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Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces

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Abstract. In this paper we prove the existence of the unique fixed point for the pair of weakly compatible self-mappings satisfying some Φ-type contractive conditions in the framework of S-metric spaces. Our results generalize, extend, unify, complement and enrich recently fixed point results in existing literature.

1 Introduction and preliminaries

In 1922. Banach [2] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become

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a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups. Considering different mappings etc. Many mathematic problems require one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [4, 5, 6, 8, 9, 13, 20]. Sedghi et al [18] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In this paper, we prove a coupled coincidence fixed point theorem in the setting of a generalized metric space. First, we present some basic properties of S-metric spaces.

Following is the definition of generalized metric spaces or S-metric spaces.

Definition 1 [19] *Let X be a nonempty set. An S-metric on X is a function $S : X\times X\times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ *for all* $x, y, z, a \in X$.

The pair (X, S) *is called an S-metric space.*

Some examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

(3) Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, y) + d(y, z)$ is an S-metric on X.

Lemma 1 [19], [7] *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z) \text{ and } S(x, x, z) \leq 2S(x, x, y) + S(z, z, y) \text{ for all } x, y, z \in X.$$

Also, $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2 [19] *Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows respectively:*

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(y, y, x) \leq r\}. \end{aligned}$$

Example 1 [19] *Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Thus $B_s(1, 2) = \{y \in \mathbb{R} : S(y, y, 1) < 2\} = (0, 2)$.*

Definition 3 [19] *Let (X, S) be an S-metric space, and $A \subseteq X$.*

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subseteq A$, then the subset A is called open subset of X.

(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \varepsilon$ whenever $n \geq n_0$ and we denote this $\lim_{n \longrightarrow \infty} x_n = x$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

(5) The S-metric spaces (X, S) is said to be complete if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X(induced by the S-metric S).

Definition 4 [1] *Let f and g be single-valued self mappings on a set X. If $\omega = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and ω is called a point of coincidence of f and g.*

Definition 5 [10] *Let* f *and* g *be a single-valued self mappings on a set* X*. Mappings* f *and* g *are said to be weakly compatible if* $fx = gx$ *implies* $fgx = gfx$*,* $x \in X$*.*

Proposition 1 [1] *Let* f *and* g *be weakly compatible self mappings on a set* X*. If* f *and* g *have a unique point of coincidence* $\omega = fx = gx$*, then* ω *is the unique common fixed point of* f *and* g*.*

2 Common fixed point theorems

In 1977, Matkowski [12] introduced the Φ-maps as the following : let Φ be the set of all functions ϕ such that $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\lim\limits_{n \longrightarrow \infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$. If $\phi \in \Phi$, then ϕ is called a $\Phi-$map. Furthermore, if ϕ is a Φ-map, then

(i) $\phi(t) < t$ for all $t \in (0, \infty)$,

(ii) $\phi(0) = 0$.

From now on, unless otherwise stated, ϕ is meant the Φ-map.

Lemma 2 [15], [16] *Let* (X, S) *be a* $S-$*metric space and let* $\{x_n\}$ *be a sequence in it such that*

$$\lim_{n \to \infty} S(x_{n+1}, x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ *is not a Cauchy sequence, then there exist an* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$*,* $n_k > m_k > k$ *of positive integers such that the following sequences tend to* ε *when* $k \to \infty$ *:*

$$S(x_{m_k}, x_{m_k}, x_{n_k}), S(x_{m_k}, x_{m_k}, x_{n_k+1}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$
$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}), S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}), \ldots.$$

Proof. Suppose that the sequence $\{x_n\}$ is not a Cauchy. Then, there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$, such that for every $k \in \mathbb{N}$ and $n_k > m_k > k$ the following is satisfied:

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } S(x_{m_k}, x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then, using Lemma 1 and (S3) we have

$$\begin{aligned}\varepsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k}) \\ &= S(x_{n_k}, x_{n_k}, x_{m_k}) \\ &\leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}) \\ &< 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + \varepsilon,\end{aligned}$$

and

$$\varepsilon \leq \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) \leq \varepsilon.$$

