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Logarithmically complete monotonicity of a function related to the Catalan-Qi function

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Abstract. In the paper, the authors find necessary and sufficient conditions such that a function related to the Catalan-Qi function, which is an alternative generalization of the Catalan numbers, is logarithmically complete monotonic.

1 Introduction

It is stated in [11, 40] that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

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$$\frac{2}{1+\sqrt{1-4x}}=\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}C_nx^n=1+x+2x^2+5x^3+14x^4+\cdots.$$

One of explicit formulas of C_n for $n \ge 0$ reads that

$$C_n=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

is the classical Euler gamma function. In [8, 11, 40, 43], it was mentioned that there exists an asymptotic expansion

$$C_x\sim\frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac{9}{8}\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right) \tag{1}$$

for the Catalan function C_x.

A generalization of the Catalan numbers C_n was defined in [9, 10, 16] by

$${}_pd_n=\frac{1}{n}\binom{pn}{n-1}=\frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n \ge 1$. The usual Catalan numbers $C_n={}_2d_n$ are a special case with $p=2$.

In combinatorics and statistics, the Fuss-Catalan numbers $A_n(p,r)$ are defined [6, 45] as numbers of the form

$$A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}.$$

It is easy to see that

$$A_n(2,1)=C_n,\quad n\ge0\quad\text{and}\quad A_{n-1}(p,p)={}_pd_n,\quad n\ge1.$$

There have existed some literature, such as [2, 4, 5, 7, 12, 14, 18, 19, 20, 21, 41, 42, 45], on the investigation of the Fuss-Catalan numbers $A_n(p,r)$.

In [31, Remark 1], an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge0.$$

For the uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call the quantity $C(a,b;x)$ the Catalan-Qi function and, when taking

$x = n \geq 0$, call $C(a, b; n)$ the Catalan-Qi numbers. In the recent papers [13, 15, 22, 24, 25, 29, 30, 31, 32, 33, 34, 39], among other things, some properties, including the general expression and a generalization of the asymptotic expansion (1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n, the Catalan function C_x, the Catalan-Qi numbers $C(a, b; n)$, the Catalan-Qi function $C(a, b; x)$, and the Fuss-Catalan numbers $A_n(p, r)$ were established. Very recently, we discovered in [25, Theorem 1.1] a relation between the Fuss-Catalan numbers $A_n(p, r)$ and the Catalan-Qi numbers $C(a, b; n)$, which reads that

$$A_n(p, r) = r^n \frac{\prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; n\right)}{\prod_{k=1}^{p-1} C\left(\frac{k+r}{p-1}, 1; n\right)}$$

for integers $n \geq 0$, $p > 1$, and $r > 0$.

Recall from [3, 26, 28, 38] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if it satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ on I for all $k \in \mathbb{N}$.

From the viewpoint of analysis, motivated by the idea in the papers [27, 35, 36, 37] and closely-related references cited therein, the author considered in [23] the function $\mathcal{C}_{a,b;x}(t) = C(a + t, b + t; x)$ for $t, x \geq 0$ and $a, b > 0$ and obtained the following conclusions:

1. the function $\mathcal{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \leq b$ or $x \geq 1$ and $a \geq b$,
2. the function $\frac{1}{\mathcal{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \geq b$ or $x \geq 1$ and $a \leq b$.

This implies the logarithmically complete monotonicity of $[\mathcal{C}_{a,b;x}(t)]^{\pm 1}$ in $t \geq 0$ along with the ray $\begin{cases} u(t) = a + t \\ v(t) = b + t \end{cases}$ on the plane (u, v), where $x \geq 0$ and $a, b > 0$. Then one may ask a question: how about its logarithmically complete monotonicity along the ray $\begin{cases} u(t) = a + \alpha t \\ v(t) = b + \beta t \end{cases}$ for $\alpha, \beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$ when $x, t \geq 0$ and $a, b > 0$? In other words, is the function

$$\mathcal{C}_{a,b;x;\alpha,\beta}(t) = C(a + \alpha t, b + \beta t; x), \quad x \geq 0, \quad a, b > 0$$

of logarithmically complete monotonicity in $t \in [0, \infty)$? When $\alpha = \beta \neq 0$, this question has been answered essentially by the above-mentioned conclusions in [23]; when $\alpha = 0$ or $\beta = 0$, this question has been answered virtually by [34, Theorem 1.2] which states that the function $[C(a, b; x)]^{\pm 1}$ is logarithmically completely monotonic

1. with respect to $a > 0$ if and only if $x \gtreqless 1$,
2. with respect to $b > 0$ if and only if $x \lesseqgtr 1$.

In this paper, we will discuss the rest cases $\alpha, \beta > 0$ and $\alpha \neq \beta$ of the above question. Our main results can be formulated as the following theorem.

