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Slant helices of (k, m)-type in $\mathbb{E}^4$

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Abstract. In the present work, we define new type slant helices called (k,m)-type and we conclude that there are no (1,k) type $(1 \leq k \leq 4)$ slant helices. Also we obtain conditions for different type slant helices.

1 Introduction

The curve theory has been one of the most studied subject because of having many application area from geometry to the various branch of science. Especially the characterizations on the curvature and torsion play important role to define special curve types such as so-called helices. The curves of this type have drawn great attention from mathematics to natural sciences and engineering. Helices appear naturally in structures of DNA, nanosprings. They are also widely used in engineering and architecture.The concept of slant helix defined by Izumiya and Takeuchi [6] based on the property that the principal normal lines of an α curve(with non-vanishing curvature) make constant angle with a fixed direction of the ambient space. After this lightening work many researcher have characterized this type of curves in various spaces. For instance in [1] authors extended slant helix concept to $\mathbb{E}^n$ and conclude that there are no slant helices with non-zero constant curvatures in the space $\mathbb{E}^4$. The slant helix subject are also considered in 3-, 4-, and n- dimensional Eucliedan spaces, respectively in [7, 10, 12] different dimensions. Moreover different properties

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of helices are also discussed in [4, 8, 11, 13] . On the other hand in A.T Ali, R. Lopez and M. Turgut extended this study to the k-type slant helix in $\mathbb{E}_1^4$. In this study they called α curve as k-type slant helix if there exists on (non-zero) constant vector field $U \in \mathbb{E}_1^4$ such that $\langle V_{k+1}, U \rangle = const$, for $0 \leq k \leq 3$. Here V_{k+1} shows the Frenet vectors of this curve [2].

One may easily conclude that 0-type slant helices are general helices and 1-type slant helices correspond just slant helices. They consider k-type slant helices for partially null and pseudo null curves, and in hyperbolic space.

In accordance with above studies, in this work we define (k, m)-type slant helices in $\mathbb{E}^4$ and we show that there are no $(1, m)$ type slant helices in $\mathbb{E}^4$.

2 Preliminaries

In this section we will present on brief basic tools for the space curves in $\mathbb{E}^4$. A detailed information can be found in [5].

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be an arbitrary curve in Euclidean space $\mathbb{E}^4$. The standard scalar product in $\mathbb{E}^4$ given by

$$\langle x, y \rangle = \sum_{i=1}^{4} x_i y_i$$

where $x, y \in \mathbb{E}^4 \, (1 \leq i \leq 4)$, Then the curve α is said to be of unit speed (or parametrized by arclength) if it satisfies $\left\langle \alpha'(s), \alpha'(s) \right\rangle = 1$. In addition the norm of an arbitrary vector x in $\mathbb{E}^4$ is given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Let $\{T, N, B_1, B_2\}$ be the moving frame along the unit speed curve α, where T, N, B_1 and B_2 denote, the tangent, the principal normal, binormal and tri-normal vector fields, respectively. Then the Frenet formulas are given by [3]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{1}$$

Hence k_1, k_2 and $k_3$are called, the first, the second and the third curvature of α. If $k_3 \neq 0$ for each $s \in I \subseteq \mathbb{R}$, the curve lies fully in $\mathbb{E}^4$.

3 (k,m)-type slant helices in $\mathbb{E}^4$

In this section, we will define (k,m) type slant helices in $\mathbb{E}^4$.

Definition 1 *Let* α *be a regular unit speed curve in* $\mathbb{E}^4$ *with Frenet frame* $\{V_1,V_2,V_3,V_4\}$. *We call* α *is a* (k,m) *type slant helix if there exists a non-zero constant vector field* $U\in\mathbb{E}^4$ *satisfies* $\langle V_k,U\rangle=m$ $(m$ *constant) for* $1\le k\le 4$ $k\neq m$. *The constant vector* U *is on axis of* (k,m)*-type slant helix. We decompose* U *with respect to Frenet frame* $\{T,N,B_1,B_2\}$ *as* $U=u_1T+u_2N+u_3B_1+u_4B_2$, *where* $u_i=u_i(s)$ *are differentiable functions of* s. *Here we denote* $V_1=T,V_2=N,V_3=B_1,V_4=B_2$. *From now on, in sake of easinesss, we will use these notations and assume that* $k_i\neq 0$, $(1\le i\le 3)$.