Therefore $\lim_{k\to\infty} S(x_{n_k}, x_{n_k}, x_{m_k}) = \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$. Further, as

$$|S(x_{n_k}, x_{n_k}, x_{m_k}) - S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k})| \leq 2S(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k})$$

we obtain that

$$\lim_{k\to\infty} S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k}) = \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_{k+1}}) = \varepsilon.$$

Analogous, it can be proved that

$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k}\right), S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right), S\left(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}\right), \ldots.$$

tend to ε. □

Theorem 1 *Let (X, S) be a S-metric space. Suppose that the mapping $f, g : X \to X$ satisfy*

$$S(fx, fy, fz) \leq \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}), \tag{1}$$

for all $x, y, z \in X$. If the range of g contains the range of f, and one of $f(X)$ or $g(X)$ is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Assume that f and g satisfy the condition (1). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that

$gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\ &\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\ &\quad S(gx_n, gx_n, fx_n)\}) \\ &\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}) \\ &= \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\}). \end{aligned}$$

If $\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\} = S(gx_n, gx_n, gx_{n+1})$, then

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_n, gx_n, gx_{n+1})) < S(gx_n, gx_n, gx_{n+1}),$$

which leads to a contradiction. This implies that

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\ &\leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) \\ &\leq \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1})) \\ &\vdots \\ &\leq \phi^n(S(gx_0, gx_0, gx_1)). \end{aligned}$$

So we have $\lim_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S\left(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}\right) \text{ and } S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right), \tag{2}$$

Putting now in (1) $x = y = x_{m_k}, z = x_{n_k}$ we obtain

$$\begin{aligned} &S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) \\ =\ &S(fx_{m_k}, fx_{m_k}, fx_{n_k}) \\ \leq\ &\phi(\{\max\{S(gx_{m_k}, gx_{m_k}, fx_{m_k}), S(gx_{m_k}, gx_{m_k}, fx_{m_k}), S(gx_{n_k}, gx_{n_k}, fx_{n_k})\}\}) \\ =\ &\phi(\{\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\}\}). \end{aligned}$$

If $\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\} = S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})$, and since $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ we have

$$\begin{aligned} S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) &\leq \phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) \\ &< S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}). \end{aligned}$$

Letting $k \to \infty$ we obtain

$$\varepsilon \leq \lim_{k\to\infty} \phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) \leq 0.$$

A contradiction.

Analogous, if $\max\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\} = S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})$ we got a contradiction.

So, it follows that $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (1), we have

$$\begin{aligned} S(gx_n, gx_n, fp) &= S(fx_{n-1}, fx_{n-1}, fp) \\ &\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\ &\quad S(gp, gp, fp)\}) \\ &= \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\}). \end{aligned}$$

Case 1.

$$\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\} = S(gx_{n-1}, gx_{n-1}, gx_n),$$

we obtain that

$$S(gx_n, gx_n, fp) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) < S(gx_{n-1}, gx_{n-1}, gx_n).$$

By taking $n \to \infty$, we have $S(gp, gp, fp) = 0$ and so $gp = fp$.

Case 2.

$$\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\} = S(gp, gp, fp),$$

we obtain that

$$S(gx_n, gx_n, fp) \leq \phi(S(gp, gp, fp)).$$

By taking $n \to \infty$, we have $S(gp, gp, fp) \leq \phi(S(gp, gp, fp)) < S(gp, gp, fp)$, which leads to a contradiction. Therefore $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fl = gl$ for some $l \in X$. By applying (1), it follows that

$$\begin{aligned} S(gp, gp, gl) &= S(fp, fp, fl) \\ &\leq \phi(max\{S(gp, gp, fp), S(gp, gp, fp), S(gl, gl, fl)\}) \\ &= 0. \end{aligned}$$

Therefore $gp = gl$. This implies that f and g have a unique point of coincidence. By Proposition 1, we can conclude that f and g have a unique common fixed point. □

