Acta Univ. Sapientiae, Mathematica, **7**, 1 (2015) 15–26

DOI: 10.1515/ausm-2015-0002

Nonhomogeneous linear differential equations with entire coefficients having the same order and type

Berrighi Nacera
Laboratory of Pure and Applied Mathematics
University of Mostaganem, Algeria
email: berrighinacera@univ-mosta.dz

Hamouda Saada
Laboratory of Pure and Applied Mathematics
University of Mostaganem, Algeria
email: hamoudasaada@univ-mosta.dz

Abstract. In this paper we will investigate the growth of solutions of certain class of nonhomogeneous linear differential equations with entire coefficients having the same order and type. This work improves and extends some previous results in [1], [7] and [9].

1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [6]). We denote by $\sigma(f)$ the order of growth of f that defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r};$$

and the type of a meromorphic function f of finite order σ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\sigma}},$$

2010 Mathematics Subject Classification: Primary 34M10; Secondary 30D35
Key words and phrases: linear differential equations, growth of solutions, entire coefficients

where $T(r,f)$ is the Nevanlinna characteristic function of f. We remark that if f is an entire function then we have

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\sigma}}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_0(z) f = H(z), \tag{1}$$

where $A_0 \not\equiv 0, A_1, ..., A_{k-1}, H \not\equiv 0$ are entire functions. It is well known that all solutions of (1) are entire functions. The case when the coefficients are polynomials has been studied by Gundersen, Steinbart and Wang in [5] and if p is the largest integer such that A_p is transcendental, Frei proved in [3] that there exist at most p linearely independent finite order solutions of the corresponding homogeneous equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_0(z) f = 0. \tag{2}$$

Several authors studied the case when the coefficients have the same order. In 2008, Tu and Yi investigated the growth of solutions of the homogeneous equation (2) when most coefficients have the same order, see [8]. Next, in 2009, Wang and Laine improved this work to nonhomogeneous equation (1) by proving the following result.

Theorem 1 [9] *Suppose that* $A_j(z) = h_j(z) e^{P_j(z)}$ $(j = 0, ..., k-1)$, *where* $P_j(z) = a_{jn} z^n + ... + a_{j0}$ *are polynomials with degree* $n \geq 1$, $h_j(z)$ *are entire functions of order less than* n, *not all vanishing, and that* $H(z) \not\equiv 0$ *is an entire function of order less than* n. *If* a_{jn} $(j = 0, ..., k-1)$ *are distinct complex numbers, then every solution of* (1) *is of infinite order.*

Now how about the case when a_{jn} $(j = 0, ..., k-1)$ are equals? we will answer this question in this paper. For the homogeneous equation case, Huang and Sun proved the following result.

Theorem 2 [7] *Let* $A_j(z) = B_j(z) e^{P_j(z)}$ $(j = 0, ..., k-1)$, *where* $B_j(z)$ *are entire functions,* $P_j(z)$ *are non constant polynomials with* $\deg(P_j(z) - P_i(z)) \geq 1$ *and* $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z) - P_i(z))$ $(i \neq j)$. *Then every transcendental solution* f *of* (2) *satisfies* $\sigma(f) = \infty$.

The nonhomogeneous case of this result is improved later in Theorem 4. Recentely, in [1] the authors investigated the order and hyper-order of solutions of the linear differential equation

$$f^{(k)}+\left(A_{k-1}(z)e^{P_{k-1}(z)}e^{\lambda z^m}+B_{k-1}(z)\right)f^{(k-1)}+...+\left(A_0(z)e^{P_0(z)}e^{\lambda z^m}+B_0(z)\right)f=0,$$

where $\lambda \in \mathbb{C}-\{0\}$, $m \geq 2$ is an integer and $\max_{j=0,...,k-1}\{\deg P_j(z)\} < m$, A_j, B_j $(j=0,...,k-1)$ are entire functions of order less than m.

In this paper we will investigate certain class of nonhomogeneous linear differential equations with entire coefficients having the same order and type. In fact we will prove the following results.

