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Finite groups with a certain number of cyclic subgroups II

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Abstract. In this note we describe the finite groups G having $|G|-2$ cyclic subgroups. This partially solves the open problem in the end of [3].

Let G be a finite group and $C(G)$ be the poset of cyclic subgroups of G. The connections between $|C(G)|$ and $|G|$ lead to characterizations of certain finite groups G. For example, a basic result of group theory states that $|C(G)| = |G|$ if and only if G is an elementary abelian 2-group. Recall also the main theorem of [3], which states that $|C(G)| = |G|-1$ if and only if G is one of the following groups: $\mathbb{Z}_3$, $\mathbb{Z}_4$, S_3 or D_8.

In what follows we shall continue this study by describing the finite groups G for which

$$|C(G)| = |G| - 2. \qquad (*)$$

First, we observe that certain finite groups of small orders, such as $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_{12} and $\mathbb{Z}_2 \times D_8$, have this property. Our main theorem proves that in fact these groups exhaust all finite groups G satisfying $(*)$.

Theorem 1 *Let G be a finite group. Then $|C(G)| = |G| - 2$ if and only if G is one of the following groups: $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_{12} or $\mathbb{Z}_2 \times D_8$.*

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Proof. We will use the same technique as in the proof of Theorem 2 in [3]. Assume that G satisfies $(*)$, let $n = |G|$ and denote by $d_1 = 1, d_2, \ldots, d_k$ the positive divisors of n. If $n_i = |\{H \in C(G) \mid |H| = d_i\}|$, $i = 1, 2, \ldots, k$, then

$$\sum_{i=1}^{k} n_i \phi(d_i) = n.$$

Since $|C(G)| = \sum_{i=1}^{k} n_i = n - 2$, one obtains

$$\sum_{i=1}^{k} n_i (\phi(d_i) - 1) = 2,$$

which implies that we have the following possibilities:

Case 1. There exists $i_0 \in \{1, 2, \ldots, k\}$ such that $n_{i_0}(\phi(d_{i_0}) - 1) = 2$ and $n_i(\phi(d_i) - 1) = 0, \forall\, i \neq i_0$.

Since the image of the Euler's totient function does not contain odd integers > 1, we infer that $n_{i_0} = 2$ and $\phi(d_{i_0}) = 2$, i.e. $d_{i_0} \in \{3, 4, 6\}$. We remark that d_{i_0} cannot be equal to 6 because in this case G would also have a cyclic subgroup of order 3, a contradiction. Also, we cannot have $d_{i_0} = 3$ because in this case G would contain two cyclic subgroups of order 3, contradicting the fact that the number of subgroups of a prime order p in G is $\equiv 1 \pmod{p}$ (see e.g. the note after Problem 1C.8 in [1]). Therefore $d_{i_0} = 4$, i.e. G is a 2-group containing exactly two cyclic subgroups of order 4. Let $n = 2^m$ with $m \geq 3$. If $m = 3$ we can easily check that the unique group G satisfying $(*)$ is $\mathbb{Z}_2 \times \mathbb{Z}_4$. If $m \geq 4$ by Proposition 1.4 and Theorems 5.1 and 5.2 of [2] we infer that G is isomorphic to one of the following groups:

- M_{2^m};
- $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^8 = 1,\ a^b = a^{-1},\ a^{2^{m-3}} = b^4 \rangle$, where $m \geq 5$;
- $\mathbb{Z}_2 \times D_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^2 = 1,\ a^b = a^{-1+2^{m-4}} c,\ c^2 = [c, b] = 1,\ a^c = a^{1+2^{m-3}} \rangle$, where $m \geq 5$.

All these groups have cyclic subgroups of order 8 for $m \geq 5$ and thus they do not satisfy $(*)$. Consequently, $m = 4$ and the unique group with the desired property is $\mathbb{Z}_2 \times D_8$.

Case 2. There exist $i_1, i_2 \in \{1, 2, \ldots, k\}$, $i_1 \neq i_2$, such that $n_{i_1}(\phi(d_{i_1})-1) = n_{i_2}(\phi(d_{i_2}) - 1) = 1$ and $n_i(\phi(d_i) - 1) = 0, \forall\, i \neq i_1, i_2$.

Then $n_{i_1} = n_{i_2} = 1$ and $\phi(d_{i_1}) = \phi(d_{i_2}) = 2$, i.e. $d_{i_1}, d_{i_2} \in \{3, 4, 6\}$. Assume that $d_{i_1} < d_{i_2}$. If $d_{i_2} = 4$, then $d_{i_1} = 3$, that is G contains normal cyclic subgroups of orders 3 and 4. We infer that G also contains a cyclic subgroup of order 12, a contradiction. If $d_{i_2} = 6$, then we necessarily must have $d_{i_1} = 3$. Since G has a unique subgroup of order 3, it follows that a Sylow 3-subgroup of G must be cyclic and therefore of order 3. Let $n = 3 \cdot 2^m$, where $m \geq 1$. Denote by n_2 the number of Sylow 2-subgroups of G and let H be such a subgroup. Then H is elementary abelian because G does not have cyclic subgroups of order 2^i with $i \geq 2$. By Sylow's Theorems,

$$n_2 | 3 \text{ and } n_2 \equiv 1 \pmod 2,$$

implying that either $n_2 = 1$ or $n_2 = 3$. If $n_2 = 1$, then $G \cong \mathbb{Z}_2^m \times \mathbb{Z}_3$, a group that satisfies $(*)$ if and only if $m = 1$, i.e. $G \cong \mathbb{Z}_6$. If $n_2 = 3$, then $|\mathrm{Core}_G(H)| = 2^{m-1}$ because $G/\mathrm{Core}_G(H)$ can be embedded in S_3. It follows that G contains a subgroup isomorphic with $\mathbb{Z}_2^{m-1} \times \mathbb{Z}_3$. If $m \geq 3$ this has more than one cyclic subgroup of order 6, contradicting our assumption. Hence either $m = 1$ or $m = 2$. For $m = 1$ one obtains $G \cong S_3$, a group that does not have cyclic subgroups of order 6, a contradiction, while for $m = 2$ one obtains $G \cong D_{12}$, a group that satisfies $(*)$. This completes the proof. □

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