Corollary 1 *Let* (X, S) *be a* S*-metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq k \max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\},$$

for all $x, y, z \in X$ *where* $0 \leq k < 1$*. If the range of* g *contains the range of* f *and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Putting $\phi(t) = kt, t \geq 0, 0 \leq k < 1$ in (1), the result follows. □

Example 2 *Let* $X = [0, 2]$ *and* $S(x, y, z) = max\{|x - y|, |y - z|, |x - z|\}$ *and* $\phi \in \Phi$*. Define* $f, g : X \to X$ *by*

$$fx = 1 \qquad \text{and} \qquad gx = 2 - x.$$

We obtain that f *and* g *satisfy (1) in Theorem 1. Indeed, we have*

$$S(fx, fy, fz) = 0,$$
$$\phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}) = \phi(\max\{|1 - x|, |1 - y|, |1 - z|\}).$$

It is obvious that the range of g *contains the range of* f *and* $g(X)$ *is a complete subspace of* (X, S)*. Furthermore,* f *and* g *are weakly compatible. Thus all assumptions in Theorem 1 are satisfied. This implies that* f *and* g *have a unique common fixed point fixed point which is* $x = 1$*.*

Theorem 2 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \max\{\phi(S(gx, gx, fx)), \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\},$$

for all $x, y, z \in X$*. If the range of* g *contains the range of* f*, and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. The proof is very similar to the proof of Theorem 1 so we omitted it. □

Theorem 3 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \phi(S(gx, gy, gz)), \tag{3}$$

for all $x, y, z \in X$*, where* ϕ *satisfies* $\lim_{s \to t+} \phi(s) < t$ *for all* $t > 0$*. If the range of* g *contains the range of* f*, and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(fx_{n-1}, fx_{n-1}, fx_n) \\ &\leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) \\ &\leq \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1})) \\ &\vdots \\ &\leq \phi^n(S(gx_0, gx_0, gx_1)). \end{aligned}$$

This implies that $\lim_{n \to \infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S\left(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}\right) \text{ and } S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right), \tag{4}$$

Putting now in (3) $x = y = x_{m_k}, z = x_{n_k}$, and since $S(gx_{m_k}, gx_{m_k}, gx_{n_k}) > 0$ we obtain

$$\begin{aligned} S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) &= S(fx_{m_k}, fx_{m_k}, fx_{n_k}) \\ &\leq \phi(S(gx_{m_k}, gx_{m_k}, gx_{n_k})). \end{aligned}$$

Letting $k \to \infty$ and using the assumption of the mapping ϕ we obtain

$$\begin{aligned} \varepsilon \leq \lim_{k\to\infty} \phi\left(S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\right) &= \lim_{S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\to\varepsilon^+} \phi\left(S\left(gx_{m_k}, gx_{m_k}, gx_{n_k}\right)\right) \\ &= \lim_{t\to\varepsilon^+} \phi\left(t\right) < \varepsilon. \end{aligned}$$

A contradiction. Therefore, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By (3) we have

$$\begin{aligned} S(gp, gp, fp) &\leq 2S(gp, gp, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, fp) \\ &\leq 2S(gp, gp, gx_{n+1}) + \phi(S(gx_n, gx_n, gp)) \\ &\leq 2S(gp, gp, gx_{n+1}) + S(gx_n, gx_n, gp). \end{aligned}$$

By taking $n \to \infty$, we have $S(gp, gp, fp) = 0$ and so $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. Assume that $gp \neq gq$. By applying (3), it follows that

$$\begin{aligned} S(gp, gp, gq) &= S(fp, fp, fq) \\ &\leq \phi(S(gp, gp, gq)) \\ &< S(gp, gp, gq), \end{aligned}$$

which leads to a contradiction. Therefore $gp = gq$. This implies that f and g have a unique point of coincidence. By Proposition 1, we can conclude that f and g have a unique common fixed point. □

By setting g to be the identity function on X, we immediately have the following corollary. This result extends and generalizes Boyd-Wong theorem from the metric spaces to the S-metric spaces. We do not need upper semi-continuity of the comparison function, we only use $\phi \in \Phi$ with $\lim_{s\to t^+} \phi\left(s\right) < t$, $t > 0$.

