 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following statements are equivalent:*

1. F *is lower nearly* ω*-continuous.*
2. $F^{-}(V)$ *is* ω*-open for each open set* V *of* Y *having* N*-closed complement.*
3. $F^{+}(K)$ *is* ω*-closed for every* N*-closed and closed set* K *of* Y*.*
4. $\omega\,\mathrm{Cl}(F^{+}(B)) \subset F^{+}(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having* N*-closed closure.*
5. $F^{-}(\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^{-}(B))$ *for every subset* B *of* Y *such that* $Y \backslash \mathrm{Int}(B)$ *is* N*-closed.*

Proof. The proof is similar to that of Theorem 1. $\square$

Corollary 1 *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is upper nearly* ω*-continuous (resp. lower nearly* ω*-continuous) if* $F^{-}(K)$ *is* ω*-closed (resp.* $F^{+}(K)$ *is* ω*-closed) for every* N*-closed set* K *of* Y*.*

Proof. Let G be any open set of Y having N-closed complement. Then $Y \backslash G$ is N-closed. By the hypothesis, $X \backslash F^{+}(G) = F^{-}(Y \backslash G) = \omega\,\mathrm{Int}(F^{-}(Y \backslash G)) = \omega\,\mathrm{Cl}(X \backslash F^{+}(G)) = X \backslash \omega\,\mathrm{Int}(F^{+}(G))$ and hence, $F^{+}(G) = \omega\,\mathrm{Int}(F^{+}(G))$. It follows from Theorem 1, that F is upper nearly ω-continuous. The proof of lower nearly ω-continuity is entirely similar. $\square$

Theorem 3 *Let* (Y, σ) *be a regular space. For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

1. F *is upper nearly* ω*-continuous;*
2. $F^{-}(\mathrm{Cl}_{\theta}(B))$ *is an* ω*-closed set in* X *for every subset* B *of* Y *such that* $\mathrm{Cl}_{\theta}(B)$ *is* N*-closed;*
3. $F^{-}(K)$ *is an* ω*-closed set in* X *for every* θ*-closed and* N*-closed set* K *of* Y*;*
4. $F^{+}(V)$ *is an* ω*-open set in* X *for every* θ*-open set* V *of* Y *having* N*-closed complement.*

Proof. (1)⇒(2): Let B be any subset of Y such that $\mathrm{Cl}_\theta(V)$ is N-closed. Then $\mathrm{Cl}_\theta(B)$ is closed and N-closed. By Theorem 1, $F^-(\mathrm{Cl}_\theta(B))$ is an ω-closed set in X.
(2)⇒(3): Let K be any N-closed and θ-closed set of Y. Then $K = \mathrm{Cl}_\theta(K)$ is N-closed. By (2), it follows that $F^-(K)$ is an ω-closed set in X
(3)⇒(4): Let V be any θ-open set of Y having N-closed complement. Then $Y\backslash V$ is θ-closed and N-closed and by (3), $F^-(Y\backslash V) = \omega\,\mathrm{Cl}(F^-(Y\backslash V))$. Therefore, $X\backslash F^+(V) = \omega\,\mathrm{Cl}(X\backslash F^+(V)) = X\backslash \omega\,\mathrm{Int}(F^+(V))$. Then $F^+(V)$ is an ω-open set in X.
(4)⇒(1): Let V be any open set of Y having N-closed complement. Since Y is regular, V is a θ-open set in Y having N-closed complement and by (4), we have $F^+(V)$ is an ω-open set in X. By Theorem 1, F is upper nearly ω-continuous. □

Theorem 4 *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. F *is lower nearly ω-continuous;*
2. $F^+(\mathrm{Cl}_\theta(B))$ *is an ω-closed set in X for every subset B of Y such that $\mathrm{Cl}_\theta(B)$ is N-closed;*
3. $F^+(K)$ *is an ω-closed set in X for every θ-closed and N-closed set K of Y;*
4. $F^-(V)$ *is an ω-open set in X for every θ-open set V of Y having N-closed complement.*

Proof. The proof is similar to that of Theorem 3. □

Definition 5 *A subset A of a topological space (X, τ) is said to be:*

(i) *α-regular [9] if for each $a \in A$ and any open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;*

(ii) *α-paracompact [13] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.*

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the multifunction $\mathrm{Cl}\,F : (X, \tau) \to (Y, \sigma)$ is defined as follows $(\mathrm{Cl}\,F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\omega\,\mathrm{Cl}\,F$.

