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A note on some relations between certain inequalities and normalized analytic functions

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Abstract. In this note, an extensive result consisting of several relations between certain inequalities and normalized analytic functions is first stated and some consequences of the result together with some examples are next presented. For the proof of the presented result, some of the assertions indicated in [5], [8] and [11] along with the results in [3] and [4] are also considered.

1 Introduction, definitions and motivation

Firstly, *here and throughout this investigation,* let $\mathbb{C}$ be the complex plane, $\mathbb{U}$ be the *unit open* disc, *i.e.,* $\{z \in \mathbb{C} : |z| < 1\}$ and also let $\mathcal{H}$ denote the class of all analytic functions in $\mathbb{U}$. Moreover, a function $f(z) \in \mathcal{H}$ is said to be a *convex function* (in $\mathbb{U}$) *if* $f(\mathbb{U})$ is a *convex* domain. In this respect, let $\mathcal{A}$ be the subclass of all functions $\mathcal{H}$ such that $f(0) = f'(0) - 1 = 0$, that is, $f(z) \in \mathcal{A}$ is of the form $f(z) = z + a_1 z + a_2 z^2 + \cdots$, *where* $z \in \mathbb{U}$ and $a_i \in \mathbb{C}$

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for all $i = 1, 2, 3, \cdots$. In general, the subclass of $\mathcal{A}$ consisting of all univalent functions is denoted by $\mathcal{S}$. At the same time, $f(z) \in \mathcal{A}$ is *convex function* iff $\mathfrak{Re}\{1 + zf''(z)/f'(z)\} > 0$ for all $z \in \mathbb{U}$. Furthermore, $f(z) \in \mathcal{H}$ is said to be *starlike* if $f(z)$ is *univalent* and $f(\mathbb{U})$ is a *starlike* domain (with respect to $z = 0$). It is well-known that $f(z) \in \mathcal{A}$ is *starlike* iff $\mathfrak{Re}\{zf'(z)/f(z)\} > 0$ for all $z \in \mathbb{U}$. The classes $\mathcal{K}$ and $\mathcal{S}^*$ denote the *normalized* functions' class of the functions $f(z)$ in $\mathcal{S}$, *when* $f(\mathbb{U})$ is *convex* and $f(\mathbb{U})$ is *starlike,* respectively. The class $\mathcal{S}^*(\alpha)$ denotes the class of all s*tarlike* functions $f(z)$ of order α $(0 \leq \alpha < 1)$ if $f(z) \in \mathcal{A}$ and $\mathfrak{Re}\{zf'(z)/f(z)\} > \alpha$ for all $z \in \mathbb{U}$. Besides, the class $\mathcal{K}(\alpha)$ denotes the class of all s*convex* functions $f(z)$ of order α $(0 \leq \alpha < 1)$ if $f(z) \in \mathcal{A}$ and $\mathfrak{Re}\{1 + zf'(z)/f(z)\} > \alpha$ for all $z \in \mathbb{U}$. Namely, $\mathcal{K}(\alpha)$ is the class of all *convex* functions $f(z) \in \mathcal{A}$ satisfying the condition $\mathfrak{Re}\{1 + zf''(z)/f'(z)\} > \alpha$ for all $z \in \mathbb{U}$ and for some α $(0 \leq \alpha < 1)$. In addition, let $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$, *which* are the subclasses of *starlike* and *convex* functions with respect to the origin $(z = 0)$ in $\mathbb{U}$, respectively. (See, for the details of the related definitions (and also information), [1], [2], and see also (for novel examples) [3], [4], [6], [7].)

The literature presents us several works including important *or* interesting results between certain inequalities and certain classes of the functions which are *analytic* and *univalent* in the disc $\mathbb{U}$. For those, one may look over the earlier results presented in [3], [8], [9], [10] and [11]. In particularly, in [8], the problem of finding $\lambda > 0$ such that the condition $|f''(z)|$, *where* $f(z) \in \mathcal{A}$ and $z \in \mathbb{U}$, implies $f(z) \in \mathcal{S}^*$, was firstly considered by P. T. Mocanu for $\lambda = 2/3$. Later, in [9], S. Ponnusamy and V. Singh considered the problem for $\lambda = 2/\sqrt{3}$. Afterwards, in [10], M. Obradović focused on the problem for $\lambda = 1$ by proving that his result is sharp. In [11], N. Tuneski also obtained certain results dealing with the same problems, which are also generalizations of the results of M. Obradović in [10].

In this investigation, by using a different technique, developed by S. S. Miller and P. T. Mocanu in [5], certain results determined by the functions $f(z) \in \mathcal{A}$ relating to both condition $|f''(z)| \leq \lambda$ for some values of $\lambda > 0$ and the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are restated and then their certain consequences which will be important for (analytic and) geometric function theory are given. In addition, only for the proofs of these consequences of our main results derived in the Section 2 of this paper, both the assertion of S. S. Miller and P. T. Mocanu given in [3] and the results of N. Tuneski given in [11] are also used.

The following two assertions (Lemma 1 in [3] and Lemma 2 in [11] below) will be required to prove the main results.

Lemma 1 *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ *and* $0 \leq \alpha < 1$. *Then,*

$$(2-\alpha)\left|f''(z)\right| \leq 2(1-\alpha) \;\Rightarrow\; f(z) \in \mathcal{S}^*(\alpha).$$

The result is sharp.