Theorem 3 *Consider the linear differential equation*

$$f^{(k)}+B_{k-1}(z)\,e^{P_{k-1}(z)}e^{\lambda z^m}f^{(k-1)}+...+B_0(z)\,e^{P_0(z)}e^{\lambda z^m}f=H(z)\,, \qquad (3)$$

where $\lambda \neq 0$ *is a complex constant,* $m \geq 2$ *is an integer,* $P_j(z) = a_{jn}z^n + ... + a_{j0}$ $(j=0,...,k-1)$ *be non constant polynomials such that* $n < m$; $B_0 \not\equiv 0, B_1, ..., B_{k-1}$, $H \not\equiv 0$ *are entire functions of order smaller than* n. *If one of the following occurs:*

(1) a_{jn} $(j=0,...,k-1)$ *are distinct complex numbers;*

(2) *there exist* $s, t \in \{0,1,...,k-1\}$ *such that* $\arg a_{sn} \neq \arg a_{tn}$ *and for* $j \neq s, t$ $a_{jn} = c_j a_{sn}$ *or* $a_{jn} = c_j a_{tn}$ *with* $0 < c_j < 1$, $B_s B_t \not\equiv 0$;

then every solution of (3) *is of infinite order.*

Corollary 1 *Consider the linear differential equation*

$$f^{(k)}+B_{k-1}(z)\,e^{\lambda z^3+a_{k-1}z^2+b_{k-1}z}f^{(k-1)}+...+B_0(z)\,e^{\lambda z^3+a_0z^2+b_0z}f=H(z)$$

where $\lambda \in \mathbb{C}-\{0\}$, a_j *are distinct complex numbers (or satisfy the condition (2) of Theorem 3) and* $B_0 \not\equiv 0, B_1, ..., B_{k-1}, H \not\equiv 0$ *are entire functions of order smaller than* 2. *Then every solution* f *of this differential equation is of infinite order.*

Theorem 4 *Let* $A_j(z) = B_j(z)\,e^{P_j(z)}$ $(j=0,...,k-1)$, *where* $B_j(z)$ *are entire functions,* $P_j(z)$ *be non constant polynomials with*
$\deg(P_j(z)-P_i(z)) \geq 1$ *and* $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z)-P_i(z))$ $(i \neq j)$, *and let* $H(z) \not\equiv 0$ *be an entire function of order less than* 1. *Then every solution of* (1) *is of infinite order.*

Example 1 *Consider the linear differential equation*

$$f^{(4)}+B_3(z)\,e^{z^2+z}f^{(3)}+B_2(z)\,e^{2z^2+z}f''+B_1(z)\,e^{2z^2+iz}f'+B_0(z)\,e^{z^2+iz}f=H(z)\,,$$

where $B_0 \not\equiv 0, B_1, B_2, H \not\equiv 0$ *are entire functions of order less than* 1. *By Theorem 4, every solution of this differential equation is of infinite order.*

Theorem 5 *Let* $A_j(z)=B_j(z)\,P_j\left(e^{R(z)}\right)+G_j(z)\,Q_j(e^{-R(z)})$ *for* $j=0,1,...,k-1$ *where* $P_j(z)$, $Q_j(z)$ *and* $R(z)=c_dz^d+...+c_1z+c_0$ $(d\geq 1)$ *are polynomials; and let* $B_j(z)$, $G_j(z)$, $H(z)\not\equiv 0$ *be entire functions of order less than* d. *Suppose that* $B_0(z)\,P_0(z)+G_0(z)\,Q_0(z)\not\equiv 0$ *and there exists* s $(0\leq s\leq k-1)$ *such that for* $j\neq s$, $\deg P_s>\deg P_j$ *and* $\deg Q_s>\deg Q_j$. *Then every solution* f *of* (1) *is of infinite order.*

Example 2 *By Theorem 5, every solution of the differential equation*

$$f''+\sin\left(2z^2\right)f'+\cos\left(z^2\right)f=\sin z$$

is of infinite order.

2 Preliminaries Lemmas

We need the following lemmas for our proofs.

Lemma 1 [4] *Let* $f(z)$ *be a transcendental meromorphic function of finite order* σ, *and let* $\varepsilon>0$ *be a given constant. Then there exists a set* $E\subset[0,2\pi)$ *of linear measure zero such that for all* $z=re^{i\theta}$ *with* $|z|$ *sufficiently large and* $\theta\in[0,2\pi)\setminus E$, *and for all* k,j, $0\leq j\leq k$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq|z|^{(k-j)(\sigma-1+\varepsilon)}\,.$$