Lemma 1 [14] *If* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*paracompact and* α-*regular for each* $x \in X$*, then for each open set* V *of* Y*,* $(\mathrm{Cl}\, F)^+(V) = (\omega\, \mathrm{Cl}\, F)^+(V) = F^+(V)$*.*

Theorem 5 *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*regular and* α-*paracompact for each* $x \in X$*. Then* F *is upper nearly* ω-*continuous if and only if* $G : (X, \tau) \to (Y, \sigma)$ *is upper nearly* ω-*continuous, where* G *denotes* $\mathrm{Cl}\, F$ *or* $\omega\, \mathrm{Cl}\, F$*.*

Proof. Suppose that F is upper nearly ω-continuous multifunction. Let V be any open set of Y having N-closed complement. Then by Lemma 1 and Theorem 1, we have $G^+(V) = F^+(V) = \omega\, \mathrm{Int}(F^+(V)) = \omega\, \mathrm{Int}(G^+(V))$. This shows that G is upper nearly ω-continuous. Conversely, suppose that G is upper nearly ω-continuous. Let V be any open set of Y having N-closed complement. Then by Lemma 1 and Theorem 1, we have $F^+(V) = G^+(V) = \omega\, \mathrm{Int}(G^+(V)) = \omega\, \mathrm{Int}(F^+(V))$. By Theorem 1, F is upper nearly ω-continuous. □

Lemma 2 [14] *If* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α-*paracompact* α-*regular for each* $x \in X$*, then for each open set* V *of* Y*,* $(\mathrm{Cl}\, F)^-(V) = (\omega\, \mathrm{Cl}\, F)^-(V) = F^-(V)$*.*

Theorem 6 *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is lower nearly* ω-*continuous if and only if* $G : (X, \tau) \to (Y, \sigma)$ *is lower nearly* ω-*continuous, where* G *denotes* $\mathrm{Cl}\, F$ *or* $\omega\, \mathrm{Cl}\, F$*.*

Proof. By using Lemma 2, this shown similarly as in Theorem 1. □

Remark 1 *It is well known that every upper (lower)* ω-*continuous multifunction is upper (lower) nearly* ω-*continuous, but the converse is not true in general as we can see in the following example.*

Example 4 *Let* $\mathfrak{R}$ *be the set of real numbers with the finite complement topology* τ_f *and the discrete topology* τ_d*. Consider the multifunction* $F : (\mathfrak{R}, \tau_f) \to (\mathfrak{R}, \tau_d)$*, defined by* $F(x) = \{x\}$*. Observe that* F *is an upper (lower) nearly* ω-*continuous multifunction in* $\mathfrak{R}$ *but* F *is not upper (lower)-continuous multifunction*

Now if we consider some additional condition, we can proof the converse.

Theorem 7 *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* (Y, σ) *has a base of sets having* N-*closed complements. If* F *is lower nearly* ω-*continuous, then* F *is lower* ω-*continuous.*

Proof. Let V be any open set of Y. By the hypothesis, $V = \underset{i\in I}{\cup} V_i$, where V_i is an open set having N-closed complement for each $i \in I$. By Theorem 1, $F^-(V_i)$ is ω-open in X for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V)$ is ω-open in X. Hence F is lower ω-continuous. □

Suppose that (X, τ), (Y, σ) and (Z, θ) are topological spaces. If $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 8 *Let $F : (X, \tau) \to (Y, \sigma)$ and $G : (Y, \sigma) \to (Z, \theta)$ be multifunctions. If F is upper ω-continuous (resp. lower ω-continuous) and G is upper nearly continuous (resp. lower nearly continuous), then $G \circ F : (X, \tau) \to (Z, \theta)$ is upper nearly ω-continuous (resp. lower nearly ω-continuous).*

Proof. Let V be any open set of V having N-closed complement. Since G is upper nearly continuous (resp. lower nearly continuous), by Theorem 2 of [5], $F^+(V)$ (resp. $F^-(V)$) is an open set of y. Since F is upper ω-continuous (resp. lower ω-continuous), $(G \circ F)^+(V) = F^+(G^+(V)) = \omega\,\mathrm{Int}(F^+(G^+(V))) = \omega\,\mathrm{Int}((G \circ F)^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V)) = \omega\,\mathrm{Int}(F^-(G^-(V))) = \omega\,\mathrm{Int}((G \circ F)^-(V))$). By Theorem 1 (resp. Theorem 2), F is upper nearly ω-continuous (resp. lower nearly ω-continuous). □

Definition 6 *A topological space (Y, σ) is said to be N-normal [5] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U$, $H \subset V$ and $U \cap V = \emptyset$.*

Definition 7 *A topological space (X, τ) is said to be ω-T_2 [2] if for each distinct points $x, y \in X$, there exist ω-open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Theorem 9 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper nearly ω-continuous multifunction satisfying the following conditions:*

1. *$F(x)$ is closed in Y for each $x \in X$,*
2. *$F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,*
3. *(Y, σ) is an N-normal space,*

then (X, τ) is ω-T_2.

