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Study on subclasses of analytic functions

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Abstract. By making use of new linear fractional differential operator, we introduce and study certain subclasses of analytic functions associated with Symmetric Conjugate Points and defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Inclusion relationships are established and convolution properties of functions in these subclasses are discussed.

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is called starlike if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq 0, (z \in \mathbb{U}).$$

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The class of starlike functions is denoted by S.

A function $f \in A$ is called convex if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 0, (z \in \mathbb{U}).$$

The class of convex functions is denoted by K.

A function $f \in A$ is called starlike of order ρ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \rho, (\rho > 0, z \in \mathbb{U}). \tag{2}$$

The class of starlike functions of order ρ is denoted by $SV^\star(\rho)$.

Similarly a function $f \in A$ is called convex of order ρ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \rho, (\rho > 0, z \in \mathbb{U}). \tag{3}$$

The class of starlike functions of order ρ is denoted by $KV(\rho)$.

It follows from (2) and (3) that $f \in KV(\rho)$ if and only if $zf'(z) \in SV^\star(\rho)$.

Let $f \in A$ and $g \in SV^\star(\rho)$, then $f \in A$ is called close-to-convex of order θ and type ρ if and only if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) \geq \theta, (0 \leq \theta, \rho < 1, z \in \mathbb{U}).$$

The class of close-to-convex of order θ and type ρ is denoted by $CV(\theta, \rho)$.

In 1959, Sakaguchi [1] introduced the following class of analytic functions:

A function $f \in A$ is called starlike with respect to symmetrical points, and its class is denoted by SV_s, if it satisfies the analytic criterion

$$\Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0, (z \in \mathbb{U}).$$

For more details we refer to study Shanmugam et al. [2], Chand and Singh [3] and Das and Singh [4] respectively.

In 1987, El-Ashwah and Thomas [5] introduced the following class of analytic functions:

A function $f \in A$ is called starlike with respect to symmetric conjugate points, and its class is denoted by SV_{sc}, if it satisfies the analytic criterion

$$\Re\left(\frac{zf'(z)}{f(z)-\overline{\overline{f(-z)}}}\right) > 0, \ (z \in \mathbb{U}).$$

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z)\,(z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$f(z) = g(w(z))\ z \in \mathbb{U}.$$

Indeed it is known that

$$f(z) \prec g(z)\ z \in \mathbb{U} \Rightarrow f(0) = g(0)\ and\ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z)\ z \in \mathbb{U} \Leftrightarrow f(0) = g(0)\ and\ f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ the Hadamard product (or convolution) $f * g$ is defined as usual by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let

$$\varphi(a, c; z) = z_2F(1, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1},\ (z \in \mathbb{U}; c \neq 0, -1, -2, .s),$$

where $(a)_k$ is the pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(k+a)}{\Gamma(a)} = \left\{ \begin{array}{ll} 1 & \text{if } k = 0 \\ a(a+1)(a+2).s(a+k-1) & \text{if } k \in N \end{array} \right\}.$$

In 1987, Owa and Srivastava [7] introduced the operator as follow

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z) = \varphi(2, 2-\alpha; z) * f(z), \\ \alpha &\neq 2, 3, 4, \ldots. \end{aligned} \tag{4}$$

Also note that $\Omega^0 f(z) = f(z)$ and $D_z^{\alpha} f(z)$ is the fractional derivative of order α given in [6].

For $f \in A$, we define the linear fractional differential operator as follow:

$$\begin{aligned}
I_{\lambda}^{0,\nu}(\alpha,\beta,\mu)f(z) &= f(z)\\
I_{\lambda}^{1,\nu}(\alpha,\beta,\mu)f(z) &= \left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)\Omega^{\alpha}f(z) + \left(\frac{\mu+\lambda}{\nu+\beta}\right) z(\Omega^{\alpha}f(z))'\\
I_{\lambda}^{2,\nu}(\alpha,\beta,\mu)f(z) &= I_{\lambda}^{\alpha}(I_{\lambda}^{1,\nu}(\alpha,\beta,\mu)f(z))\\
&\vdots\\
I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) &= I_{\lambda}^{\alpha}(I_{\lambda}^{n-1,\nu}(\alpha,\beta,\mu)f(z)).
\end{aligned} \tag{5}$$