Lemma 2 [2] *Let* $P(z)=a_nz^n+...+a_0$, $(a_n=\alpha+i\beta\neq 0)$ *be a polynomial with degree* $n\geq 1$ *and* $A(z)$ $(\not\equiv 0)$ *be entire function with* $\sigma(A)<n$. *Set* $f(z)=A(z)\,e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. *Then for any given* $\varepsilon>0$, *there exists a set* $E\subset[0,2\pi)$ *that has linear measure zero, such that for any* $\theta\in[0,2\pi)\setminus E\cup H$, *where* $H=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ *is a finite set, there is* $R>0$ *such that for* $|z|=r>R$, *we have*

(i) *if* $\delta(P,\theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\} \le |f(z)| \le \exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\},$$

(ii) *if* $\delta(P,\theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\} \le |f(z)| \le \exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\}.$$

Lemma 3 [7] *Let* $n \ge 2$ *and* $A_j(z) = B_j(z)\,e^{P_j(z)}$ $(1 \le j \le n)$, *where each* $B_j(z)$ *is an entire function and* $P_j(z)$ *is a nonconstant polynomial. Suppose that* $\deg(P_j(z) - P_i(z)) \ge 1$, $\max\{\sigma(B_j), \sigma(B_i)\} < \deg(P_j(z) - P_i(z))$ *for* $i \ne j$. *Then there exists a set* $H_1 \subset [0, 2\pi)$ *that has linear measure zero, such that for any given constant* $M > 0$ *and* $z = re^{i\theta}$, $\theta \in [0,2\pi) - (H_1 \cup H_2)$, *we have some integer* $s = s(\theta) \in \{1, 2, ..., n\}$, *for* $j \ne s$,

$$\frac{\left|A_j\left(re^{i\theta}\right)\right| |z|^M}{\left|A_s\left(re^{i\theta}\right)\right|} \to 0, \quad \text{as } r \to \infty,$$

where $H_2 = \{\theta \in [0,2\pi) : \delta(P_j,\theta) = 0$ *or* $\delta(P_i - P_j, \theta) = 0,\ i \ne j\}$ *is a finite set.*

Lemma 4 [9] *Let* $f(z)$ *be an entire function and suppose that*

$$G(z) = \frac{\log^+\left|f^{(k)}(z)\right|}{|z|^\sigma}$$

is unbounded on some ray $\arg z = \theta$ *with constant* $\sigma > 0$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, ...)$, *where* $r_n \to \infty$, *such that* $G(z_n) \to \infty$ *and*

$$\frac{\left|f^{(j)}(z_n)\right|}{\left|f^{(k)}(z_n)\right|} \le \frac{1}{(k-j)!}\,(1+o(1))\,r_n^{k-j}, \qquad j = 0, 1, ..., k-1$$

as $n \to \infty$.

Lemma 5 [9] *Let* $f(z)$ *be an entire function with finite order* $\sigma(f)$. *Suppose that there exists a set* $E \subset [0,2\pi)$ *which has linear measure zero, such that* $\log^+|f\left(re^{i\theta}\right)| \le Mr^\sigma$ *for any ray* $\arg z = \theta \in [0,2\pi)\setminus E$, *where* M *is a positive constant depending on* θ, *while* σ *is a positive constant independent of* θ. *Then* $\sigma(f) \le \sigma$.

Lemma 6 [10] *Suppose that* $f_1(z), f_2(z), ..., f_n(z)$ $(n \geq 2)$ *are linearly independent meromorphic functions and* $g_1(z), g_2(z), ..., g_n(z)$ *are entire fuctions satisfying the following conditions*
(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0.$
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n.$
(iii) For $1 \leq i \leq n,\ 1 \leq j < k \leq n,$

$$T(r, f_j) = o\left\{T\left(r, e^{g_j(z) - g_k(z)}\right)\right\}, \quad (r \to \infty,\ r \notin E)$$

where E *is a set with finite linear measure.*
Then $f_j \equiv 0,\ 1 \leq j \leq n.$

3 Proof of main results

Proof. [Proof of Theorem 3] We will prove the two cases together. If we suppose that f is a solution of (3) of finite order $\sigma(f) = \sigma < \infty$, (contrary to the assertion), then $\sigma \geq n$. Indeed, if $\sigma < n$ then we get the following contradiction. From (3), we can write

$$\left(B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f\right) e^{\lambda z^m} = H(z) - f^k. \qquad (4)$$

Now for the condition (1), if $B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f \equiv 0$, then by Lemma 6, we have $B_0(z) f \equiv 0$, and since $B_0(z) \not\equiv 0$, then $f \equiv 0$, which implies that $H(z) \equiv 0$, a contradiction. So

$$B_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + B_0(z) e^{P_0(z)} f \not\equiv 0.$$