Theorem 1 *There are no* $(1,2)$ *type slant helices in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1,2)$ type slant helix. Then for a constant vector field U. $\langle T,U\rangle=a$ is const and $\langle N,U\rangle=b$ is constant. Differentiating this equation and using Frenet equations, we obtain $k_1\langle N,U\rangle=0$ means that U is orthogonal to N. □

Theorem 2 *There are no* $(1,3)$ *type slant helices in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1,3)$ type slant helix. Then we may write $\langle T,U\rangle=const=a\ \langle B,U\rangle=const=b$. Also taking account Theorem 1 we decompose U as follows

$$U=aT+bB_1+u_2B_2 \tag{2}$$

Differentiating constant vector U, we get

$$\begin{aligned} a\left(k_1N\right)+b\left(-k_2N+k_3B_2\right)+u_2'B_2+u_2\left(-k_3B_1\right)&=0\\ ak_1N-bk_2&=0 &(3)\\ -u_2k_3&=0 &(4)\\ u_2'+bk_3&=0. &(5)\end{aligned}$$

From (4) we get $u_2=0$ and hence $b=0$, which means that there are no $(1,3)$ type slant helices in $\mathbb{E}^4$. □

Theorem 3 *There are no* $(1,4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1,4)$ type slant helix. Then we may write

$$U = aT + u_1 B_1 + bB_2$$

We know that U is constant then we get

$$\begin{aligned} a\left(k_1 N\right) + u_1^{'} B_1 + u_1\left(-k_2 N + k_3 B_2\right) + b\left(-k_3 B_1\right) &= 0 \\ a k_1 - u_1 k_2 &= 0 \quad (6) \\ u_1^{'} - b k_3 &= 0 \quad (7) \\ u_1 k_3 &= 0 \quad (8) \end{aligned}$$

means that $u_1 = 0$ and from (7) we get $b = 0$, hence there are no $(1,4)$ type slant helix in $\mathbb{E}^4$. □

Corollary 1 *There are no* $(1,k)$ *type slant helix in* $\mathbb{E}^4$.

Theorem 4 *If* α *is a* $(2,3)$ *type slant helix in* $\mathbb{E}^4 \Longleftrightarrow$ *there exist a constant such that*

$$\frac{k_2(t)}{k_1(t)} - \int_0^s k_1(t)\,dt = 0.$$

Proof. Assume that α is a $(2,3)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1 T + aN + bB_1 + u_2 B_2.$$

Differentiating constant vector U, one may get

$$\begin{aligned} u_1\left(k_1 N\right) + u_1^{'} T + a\left(-k_1 T + k_2 B_1\right) & \\ + b\left(-k_2 N + k_3 B_2\right) + u_2^{'} B_2 + u_2\left(-k_3 B_1\right) &= 0 \\ u_1^{'} - a k_1 &= 0 \quad (9) \\ u_1 k_1 - b k_2 &= 0 \quad (10) \\ a k_2 - k_3 &= 0 \quad (11) \\ b k_3 + u_2^{'} &= 0 \quad (12) \end{aligned}$$

Using (9) and (12)

$$u_1 = a \int_0^s k_1(t)\,dt \quad (13)$$

and

$$u_2 = -b\int_0^s k_3(t)\,dt \tag{14}$$

From (10) we get $u_1 = b\frac{k_2}{k_1}$. Taking into account this result in (13) we get

$$c\int_0^s k_1(t)\,dt = b\frac{k_2}{k_1} = \frac{a}{b}\int_0^s k_1(t)\,dt. \tag{15}$$

We assume a and b are constants so we can denote $\frac{a}{b} = \mathrm{const}$.

Using this fact in (15) we conclude that

$$\frac{k_2(t)}{k_1(t)} - \int_0^s k_1(t)\,dt = 0.$$

This completes the proof. □

Theorem 5 *There are no* $(2, 4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(2, 4)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1T + aN + u_2B_1 + bB_2.$$

Recall that U is a constant vector, we obtain

$$\begin{aligned} u_1^{'}T + u_1(k_1N) + a(-k_1T + k_2B_1) \\ +u_2^{'}B_1 + u_2(-k_2N + k_3B_2) + b(-k_3B_1) = 0 \\ u_1^{'} - ak_1 = 0 \qquad (16) \\ u_1k_1 - u_2k_2 = 0 \qquad (17) \\ ak_2 + u_2^{'} - bk_3 = 0 \qquad (18) \\ u_2k_3 = 0 \qquad (19) \end{aligned}$$

From (19) we get $u_2 = 0$. Using this in (17) we get $u_1 = 0$ and finally we get $a = b = 0$ which means that there are no $(2, 4)$ type slant helix in $\mathbb{E}^4$. □

Theorem 6 *There are no* $(3, 4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(3,4)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1T + u_2N + aB_1 + bB_2$$

Taking into account of the constant vector U we get

$$\begin{aligned}\left(u_1^{'} - u_2k_1\right)T + \left(u_1k_1 + u_2^{'} - ak_2\right)N \\ + \left(u_2k_2 - bk_3\right)B_1 + \left(ak_3\right)B_2 &= 0 \\ u_1^{'} - u_2k_1 &= 0 \\ u_1k_1 + u_2^{'} - ak_2 &= 0 \\ u_2k_2 - bk_3 &= 0 \\ ak_3 &= 0\end{aligned}$$

$\Rightarrow a = 0$ so there is no $(3,4)$ type slant helix in $\mathbb{E}^4$. □

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