If $f(z)$ is given by (1) then from (5) we have

$$\begin{aligned}
&I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z)\\
&= z + \sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^{n} a_k z^k.
\end{aligned} \tag{6}$$

Using (4) we conclude that

$$I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)]} * f(z), \tag{7}$$

where

$$\begin{aligned}
g_{\beta,\lambda}^{\mu,\nu}(z) &= \frac{z - \left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right) z^2}{(1-z)^2}\\
&= \left(z - \frac{\nu-\mu+\beta-\lambda}{\nu+\beta} z^2\right)(1+2z+3z^2+\cdots)\\
&= z + \left(1+\frac{\mu+\lambda}{\nu+\beta}\right) z^2 + \left(1+2\frac{\mu+\lambda}{\nu+\beta}\right) z^3 + \cdots\\
&\vdots\\
g_{\beta,\lambda}^{\mu,\nu}(z) &= z + \sum_{k=2}^{\infty}\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right) z^k.
\end{aligned} \tag{8}$$

$$\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z) = \text{n-times product}.$$

Similarly, we find the following:

$$g^{\mu}_{\beta,\lambda}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\beta + (\mu + \lambda)(k-1)}{\beta} \right) z^k. \tag{9}$$

$$g^{\nu}_{\beta,\lambda}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\nu + \lambda(k-1) + \beta}{\nu + \beta} \right) z^k. \tag{10}$$

$$g^{\nu}_{\beta}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\nu + (k-1) + \beta}{\nu + \beta} \right) z^k. \tag{11}$$

$$g^{\mu}_{\lambda}(z) = z + \sum_{k=2}^{\infty} \left(1 + (\mu + \lambda)(k-1)\right) z^k. \tag{12}$$

$$g_{\lambda}(z) = z + \sum_{k=2}^{\infty} \left(1 + \lambda(k-1)\right) z^k. \tag{13}$$

Further, a straightforward calculation reveals that many differential operators introduced in other papers are special cases of the differential operator defined by (6) which generalizes some well known differential operators.

1. $\beta = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Aouf et al. differential operator [8].
2. $\alpha = 0$, we obtain differential operator of Darus and Faisal [29].
3. $\beta = 0$, $\alpha = 0$, we obtain differential operator Darus and Faisal [30].
4. $\nu = 1$, $\beta = o$, $\mu = 0$, $\alpha = 0$, we obtain, Al-Oboudi differential operator [9].
5. $\nu = 1$, $\beta = o$, $\mu = 0$, $\lambda = 1$, $\alpha = 0$, we obtain, Sălăgean's operator [10].
6. $\beta = l$, $\mu = 0$, $\alpha = p$, we obtain, A. Catas operator [26].
7. $\nu = 1$, $\beta = o$, $\mu = 0$, we obtain, Al-Oboudi–Al-Amoudi operator [14, 25].
8. $\beta = 1$, $\lambda = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Cho–Srivastava operator [12, 13].
9. $\nu = 1$, $\beta = 1$, $\lambda = 1$, $\mu = 0$, $\alpha = 0$, we obtain, Uralegaddi–Somanatha operator [11].
10. $\nu = 1$, $\beta = o$, $\mu = 0$, $\lambda = 0$, $n = 1$, we obtain, Owa–Srivastava operator [7].

11. $\beta = l$, $\mu = 0$, $\alpha = p$, $\lambda = 1$, we obtain, Kumar et al. and Srivastava et al. operators [27, 28].

For $f \in A$, we define f_m by

$$f_m(z) = \frac{1}{2m} \sum_{k=0}^{m-1} \left[\omega^{-k} f(\omega^k z) + \omega^k \overline{f(\omega^k \overline{z})} \right], \tag{14}$$

where m be a positive integer and $\omega = \exp(2\pi/(m))$.