Then the order of growth of the left side of (4) is equal m and the order of the right side is smaller than n, a contradiction. So, we have $\sigma(f) = \sigma \geq n$.
And for the condition (2), to apply Lemma 6 we may collecte terms of the same power, and we have at least two terms linearly independents: if

$$B_s(z) f^{(s)} e^{P_s(z)} + B_t(z) f^{(t)} e^{P_t(z)} + \sum_{u=1}^{p} G_u e^{c_{j_u} P_s(z)} + \sum_{v=1}^{q} L_v e^{c_{i_v} P_t(z)} \equiv 0$$

by Lemma 6, $B_s(z) f^{(s)} \equiv 0$, and since $B_s(z) \not\equiv 0$, then $f^{(s)} \equiv 0$ and so $f^{(k)} \equiv 0$, which implies that $H(z) \equiv 0$, a contradiction. So

$$B_s(z) f^{(s)} e^{P_s(z)} + B_t(z) f^{(t)} e^{P_t(z)} + \sum_{u=1}^{p} G_u e^{c_{j_u} P_s(z)} + \sum_{v=1}^{q} L_v e^{c_{i_v} P_t(z)} \not\equiv 0.$$

By similar reasoning as above we get $\sigma(f) = \sigma \geq n$.
By Lemma 1, for any given ε $(0 < \varepsilon < 1)$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero , such that if $\psi \in [0, 2\pi) \setminus E_1$, then

$$\frac{\left|f^{(j)}(z)\right|}{\left|f^{(i)}(z)\right|} \leq |z|^{k\sigma}, \quad 0 \leq i < j \leq k \tag{5}$$

as $z \to \infty$ along $\arg z = \psi$. Denote $E_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0,\ 0 \leq j \leq k\} \cup \{\theta \in [0, 2\pi) : \delta(P_j - P_i, \theta) = 0,\ 0 \leq i < j \leq k\} \cup \{\theta \in [0, 2\pi) : \delta(\lambda z^m, \theta) = 0\}$, so E_2 is a finite set. Suppose that $H_j \subset [0, 2\pi)$ is the exceptional set applying lemma 2 to $A_j(z) = B_j(z) e^{\lambda z^m + P_j(z)}$ $(j = 0, ..., k-1)$. Then $E_3 = \cup_{j=0}^{k-1} H_j$ has linear measure zero. Set $E = E_1 \cup E_2 \cup E_3$. Take $\arg z = \psi \in [0, 2\pi) - E$. We need to treat two principal cases:

Case (i): $\delta = \delta(\lambda z^m, \psi) < 0$. By lemma 2, for a given $0 < \varepsilon < 1$, we have

$$|A_j(z)| \leq \exp\{(1-\varepsilon)\delta r^m\}. \tag{6}$$

Now we prove that $\dfrac{\log^+ \left|f^{(k)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi_0$. Suppose that it is not the case. By Lemma 4, there is a sequence of points $z_i = r_i e^{i\theta}$ $(i = 1, 2, ...)$, such that $r_i \to \infty$ as $i \to \infty$, and that

$$\frac{\log^+ \left|f^{(k)}(z_i)\right|}{|z_i|^{\sigma(H)+\varepsilon}} \to \infty \tag{7}$$

and

$$\frac{\left|f^{(j)}(z_i)\right|}{\left|f^{(k)}(z_i)\right|} \leq (1 + o(1)) r_i^{t-j}, \qquad j = 0, 1, ..., k-1. \tag{8}$$

From (7) and the definition of the order $\sigma(H)$, it is easy to see that

$$\left|\frac{H(z_i)}{f^{(k)}(z_i)}\right| \to 0 \tag{9}$$

as $z_i \to \infty$. From (3), we obtain

$$1 \leq |A_{k-1}(z_i)| \left|\frac{f^{(k-1)}(z_i)}{f^{(k)}(z_i)}\right| + ... + |A_0(z_i)| \left|\frac{f(z_i)}{f^{(k)}(z_i)}\right| + \left|\frac{H(z_i)}{f^{(k)}(z_i)}\right|. \tag{10}$$