Theorem 1 *If and only if $\alpha = 0$ and $\beta > 0$, or $\alpha > 0$ and $\beta = 0$, or $\alpha = \beta > 0$, the function $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ is of some logarithmically complete monotonicity. Concretely speaking,*

1. *the function $[C(a, b; x)]^{\pm 1}$ is logarithmically completely monotonic*
   (a) *with respect to $a > 0$ if and only if $x \gtreqless 1$,*
   (b) *with respect to $b > 0$ if and only if $x \lesseqgtr 1$,*
2. *the function $\mathcal{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \le x \le 1$ and $a \le b$ or $x \ge 1$ and $a \ge b$,*
3. *the function $\frac{1}{\mathcal{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \le x \le 1$ and $a \ge b$ or $x \ge 1$ and $a \le b$.*

2 Proof of Theorem 1

Taking the logarithm of $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ and differentiating with respect to t give

$$[\ln \mathcal{C}_{a,b;x;\alpha,\beta}(t)]' = \psi(\beta t + b) - \psi(\alpha t + a) + x\left(\frac{1}{\beta t + b} - \frac{1}{\alpha t + a}\right) + \psi(\alpha t + x + a) - \psi(\beta t + x + b).$$

Making use of

$$\psi(z) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}}\right) d\,u, \quad \Re(z) > 0$$

in [1, p. 259, 6.3.21] leads to

$$\begin{aligned}
[\ln \mathcal{C}_{a,b;x;\alpha,\beta}(t)]' &= \int_0^\infty \frac{e^{-(a+\alpha t)u} - e^{-(b+\beta t)u}}{1-e^{-u}} \,\mathrm{d}\,u \\
&\quad + x \int_0^\infty \Big[e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}\Big] \,\mathrm{d}\,u \\
&\quad + \int_0^\infty \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} e^{-xu} \,\mathrm{d}\,u \\
&= \int_0^\infty \big[e^{-xu} - 1 + x\big(1-e^{-u}\big)\big] \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} \,\mathrm{d}\,u \\
&= x \int_0^\infty \bigg(\frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu}\bigg) \frac{e^{-(b+\beta t)u} - e^{-(a+\alpha t)u}}{1-e^{-u}} u \,\mathrm{d}\,u.
\end{aligned}$$

It is easy to see that the function $\frac{1-e^{-u}}{u}$ is positive and strictly decreasing on $(0,\infty)$. Hence,

$$\frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu} \gtreqless 0 \tag{2}$$

for $u \in (0,\infty)$ if and only if $x \lesseqgtr 1$.

Recall from [17, Chapter XIII], [38, Chapter 1], and [44, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. It is not difficult to see that a positive function f is logarithmically completely monotonic if and only if the function $-(\ln f)'$ is completely monotonic. The famous Bernstein-Widder theorem, [44, p. 160, Theorem 12a], states that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \le x < \infty$ is that $f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\,\alpha(t)$, where α is bounded and non-decreasing and the above integral converges for $0 \le x < \infty$. Therefore, it is sufficient to find necessary and sufficient conditions on $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha \ne \beta$ for the function

$$\begin{aligned}
e^{-(b+\beta t)u} - e^{-(a+\alpha t)u} &= \int_{(b+\beta t)u}^{(a+\alpha t)u} e^{-v} \,\mathrm{d}\,v \\
&= \int_0^1 [(a-b) + (\alpha-\beta)t] u e^{-[(1-s)(b+\beta t)+s(a+\alpha t)]u} \,\mathrm{d}\,s \\
&= \int_0^1 [(a-b) + (\alpha-\beta)t] e^{-[(1-s)\beta+s\alpha]ut} u e^{-[(1-s)b+sa]u} \,\mathrm{d}\,s
\end{aligned}$$

to be completely monotonic in $t \in [0,\infty)$ for all $u \in (0,\infty)$.

By induction, we obtain

$$\left[(A+Bt)e^{-Dt}\right]^{(k)}=(-1)^kD^{k-1}(BDt+AD-kB)e^{-Dt},\quad k\ge 0,$$

where A,B,D are real constants. Accordingly, the function $(A+Bt)e^{-Dt}$ is completely monotonic in $t\in[0,\infty)$ if and only if $A,B\ge 0$, $D>0$, and

$$D^{k-1}(BDt+AD-kB)\ge 0,\quad k\ge 0,\quad t\in[0,\infty). \tag{3}$$

Simply speaking, the function $(A+Bt)e^{-Dt}$ is completely monotonic in $t\in[0,\infty)$ if and only if $A\ge 0$, $B=0$, and $D>0$. Applying A to $a-b$, B to $\alpha-\beta$, and D to $[(1-s)\beta+s\alpha]u$ yields that the function $e^{-(b+\beta t)u}-e^{-(a+\alpha t)u}$ is completely monotonic in $t\in[0,\infty)$ if and only if $a\ge b$, $\alpha=\beta$, and $\alpha,\beta\ge 0$ with $(\alpha,\beta)\ne(0,0)$. Combining this result with the inequality (2) and with the proofs of [23, Theorem 1.1] and [34, Theorem 1.2] concludes that, if and only if $\alpha=0$ and $\beta>0$, or $\alpha>0$ and $\beta=0$, or $\alpha=\beta>0$, the function $\mathcal{C}_{a,b;x;\alpha,\beta}(t)$ is of some logarithmically complete monotonicity. The proof of Theorem 1 is thus complete.

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