A function $f \in A$ is called λ-starlike with respect to $2m$-symmetric conjugate points and its class is denoted by $SV_m(\lambda)$, if it satisfy the analytic criterion

$$\Re\left(\frac{(1-\lambda)zf'(z) + \lambda(zf'(z))'}{(1-\lambda)f_m(z) + \lambda z f'_m(z)}\right) > 0, \ \ (z \in U, \ \lambda \geq 0),$$

where f_m is given by (14). For details about $SV_m(\lambda)$, we refer to study [15, 16, 17, 18].

By using (14), we have

$$\begin{aligned} f_m(z) &= \frac{1}{2m} \sum_{k=0}^{m-1} \left\{ \omega^{-k} f(\omega^k z) + \omega^k \overline{f(\omega^k \overline{z})} \right\}, \text{ implies} \\ I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(z) &= \frac{1}{2m} \sum_{k=0}^{m-1} \{ \omega^{-k} I_\lambda^{n,\nu}(\alpha, \beta, \mu) f(\omega^k z) \\ &\quad + \omega^k \overline{I_\lambda^{n,\nu}(\alpha, \beta, \mu) f(\omega^k \overline{z})} \}, \\ z(I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(z))' &= \frac{1}{2m} \sum_{k=0}^{m-1} \{ \omega^{-k} z (I_\lambda^{n,\nu}(\alpha, \beta, \mu) f(\omega^k z))' \\ &\quad + \omega^k z \overline{(D_\lambda^{n,\nu}(\alpha, \beta, \mu) f(\omega^k \overline{z}))'} \}, \\ I_\lambda^{n,\nu}(\alpha, \beta, \mu)(z f'_m(z)) &= \frac{1}{2m} \sum_{k=0}^{m-1} \{ I_\lambda^{n,\nu}(\alpha, \beta, \mu)(z f'(\omega^k z)) \\ &\quad + \overline{I_\lambda^{n,\nu}(\alpha, \beta, \mu)(z f'(\omega^k \overline{z})} \}, \\ I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(\omega^j z) &= \omega^j I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(z), \\ I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(\overline{z}) &= \overline{I_\lambda^{n,\nu}(\alpha, \beta, \mu) f_m(z)}. \end{aligned} \tag{15}$$

Next we introduce new subclasses of analytic functions in $\mathbb{U}$ associated with linear fractional differential operator $I_\lambda^{n,\nu}(\alpha, \beta, \mu) f(z)$, as follow;

Definition 1 *For* $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ *and* $\alpha, \beta, \lambda, \mu, \nu \geq 0$, *let* $\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ *denote the class of functions* f *defined by* (1) *and satisfying the analytic criterion*

$$\frac{z(I_{\lambda}^{n,\nu}(\alpha, \beta, \mu)f(z))'}{I_{\lambda}^{n,\nu}(\alpha, \beta, \mu)f_m(z)} \prec h(z), \quad z \in \mathbb{U},$$

where h *is a convex function in* $\mathbb{U}$ *with* $h(0) = 1$.

Definition 2 *Let* $\mathcal{KV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ *denote the class of functions* f *defined by* (1) *and satisfying the analytic criterion if* $\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)g_m(z)}{z} \neq 0$ *and*

$$\frac{z(I_{\lambda}^{n,\nu}(\alpha, \beta, \mu)f(z))'}{I_{\lambda}^{n,\nu}(\alpha, \beta, \mu)g_m(z)} \prec h(z), \quad z \in \mathbb{U},$$

for some $g \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$.

Remark 1 *In 2010, F. M. Al-oboudi* [25], *introduced certain subclasses of analytic functions which contains only functions of the form given in* (13), *but there were infinite analytic functions in the open unit disk* $\mathbb{U}$, *of the form given in* (8), (9), (10), (11) *and* (12) *respectively, which were out of range of the classes given in [25]. Therefore it was necessary to find out or to introduce a new differential operator of the form* (6). *We introduce the subclasses* $\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ *and* $\mathcal{KV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ *of analytic functions by using such differential operator, with a different approach that includes all functions of the form given in* (8) *to* (13).

2 Main results

In this section, we have discussed inclusion relations as well as convolution properties for the function belonging to the classes $\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ and $\mathcal{KV}_{m,\lambda}^{n,\nu}(\alpha, \beta, \mu)(h)$ respectively.