Using (5)-(9) in (10), we get

$$1 \leq r_i^k \exp\{(1-\varepsilon)\delta r_i^m\}.$$

This is impossible since $\delta < 0$. Therefore $\dfrac{\log^+|f^{(k)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi$. Assume that $\dfrac{\log^+|f^{(k)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}} \leq M_1$ (M_1 is a constant) and so

$$|f^{(k)}(z)| \leq M_1 \exp\{r^{\sigma(H)+\varepsilon}\}. \tag{11}$$

Using the elementary triangle inequality for the well know equality

$$f(z) = f(0) + f'(0)z + ... + \frac{1}{(k-1)!} f^{(k-1)}(0) z^{k-1} + \int_0^z ... \int_0^{\xi_1} f^{(k)}(\xi) \, d\xi d\xi_1 ... d\xi_{k-1},$$

and (11), we obtain

$$|f(z)| \leq (1 + o(1)) r^k |f^{(k)}(z)| \leq (1 + o(1)) M_1 r^k \exp\{r^{\sigma(H)+\varepsilon}\} \leq \exp\{r^{\sigma(H)+2\varepsilon}\}, \tag{12}$$

on any ray $\arg z = \psi \in [0, 2\pi) - E$.

Case (ii): $\delta = \delta(\lambda z^m, \psi) > 0$. Now we pass to $\delta_j = \delta(P_j, \psi)$. For the condition (1), since a_{jn} $(j = 0, ..., k-1)$ are distinct complex numbers, then there exists some $s \in \{0, 1, 2..., k-1\}$ such that $\delta_s > \delta_j$ for all $j \neq s$. For the condition (2), set $\delta' = \max\{\delta_s, \delta_t\}$ and without loss of generality we may assume that $\delta' = \delta_s$. In both cases, we have

$$\left|\frac{A_j(z)}{A_s(z)}\right| |z|^M \to 0, \quad \text{and} \quad \frac{|z|^M}{|A_s(z)|} \to 0, \tag{13}$$

as $|z| \to \infty$, for any $M > 0$. Suppose that $\dfrac{\log^+ |f^{(s)}(z)|}{|z|^{\sigma(H)+\varepsilon}}$ is unbounded on the ray $\arg z = \psi$. Then by lemma 4 there is a sequence of points $z_i = r_i e^{i\psi}$, such that $r_i \to \infty$, and

$$\frac{\log^+ |f^{(s)}(z_i)|}{|z_m|^{\sigma(H)+\varepsilon}} \to \infty, \tag{14}$$

and

$$\frac{|f^{(j)}(z_i)|}{|f^{(s)}(z_i)|} \leq (1 + o(1)) r_i^{s-j}, \qquad j = 0, 1, ..., s-1. \tag{15}$$

From (14) and the definition of order, it is easy to see that

$$\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right| \to 0 \tag{16}$$

as $r_i \to \infty$. From (3), we can write

$$1 \leq \frac{1}{|A_s(z_i)|}\left|\frac{f^{(k)}(z_i)}{f^{(s)}(z_i)}\right| + \left|\frac{f^{(k-1)}(z_i)}{f^{(s)}(z_i)}\right|\frac{|A_{k-1}(z_i)|}{|A_s(z_i)|} + \dots \tag{17}$$
$$+\left|\frac{f^{(s+1)}(z_i)}{f^{(s)}(z_i)}\right|\frac{|A_{s+1}(z_i)|}{|A_s(z_i)|} + \left|\frac{f^{(s-1)}(z_i)}{f^{(s)}(z_i)}\right|\frac{|A_{s-1}(z_i)|}{|A_s(z_i)|} + \dots$$
$$+\left|\frac{f(z_i)}{f^{(s)}(z_i)}\right|\frac{|A_0(z_i)|}{|A_s(z_i)|} + \frac{1}{|A_s(z_i)|}\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right|;$$

and by using (5), (13), (15) and (16) in (17) a contradiction follows as $z_i \to \infty$. Then $\dfrac{\log^+|f^{(s)}(z_i)|}{|z_i|^{\sigma(H)+\varepsilon}}$ is bounded and we have $|f^{(s)}(z)| \leq M_2 \exp\{r^{\sigma(H)+\varepsilon}\}$ on the ray $\arg z = \psi$. This implies, as in Case (i), that

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}.$$

We conclude that in all cases we have

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}$$

on any ray $\arg z = \psi \in [0, 2\pi) - E$, provided that r is large enough. Then by Lemma 5, $\sigma(f) \leq \sigma(H) + 2\varepsilon < n$ $(0 < 2\varepsilon < n - \sigma(H))$, a contradiction. Hence, every solution of (3) must be of infinite order. □