Corollary 2 *Let* (X, S) *be a complete* S*-metric space. Suppose that the mapping* $f : X \to X$ *satisfies*

$$S(fx, fy, fz) \leq \phi(S(x, y, z)),$$

for all $x, y, z \in X$*. Then* f *has a unique fixed point.*

Theorem 4 *Let* (X, S) *be a* S*-metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \le k_1\phi(S(gx, gx, fx)) + k_2\phi(S(gy, gy, fy)) + k_3\phi(S(gz, gz, fz)) \quad (5)$$

for all $x, y, z \in X$, $k_1 + k_2 + k_3 < 1$. *If the range of* g *contains the range of* f, *and one of* $f(X)$ *or* $g(X)$ *is complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Assume that f and g satisfy the condition (5). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&S(gx_n, gx_n, gx_{n+1}) = S(fx_{n-1}, fx_{n-1}, fx_n)\\
&\le k_1\phi(S(gx_{n-1}, gx_{n-1}, fx_{n-1})) + k_2\phi(S(gx_{n-1}, gx_{n-1}, fx_{n-1}))\\
&\quad + k_3\phi(S(gx_n, gx_n, fx_n))\\
&= k_1\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_2\phi(S(gx_{n-1}, gx_{n-1}, gx_n))\\
&\quad + k_3\phi(S(gx_n, gx_n, gx_{n+1}))\\
&< (k_1 + k_2)\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_3 S(gx_n, gx_n, gx_{n+1}).
\end{aligned}$$

Now we have,

$$S(gx_n, gx_n, gx_{n+1}) < \frac{k_1 + k_2}{1 - k_3}\phi(S(gx_{n-1}, gx_{n-1}, gx_n)).$$

Let $r = \frac{k_1+k_2}{1-k_3} < 1$. Then

$$\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) &< r\phi(S(gx_{n-1}, gx_{n-1}, gx_n))\\
&< \phi(S(gx_{n-1}, gx_{n-1}, gx_n)) < \cdots < \phi^n S(gx_0, gx_0, gx_1)
\end{aligned}$$

This implies that $\lim\limits_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) \text{ and } S(gx_{m_k}, gx_{m_k}, gx_{n_k}). \quad (6)$$

Putting now in (3) $x = y = x_{m_k}, z = x_{n_k}$, and using the fact that $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ and $S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}) > 0$ we obtain

$$\begin{aligned}
&S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1})\\
&= S(fx_{m_k}, fx_{m_k}, fx_{n_k})\\
&\leq k_1\phi(S(gx_{m_k}, gx_{m_k}, fx_{m_k})) + k_2\phi(S(gx_{m_k}, gx_{m_k}, fx_{m_k}))\\
&\quad + k_3\phi(S(gx_{n_k}, gx_{n_k}, fx_{n_k}))\\
&= k_1\phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) + k_2\phi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}))\\
&\quad + k_3\phi(S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}))\\
&< k_1 S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) + k_2 S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})\\
&\quad + k_3 S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})
\end{aligned}$$

Letting $k \to \infty$ we obtain $\varepsilon \leq 0$.

A contradiction. So, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (5), we have

$$\begin{aligned}
&S(gx_n, gx_n, fp) = S(fx_{n-1}, fx_{n-1}, fp)\\
&\leq k_1\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_2\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) + k_3\phi(S(gp, gp, fp)).
\end{aligned}$$

Letting $n \to \infty$ we have

$$S(gp, gp, fp) \leq k_3\phi(S(gp, gp, fp)) < k_3 S(gp, gp, fp) < S(gp, gp, fp)$$

we got a contradiction. So, $gp = fp$. The proof that f and g have a unique point of coincidence is as in Theorem 1 so we omitted it. $\square$

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