Lemma 1 [19] *Let* f *and* g *be starlike functions of order* $1/2$ *then so is* $f * g$.

Lemma 2 [20] *Let* P *be a complex function in* $\mathbb{U}$ *with* $\Re(P(z)) > 0$ *for* $z \in \mathbb{U}$ *and let* h *be a convex function in* $\mathbb{U}$. *If* p *is analytic in* $\mathbb{U}$ *with* $p(0) = h(0)$ *and if*

$$p(z) + P(z)zp'(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Lemma 3 [21] *Let* $c > -1$ *and let* $I_c : A \to A$ *be the integral operator defined by* $F = I_c(f)$, *where*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

Let h *be a convex function, with* $h(0) = 1$ *and then* $\Re(h(z) + c) > 0$, $z \in \mathbb{U}$. *If* $f \in A$ *and* $\frac{zf'(z)}{f(z)} \prec h(z)$, *then*

$$\frac{zF'(z)}{F(z)} \prec q(z) \prec h(z),$$

where q *is univalent and satisfies the differential equation*

$$q(z) + \frac{zq'(z)}{q(z) + c} = h(z).$$

Lemma 4 [22] *Let* f *and* g, *respectively be in the classes* K *and* S, *then for every function* $F \in A$, *we have*

$$\frac{(f(z) * g(z)F(z))}{(f(z) * g(z))} \in \overline{co}(F(\mathbb{U})), \;\; z \in \mathbb{U},$$

where $\overline{co}$ *denotes the closed convex hull.*

Lemma 5 [22] *Let* f *and* g *be univalent starlike of order* $\frac{1}{2}$ *for every function* $F \in A$, *we have*

$$\frac{(f(z) * g(z)F(z))}{(f(z) * g(z))} \in \overline{co}(F(\mathbb{U})), \;\; z \in \mathbb{U},$$

where $\overline{co}$ *denotes the closed convex hull.*

Theorem 1 *Let* h *be a convex function in* $\mathbb{U}$ *with* $h(0) = 1$, $\overline{h(\overline{z})} = h(z)$ *and let* $\mu + \lambda \geq \nu + \beta$, *if* $f \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$ *then*

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu) f_m(z))'}{I^{n,\nu}_{\lambda}(\alpha, \beta, \mu) f_m(z)} \prec h(z), \, z \in \mathbb{U}. \tag{16}$$

Moreover, if $\Re\left(h(z) + \frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\right) > 0$ *in* $\mathbb{U}$ *then*

$$\frac{z\big(I^{n-1,\nu}_{\lambda}(\alpha, \beta, \mu((\Omega^{\alpha} f_m(z))\big)'}{I^{n-1,\nu}_{\lambda}(\alpha, \beta, \mu)(\Omega^{\alpha} f_m(z))} \prec q(z) \prec h(z), \, z \in \mathbb{U}. \tag{17}$$

Where q *is the univalent solution of the differential equation*

$$q(z)+\frac{zq'(z)}{h(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}}=h(z),\ q(0)=1. \tag{18}$$

Proof. Because

$$\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)=\left\{f\in A:\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z),\ z\in\mathbb{U}\right\}.$$

It is remaining to show that $f_m\in\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)$, since

$$f_m(z)=\frac{1}{2m}\sum_{k=0}^{m-1}\left\{\omega^{-k}f(\omega^k z)+\omega^k\overline{f(\omega^k\overline{z})}\right\}.$$

As

$$\begin{aligned}f\in\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha,\beta,\mu)&\Rightarrow\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z)\\&\Rightarrow\frac{(D^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z).\end{aligned}$$

After replacing z by $\omega^j z$ and $\omega^j\overline{z}$, we get

$$\omega^{-j}\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\prec h(z),\ \text{and}\ \omega^j\overline{\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j\overline{z}))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z), \tag{19}$$

because

$$\overline{h(\overline{z})}=h(z),\ I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(\overline{z})=\overline{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)},\ \text{and}$$

$$I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(\omega^j z)=\omega^j I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z).$$

Using (19) we have

$$\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j z))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}+\omega^j\overline{\frac{(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)zf'(\omega^j\overline{z}))}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z),$$

implies

$$\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(\omega^j z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}+\omega^j\overline{\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(\omega^j\overline{z}))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}}\prec h(z),$$

since we have

$$z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))' = \frac{1}{2m}\sum_{k=0}^{m-1}\left\{\omega^{-k}z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k z))' + \omega^k z(\overline{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^k\overline{z})})'\right\}$$

and this implies

$$\frac{1}{2m}\sum_{k=0}^{k-1}\omega^{-j}\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^j z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} + \omega^j\frac{\overline{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(\omega^j\overline{z}))'}}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} = \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)},$$

therefore

$$\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \prec h(z).$$

Hence (16) is satisfied.