Proof. [Proof of Theorem 4] We suppose contrary to the assertion that f is a solution of (1) of finite order $\sigma(f) = \sigma < \infty$. First we prove that $\sigma \geq 1$. Indeed, if $\sigma < 1$ then we will have the following contradiction. From (1), we can write

$$B_{k-1}(z)\, e^{P_{k-1}(z)} f^{(k-1)} + \dots + B_0(z)\, e^{P_0(z)} f = H(z) - f^{(k)}. \tag{18}$$

By the same rasoning as in the proof of Theorem 3, we get that the order of the left side of (18) is greather than or equal to 1 and the order of the right side of (18) is smaller than 1, a contradiction. Therefore $\sigma \geq 1$.
Take $\arg z = \psi \in [0, 2\pi) - E$ where E has linear measure zero and set $\delta_j = \delta(P_j, \psi)$ $(j = 0, \dots, k-1)$. By Lemma 3, there exists some $s \in \{0, 1, 2\dots, k-1\}$ such that for $j \neq s$, $M > 0$, we have

$$\left|\frac{A_j(z)}{A_s(z)}\right| |z|^M \to 0, \quad \text{as } z \to \infty. \tag{19}$$

We need to treat two cases:
Case (i): $\delta_s > 0$. In this case we have also

$$\frac{1}{|A_s(z)|}|z|^M \to 0, \quad \text{as } z \to \infty. \tag{20}$$

We prove that $\dfrac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi$. Suppose that it is not the case. Then by lemma 4 there is a sequence of points $z_i = r_i e^{i\psi_0}$, such that $r_i \to \infty$, and (14), (15), (16) hold. As in the proof of Theorem 3, by using (17) we get a contradiction. Therefore, $\dfrac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded and so we conclude that

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}. \tag{21}$$

Case (ii): $\delta_s < 0$. Obsiouly in this case $\delta_j < 0$ for all j and we have

$$|A_j(z)| \leq \exp\left\{(1-\varepsilon)\,\delta_j r^{d_j}\right\},$$

where $d_j = \deg(P_j)$; which implies that

$$|A_j(z)|\,|z|^M \to 0, \quad \text{as } z \to \infty.$$

We use the same reasoning as in Case (i) in the proof of Theorem 3, we prove that $\dfrac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z = \psi$ and we conclude that

$$|f(z)| \leq \exp\{r^{\sigma(H)+2\varepsilon}\}.$$

Then by Lemma 5, $\sigma(f) \leq \sigma(H) + 2\varepsilon < 1$ $(0 < 2\varepsilon < 1 - \sigma(H))$, a contradiction. So, every solution of (1) must be of infinite order. □

Proof. [Proof of Theorem 5] Suppose that f is a solution of (1) of finite order $\sigma(f) = \sigma < \infty$. By the same reasoning as in the proof of Theorem 4 and taking account the assumption that $B_0(z)P_0(z) + G_0(z)Q_0(z) \not\equiv 0$ and there exists s $(0 \leq s \leq k-1)$ such that for $j \neq s$, $\deg P_s > \deg P_j$ and $\deg Q_s > \deg Q_j$, we can prove that $\sigma \geq d$.
Set $\delta(R,\theta) = \text{Real}\left(c_d e^{id\theta}\right)$ and

$$P_j\left(e^{R(z)}\right) = a_{jm_j}e^{m_j R(z)} + a_{j(m_j-1)}e^{(m_j-1)R(z)} + \ldots + a_{j1}e^{R(z)} + a_{j0},$$

$$Q_j\left(e^{-R(z)}\right)=b_{jn_j}e^{-n_jR(z)}+b_{j(n_j-1)}e^{-(n_j-1)R(z)}+\dots+b_{j1}e^{-R(z)}+b_{j0}.$$

By Lemma 2, it is easy to get the following
(i) If $\delta(R,\theta)>0$, then

$$\exp\left\{(1-\varepsilon)m_j\delta(R,\theta)r^d\right\}\le|A_j(z)|\le\exp\left\{(1+\varepsilon)m_j\delta(R,\theta)r^d\right\},\qquad(22)$$

(ii) if $\delta(R,\theta)<0$, then

$$\exp\left\{-(1-\varepsilon)n_j\delta(R,\theta)r^d\right\}\le|A_j(z)|\le\exp\left\{-(1+\varepsilon)n_j\delta(R,\theta)r^d\right\}.\qquad(23)$$