Proof. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Theorem 1, we obtain, an ω-open set $F^+(U)$ in X containing x and an ω-open set $F^+(V)$ in X containing y and $F^+(U) \cap F^+(V) = \emptyset$. This shows that X is ω-T_2. □

Theorem 10 *Let (X, τ) be a topological space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F form (X, τ) into an N-normal space (Y, σ) satisfying the following conditions:*

1. *$F(x_1)$ and $F(x_2)$ are closed in Y,*

2. *F is upper nearly ω-continuous at x_1 and x_2, and*

3. *$F(x_1) \cap F(x_2) = \emptyset$,*

then (X, τ) is ω-T_2.

Proof. Let x_1 and x_2 be distinct points of X. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(x_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is upper nearly ω-continuous at x_1 and x_2, there exist U_1 and U_2 ω-open sets in X containing x_1 and x_2 respectively, such that $F(U_1) \subset V_1$ and $F(U_2) \subset V_2$. This implies that $U_1 \cap U_2 = \emptyset$. Hence (X, τ) is an ω-T_2-space. □

Theorem 11 *Let F and G be upper nearly ω-continuous and point closed multifunctions from a topological space X to a N-normal topological space Y. Then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is ω-closed in X.*

Proof. Let $x \in X \backslash A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a N-normal space, it follows that there exists disjoint open sets U and V having N-closed complements containing $F(x)$ and $G(x)$, respectively. Since F and G are upper nearly ω-continuous, then the sets $F^+(U)$ and $G^+(V)$ are open and contain x. Let $H = F + (U) \cup G + (V)$. Then H is an ω-open set containing x and $H \backslash A = \emptyset$. Hence, A is ω-closed in X. □

Definition 8 *A topological space (X, τ) is said to be N-connected [6] if X cannot be written as the union of two disjoint nonempty open sets having N-closed complements.*

Definition 9 *A topological space* (X, τ) *is said to be* ω*-connected* [2] *if* X *cannot be written as the union of two disjoint nonempty* ω*-open sets.*

Theorem 12 *Let* (X, τ) *be a topological space. If* $F : (X, \tau) \to (Y, \sigma)$ *is an upper nearly* ω*-continuous or lower nearly* ω*-continuous surjective multifunction such tat* $F(x)$ *is connected for each* $x \in x$ *and* (X, τ) *is* ω*-connected, then* (Y, σ) *is* N*-connected.*

Proof. Suppose that (Y, σ) is not N-connected. There exist nonempty open sets U and V of Y having N-closed complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

1. $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,
2. $F^+(U) \cap F^+(V) = \emptyset$,
3. $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are ω-open sets in X.
(i) In case F is upper nearly ω-continuous by Theorem 1, $F^+(U)$ and $F^+(V)$ are ω-open sets in X.
(ii) In case F is lower nearly ω-continuous by Theorem 2, $F^+(V)$ is ω-closed set in X because U is clopen in (Y, σ), therefore, $F^+(V)$ is ω-open in X. Similarly $F^+(U)$ is ω-open in X. Therefore (X, τ) is not ω-connected. □

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we define $D^+_{n\omega}(F)$ and $D^-_{n\omega}(F)$ as follows:
$D^+_{n\omega}(F) = \{x \in X : F$ is not upper nearly ω-continuous at $x\}$.
$D^-_{n\omega}(F) = \{x \in X : F$ is not lower nearly ω-continuous at $x\}$.