From (5) we have

$$I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) = I_\lambda^{\alpha}(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)f(z)),$$

implies

$$I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z) = \left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) + \left(\frac{\mu+\lambda}{\nu+\beta}\right)z(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))',$$

implies

$$(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) = \frac{\nu+\beta}{(\mu+\lambda)z^{(\frac{\mu+\lambda}{\nu+\beta})-1}}\int_0^z t^{(\frac{\mu+\lambda}{\nu+\beta})-1} I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(t)\,dt.$$

Applying Lemma 3, we have

$$f(z) = I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z),\ \ F(z) = (I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)),$$

and

$$\frac{zf'(z)}{f(z)} = \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \prec h(z)(\text{proved})$$

with

$$\Re\big(h(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big)>0.$$

Therefore

$$\frac{z\big(I_\lambda^{n-1,\nu}(\alpha,\beta,\mu((\Omega^\alpha f_m(z))\big)'}{I_\lambda^{n-1,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}\prec q(z),$$

and q satisfied the equation

$$q(z)+\frac{zq'(z)}{h(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}}=h(z).$$

Hence proved. □

Corollary 1 *Let* $\Re\big(h(z)\big)>0$, *if* $f\in\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$ *then* $I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m\in S$ *and hence* $I_\lambda^{n,\nu}(\alpha,\beta,\mu)f$ *is close to convex function.*

Theorem 2 *Let* $\Re\big(h(z)\big)>0$ *and* $\overline{h(\overline{z})}=h(z)$ *then the following inclusions hold*

$$\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}_{m,\lambda}^{n-1,\nu}(\alpha,\beta,\mu)(h).$$

Proof. Let $f\in\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)$. To prove $\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}_{m,\lambda}^{n,\nu}$ $(\alpha,\beta,\mu)(h)$, it is enough to show that $f\in\mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$.

Applying Theorem 1, if $f\in\mathcal{SV}_{m,\lambda}^{n+1,\nu}(\alpha,\beta,\mu)(h)$ then

$$\frac{z(I_\lambda^{n+1,\nu}(\alpha,\beta,\mu)f_m(z))'}{I_\lambda^{n+1,\nu}(\alpha,\beta,\mu)f_m(z)}\prec h(z),\ z\in\mathbb{U},$$

if Moreover, if $\Re\big(h(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big)>0$ in $\mathbb{U}$ then

$$\frac{z\big(I_\lambda^{n,\nu}(\alpha,\beta,\mu((\Omega^\alpha f_m(z))\big)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}\prec q(z)\prec h(z),\ z\in\mathbb{U},$$

and $\Re\big(q(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}\big)>0$ in $\mathbb{U}$. Let

$$p(z)=\frac{z\big(I_\lambda^{n,\nu}(\alpha,\beta,\mu((\Omega^\alpha f(z))\big)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}.$$

Then p is analytic in U and satisfies

$$p(z)+\frac{zp'(z)}{q(z)+\frac{\nu+\beta-\mu-\lambda}{\mu+\lambda}}\prec h(z),$$

by using Lemma 2 $p(z)\prec h(z)$, implies $\Omega^{\alpha}f(z)\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$, applying Theorem 4 implies $f(z)\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$. Hence

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h).$$

Similarly we can show that $\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)$, and $\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-2,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)$ and so on, therefore

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(h).s,$$

which generalized the Al-Amiri et al. [16] results. □

Corollary 2 *Taking* $h(z)=\frac{1+z}{1-z}$, *in Theorem 2, then*

$$\mathcal{SV}^{n+1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})\subseteq\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})\subseteq\mathcal{SV}^{n-1,\nu}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z}).$$

Implies that $\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(\frac{1+z}{1-z})$ *are starlike functions with respect to symmetric conjugate points.*

Theorem 3 *If* $f\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$ *then* $f*g\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$ *where* $\Re\big(h(z)\big)>0$ *and* g *is a convex function with real coefficients in* $\mathbb{U}$.