Take $\arg z=\psi\in[0,2\pi)-E$ where E has linear measure zero. We prove that $\dfrac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on the ray $\arg z=\psi$. Suppose that it is not the case. Then by lemma 4 there is a sequence of points $z_i=r_ie^{i\psi_0}$, such that $r_i\to\infty$, and (14), (15), (16) hold. From (1) we can write

$$\begin{aligned}|A_s(z_i)|\le{}&\left|\frac{f^{(k)}(z_i)}{f^{(s)}(z_i)}\right|+|A_{k-1}(z_i)|\left|\frac{f^{(k-1)}(z_i)}{f^{(s)}(z_i)}\right|+\dots\\&+|A_{s+1}(z_i)|\left|\frac{f^{(s+1)}(z_i)}{f^{(s)}(z_i)}\right|+|A_{s-1}(z_i)|\left|\frac{f^{(s-1)}(z_i)}{f^{(s)}(z_i)}\right|+\dots\\&+|A_0(z_m)|\left|\frac{f(z_i)}{f^{(s)}(z_i)}\right|+\left|\frac{H(z_i)}{f^{(s)}(z_i)}\right|.\end{aligned}\qquad(24)$$

If $\delta(R,\theta)>0$, then by using (14), (15), (16) and (22) in (24), we obtain

$$\exp\left\{(1-\varepsilon)m_s\delta(R,\theta)r_i^d\right\}\le r_i^M\exp\left\{(1+\varepsilon)(m_s-1)\delta(R,\theta)r_i^d\right\},$$

where $M>0$ is a constant. A contradiction follows by taking $0<\varepsilon<\frac{1}{2m_s-1}$.
Now if $\delta(R,\theta)<0$, by using (23) instead of (22) in (24), we obtain

$$\exp\left\{-(1-\varepsilon)n_s\delta(R,\theta)r^d\right\}\le r_i^M\exp\left\{-(1+\varepsilon)(n_s-1)\delta(R,\theta)r_i^d\right\},$$

a contradiction follows by taking $0<\varepsilon<\frac{1}{2n_s-1}$.
Therefore, $\dfrac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\sigma(H)+\varepsilon}}$ is bounded on any ray $\arg z=\psi\in[0,2\pi)-E$ and so as the previous reasoning we conclude that

$$|f(z)|\le\exp\{r^{\sigma(H)+2\varepsilon}\}.$$

Then by Lemma 5, $\sigma(f)\le\sigma(H)+2\varepsilon<d$ $(0<2\varepsilon<d-\sigma(H))$, a contradiction. So, every solution of (1) must be of infinite order. $\square$

References

[1] N. Berrighi, S. Hamouda, Linear differential equations with entire coefficients having the same order and type, *Electron. J. Differential Equations*, **157** (2011), pp. 1–8.

[2] Z. X. Chen, K. H. Shon, On the growth of solutions of a class of higher order linear differential equations, *Acta Math. Sci. Ser. B Engl. Ed.*, **24** B (1) (2004), 52–60.

[3] M. Frei, Sur l'ordre des solutions entières d'une équation différentielle linéaire, *C. R. Acad. Sci. Paris*, **236** (1953), 38–40.

[4] G. G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, *J. Lond. Math. Soc.* **37** (2) (1988), 88–104.

[5] G. G. Gundersen, M. Steinbart, S. Wang, Growth and oscillation theory of non-homogeneous linear differential equations, *Proc. Edinb. Math. Soc. (2)*, **43** (2000), 343–359.

[6] W.K. Hayman, *Meromorphic functions,* Clarendon Press, Oxford, 1964.

[7] Z. G. Huang, G. R. Sun, Oscillation of higher-order linear differential equations with entire coefficients, *Electron. J. Differential Equations*, **81** (2010), pp. 1–11.

[8] J. Tu, C.-F. Yi, On the growth of solutions of a class of higher order linear differential equations with entire coefficients having the same order, *J. Math. Anal. Appl.* **340** (2008), 487–497.

[9] J. Wang, I. Laine, Growth of solutions of nonhomogeneous linear differential equations, *Abstr. Appl. Anal.* (2009), ID 363927, 11 pages.

[10] H. X. Yi, C. C. Yang; *The uniqueness theory of meromorphic functions,* Science Press, Beijing, 1995.

Received: 20 June 2014