Theorem 13 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^+_{n\omega} &= \bigcup_{G \in \sigma NC} \{F^+(G) \backslash \omega \operatorname{Int}(F^+(G))\} \\ &= \bigcup_{B \in iNC} \{F^+(\operatorname{Int}(B)) \backslash \omega \operatorname{Int}(F^+(B))\} \\ &= \bigcup_{B \in NC} \{\omega \operatorname{Cl}(F^-(B)) \backslash F^-(\operatorname{Cl}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\omega \operatorname{Cl}(F^-(H)) \backslash F^-(H)\}, where \end{aligned}$$

σNC *is the family of all* σ*-open sets of* Y *having* N*-closed complement,*
iNC *is the family of all subsets* B *of* Y *such that* $Y \backslash \operatorname{Int}(B)$ *is* N*-closed,*
NC *is the family of all subsets* B *of* Y *having the* N*-closed closure,*
$\mathcal{F}$ *is the family of all closed and* N*-closed sets of* (Y, σ).

Proof. We shall only proof the first equality and the last equality since the proofs of other are similar to the first.
Let $x \in D^{+}_{n\omega}(F)$. Then, by Theorem 1, there exists an open set V of Y containing $F(x)$ and having N-closed complement such that $x \in \omega \operatorname{Int}(F^{+}(V))$. Therefore, $x \in F^{+}(V) \backslash \omega \operatorname{Int}(F^{+}(V)) \subset \bigcup\limits_{G \in \sigma NC} \{F^{+}(G) \backslash \omega \operatorname{Int}(F^{+}(G))\}$.
Conversely, let $x \in \bigcup\limits_{G \in \sigma NC} \{F^{+}(G) \backslash \omega \operatorname{Int}(F^{+}(G))\}$. Then there exists an open set V of Y having N-closed complement such that $x \in F^{+}(V) \backslash \omega \operatorname{Int}(F^{+}(V))$. By Theorem 1, $x \in D^{+}_{n\omega}(F)$. We prove the last equality.
$\bigcup\limits_{H \in \mathcal{F}} \{\omega \operatorname{Cl}(F^{-}(H)) \backslash F^{-}(H)\} \subset \bigcup\limits_{B \in NC} \{\omega \operatorname{Cl}(F^{-}(B)) \backslash F^{-}(\operatorname{Cl}(B))\} = D^{+}_{n\omega}(F)$.
Conversely, we have $D^{+}_{n\omega}(F) = \bigcup\limits_{B \in NC} \{\omega \operatorname{Cl}(F^{-}(B)) \cup \bigcup\limits_{H \in \mathcal{F}} \{\omega \operatorname{Cl}(F^{-}(H)) \backslash F^{-}(H)\}$. □

Theorem 14 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^{-}_{n\omega} &= \bigcup_{G \in \sigma NC} \{F^{-}(G) \backslash \omega \operatorname{Int}(F^{-}(G))\} \\ &= \bigcup_{B \in iNC} \{F^{-}(\operatorname{Int}(B)) \backslash \omega \operatorname{Int}(F^{-}(B))\} \\ &= \bigcup_{B \in NC} \{\omega \operatorname{Cl}(F^{+}(B)) \backslash F^{+}(\operatorname{Cl}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\omega \operatorname{Cl}(F^{+}(H)) \backslash F^{+}(H)\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 13 □

Definition 10 *Let* (X, τ) *be a bitopological space and* A *be a subset of* X*. The* ω*-frontier of* A*,* $\omega Fr(A)$*, is defined by* $\omega Fr(A) = \omega \operatorname{Cl}(A) \cap \omega \operatorname{Cl}(X \backslash A) = \omega \operatorname{Cl}(A) \backslash \omega \operatorname{Int}(A)$.

Theorem 15 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*,* $D^{+}_{n}\omega(F)$ *(resp.* $D^{-}_{n\omega}(F)$*) is identical with the union of* ω*-frontiers of the upper (resp. lower) inverse images of* σ_i *open sets containing (resp. meeting)* $F(x)$ *and having* N*-closed complement.*

Proof. We shall prove the first case since the proof of the second is similar.
Let $x \in D^{+}_{n\omega}(F)$. Then, there exists an open set V of Y containing $F(x)$ and

having N-closed complement such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every open set U containing x. Then $x \in \omega\,\mathrm{Cl}(X \backslash F^+(V))$. On the other hand, since $x \in F^+(V) \subset \omega\,\mathrm{Cl}(F^+(V))$ and hence $x \in \omega\mathrm{Fr}(F^+(V))$. Conversely, suppose that F is upper nearly ω-continuous at $x \in X$. Then, for any open set V of Y containing $F(x)$ and having N-closed complement, there exists an ω-open set containing x such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Therefore, we have $x \in U \subset \omega\,\mathrm{Int}(F^+(V))$. This contradicts to the fact that $x \in \omega\mathrm{Fr}(F^+(V))$. □

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Received: August 24, 2016