Proof. Since $f\in\mathcal{SV}^{n,\gamma}_{m,\lambda}(\alpha,\beta,\mu)(h)$, applying Theorem 1, we get $I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)$ $f_m(z)\in S$. Using the convolution properties we have

$$\begin{aligned}
&\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f*g)(z))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f_m*g)(z)}=\frac{z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f(z)*g(z)))'}{I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)(f_m(z)*g(z))}\\
&=\frac{g(z)*(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\\
&=\frac{g(z)*(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'/I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z))I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}\\
&=\frac{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)(z(I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f(z))'/I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z))}{g(z)*I^{n,\nu}_{\lambda}(\alpha,\beta,\mu)f_m(z)}.
\end{aligned}$$

Implies that

$$\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(f*g)(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(f_m*g)(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m}(U)\right) \subseteq h(U), \quad z \in U.$$

Hence

$$f*g \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h).$$

□

Theorem 4 *If* $\Omega^\alpha f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$ *then* $f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$, *where* $\Re\big(h(z)\big) > 0, \overline{h(\overline{z})} = h(z)$.

Proof. Let $\Omega^\alpha f(z) \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha,\beta,\mu)(h)$, applying Theorem 1, implies

$$\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z)} \prec h(z),$$

or

$$(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m(z) \in S.$$

Because

$$\Omega^\alpha f(z) = \varphi(2, 2-\alpha; z) * f(z)$$

and

$$I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z).s\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)]} *f(z),$$

therefore we can write that

$$\begin{aligned} I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z) &= \varphi(2-\alpha,2;z) * D_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z)) \\ z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))' &= \varphi(2-\alpha,2;z) * z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z))', \end{aligned} \tag{20}$$

where $\varphi(2-\alpha,2;z) \in K$.

Using (20) we have

$$\begin{aligned} &\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} = \frac{\varphi(2-\alpha,2;z) * z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z)))'}{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}, \\ &\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \\ &\quad = \frac{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))\frac{(z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f(z)))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}}{\varphi(2-\alpha,2;z) * I_\lambda^{n,\nu}(\alpha,\beta,\mu)(\Omega^\alpha f_m(z))}, \\ &\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)\Omega^\alpha f_m}(\mathbb{U})\right) \subseteq h(\mathbb{U}), \quad z \in \mathbb{U}. \end{aligned} \tag{21}$$

Hence

$$f(z) \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h).$$

□

Theorem 5 *Let* $0 \le \alpha_1 < \alpha < 1$, *and* $\mathrm{Re}(h(z)) > \frac{1}{2}$, *then the following inclusions hold*

$$\mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h) \subseteq \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha_1, \beta, \mu)(h)\,.$$

Proof. Let $f(z) \in \mathcal{SV}^{n,\nu}_{m,\lambda}(\alpha, \beta, \mu)(h)$, from (7) we have

$$I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z) = \underbrace{[\varphi(2, 2-\alpha; z) * g^{\mu,\nu}_{\beta,\lambda}(z).s\varphi(2, 2-\alpha; z) * g^{\mu,\nu}_{\beta,\lambda}(z)]} * f(z),$$

implies

$$I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z) = \underbrace{[\varphi(2, 2-\alpha_1; z) * g^{\mu,\nu}_{\beta,\lambda}(z).s\varphi(2, 2-\alpha_1; z) * g^{\mu,\nu}_{\beta,\lambda}(z)]} * f(z),$$

$$\begin{aligned} I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z) = &\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ &* I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z), \end{aligned}$$

implies that

$$\begin{aligned} z(I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z))' = &\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ &* z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))', \end{aligned}$$

applying same technique, we have

$$\begin{aligned} I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f_m(z) = &\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} \\ &* I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f_m(z), \end{aligned}$$

since Y. Ling and S. Ding [24] already proved that $\varphi(2-\alpha, 2-\alpha_1; z) \in S(1/2)$, therefore by Lemma 1 we get $[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)] \in S(1/2)$.

From last two equations we get

$$\frac{z(I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f(z))'}{I^{n,\nu}_{\lambda}(\alpha_1, \beta, \mu)f_m(z)} = \frac{\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} * z(I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f(z))'}{\underbrace{[\varphi(2-\alpha, 2-\alpha_1; z) * .s * \varphi(2-\alpha, 2-\alpha_1; z)]} * I^{n,\nu}_{\lambda}(\alpha, \beta, \mu)f_m(z)},$$

after simplification and using Lemma 5 we deduce that

$$\frac{z(I_\lambda^{n,\nu}(\alpha_1,\beta,\mu)f(z))'}{I_\lambda^{n,\nu}(\alpha_1,\beta,\mu)f_m(z)} \in \overline{co}\left(\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu)f)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f_m}(\mathbb{U})\right) \subseteq h(\mathbb{U}), \quad z \in \mathbb{U},$$

therefore

$$f \in \mathcal{SV}_{m,\lambda}^{n,\nu}(\alpha_1,\beta,\mu)(h).$$

Hence proved. □

Remark 2

1. *When* $\nu = 1$, $\mu = \beta = 0$, $\mathcal{SV}_{m,\lambda}^{n,1}(\alpha,0,0)(h) = \mathcal{SV}_{m,\lambda}^{n}(\alpha)(h)$, *the classes of functions related to starlike functions with respect to symmetric conjugate points, defined and studied by M. K. Al-Oboudi* [25].

2. *For* $n = m = \nu = 1$, $\mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,\lambda}^{1,1}(\alpha,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{\lambda,\alpha}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to conjugate points, defined and studied by Radha* [23].

3. *For* $n = m = \nu = \lambda = 1$, $\alpha = \mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,1}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}\left(\frac{1+z}{1-z}\right)$, *the class of starlike functions with respect to conjugate points, defined and studied by El-Ashwah and Thomas* [5].

4. *For* $n = \nu = 1$, $\alpha = \mu = \beta = 0$, *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{m,\lambda}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{m,\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to* $2m$*-symmetric conjugate points, defined and studied by Al-Amiri et al.* [15, 16].

5. *For* $m = \nu = 1$, $n = \mu = \beta = 0$, $\mathcal{SV}_{1,\lambda}^{0,1}(\alpha,0,0)(h) = \mathcal{SV}(\alpha)(h)$, *the class of functions related to starlike functions with respect to conjugate points, defined and studied by Ravichandran* [31].

6. *For* $m = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{1,\lambda}^{1,1}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{SV}^{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-starlike functions with respect to conjugate points, defined and studied by Radha* [23].

7. *For* $\lambda = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{SV}_{m,1}^{1,1}(0,0,0)(h) = \mathcal{SV}^{m}\left(\frac{1+z}{1-z}\right)$, *the class of starlike functions with respect to* $2m$*-symmetric conjugate points, defined by Al-Amiri et al.* [16].

8. *For* $\nu = 1$, $\mu = \beta = 0$, $\mathcal{KV}_{m,\lambda}^{n,1}(\alpha,0,0)(h) = \mathcal{KV}_{\lambda}^{m,n}(\alpha)(h)$, *the classes of functions related to starlike functions with respect to symmetric conjugate points, defined and studied by M.K. Al-Oboudi* [25].

9. *For* $n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{KV}^{1,1}_{m,\lambda}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{KV}^{m}_{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-close to convex functions with respect to symmetric conjugate points, defined and studied by Al-Amiri et al.* [16].

10. *For* $m = n = \nu = 1$, $\alpha = \mu = \beta = 0$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{KV}^{1,1}_{1,\lambda}(0,0,0)\left(\frac{1+z}{1-z}\right) = \mathcal{KV}^{\lambda}\left(\frac{1+z}{1-z}\right)$, *the class of* λ*-close to convex functions with respect to symmetric conjugate points, defined and studied by Radha* [23].

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