Lemma 2 *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ *and* $0 \leq \alpha < 1$. *Then,*

$$(2-\alpha)\left|f''(z)\right| \leq 1-\alpha \;\Rightarrow\; f(z) \in \mathcal{K}(\alpha).$$

The result is sharp.

The following important assertion (see, for its details and also example, [3] (p. 33-34 and $a = 0$)) will be required to prove the main results.

Lemma 3 *Let* $\Omega \subset \mathbb{C}$ *and suppose that the function* $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfies* $\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega$ *for all* $K \geq Mn$, $\theta \in \mathbb{R}$, *and* $z \in \mathbb{U}$. *If the function* $p(z)$ *is in the class:*

$$\mathcal{H}\left[0,n\right] := \left\{p(z) \in \mathcal{H} \;:\; p(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots .\ (z \in \mathbb{U})\right\}$$

and

$$\psi\left(p(z), zp'(z); z\right) \in \Omega \ ,$$

then $|p(z)| < M$, *where for some* $M > 0$ *and for all* $z \in \mathbb{U}$.

2 The main results, implications and examples

By making use of Lemma 3, we shall firstly give and then prove the main result, which is given by

Theorem 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < M$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2} \quad\Rightarrow\quad \left|f''(z)\right| < M + 2|a_2|\,.$$

Proof. Let us define $p(z)$ by

$$p(z) = f''(z) - 2a_2 \ ,$$

where $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$ and $z \in \mathbb{U}$. Clearly, $p(z)$ is in the class $\mathcal{H}[0, 1]$ (when, of course, $a_3 \neq 0$). Then, it immediately follows that

$$\frac{zp'(z)}{\delta + p(z)} = \frac{zf'''(z)}{\delta - 2a_2 + f''(z)} \qquad \left(f''(z) \neq 2a_2 - \delta;\ z \in \mathbb{U}\right).$$

Let

$$\psi(r, s; z) := \frac{s}{\delta + r}$$

and

$$\Omega := \left\{ w\ :\ w \in \mathbb{C}\ \text{ and }\ \Re\mathfrak{e}\{w\} < \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2} \right\}.$$

Then we have

$$\psi\left(p(z), zp'(z); z\right) = \left(\frac{zp'(z)}{\delta + p(z)} =\right) \frac{zf'''(z)}{\delta - 2a_2 + f''(z)} \in \Omega$$

for all z in $\mathbb{U}$. Furthermore, for any $\theta \in \mathbb{R}$, $K \geq nM \geq M$, and $z \in \mathbb{U}$, we obviously obtain that

$$\Re\mathfrak{e}\left\{\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right)\right\} = \Re\mathfrak{e}\left(\frac{Ke^{i\theta}}{\delta + Me^{i\theta}}\right) \geq \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2},$$

i.e.,

$$\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega .$$

Therefore, in respect of the Lemma 3, the definition of $p(z)$ easily yields that

$$\left|p(z)\right| = \left|f''(z) - 2a_2\right| < M \qquad (M > 0; z \in \mathbb{U}),$$

which completes the desired proof. □

Proposition 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < 1$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re\mathfrak{e}\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \Phi(\alpha, \delta, a_2) \quad \Rightarrow \quad f(z) \in \mathcal{S}^*(\alpha),$$

where

$$\Phi(\alpha, \delta, a_2) := \frac{\left[2(1-\alpha) - 2(2-\alpha)|a_2|\right]\left[2(1-\alpha) - (2-\alpha)(2|a_2| + \delta)\right]}{(2-\alpha)^2\delta^2 + \left[(2-\alpha)(\delta - 2|a_2|) + 2(1-\alpha)\right]^2}.$$

Proof. If we take

$$M + 2|a_2| := \frac{2(1-\alpha)}{2-\alpha} \quad (0 \leq \alpha < 1)$$

in Theorem 1 and just then use Lemma 1, we easily get the proof. □

By letting $\alpha := 0$ in Proposition 1, we first obtain the following corollary.

Corollary 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < 1$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e \left(\frac{z f'''(z)}{\delta - 2a_2 + f''(z)} \right) < \frac{(1 - 2|a_2|)(1 - 2|a_2| - \delta)}{(1 - 2|a_2| + \delta)^2 + \delta^2} \quad \Rightarrow \quad f(z) \in \mathcal{S}^* \, .$$

By taking $\delta := 2|a_2|$ in Corollary 1, we next have the following corollary.

Corollary 2 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < 2|a_2| < 1$, *and let* $f''(z) \neq 0$. *Then,*

$$\Re e \left(\frac{z f'''(z)}{f''(z)} \right) < \frac{1 - 2|a_2|}{1 + 4|a_2|^2} \quad \Rightarrow \; f(z) \in \mathcal{S}^* \, .$$

For this result (i.e., for Corollary 2), the following example can be easily given.

Example 1 *Take* $f(z) = z + \frac{1}{4} z^2 + a_3 z^3$ *and let* $|a_3| < \frac{1}{18}$. *Since*

$$\left| f''(z) \right| = \left| \frac{1}{2} + 6a_3 z \right| \geq \frac{1}{2} - 6|a_3| > \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0 \; ,$$

we arrive at $f''(z) \neq 0$. *Besides, it is obvious that* $|a_2| = \frac{1}{4} < \frac{1}{2}$. *At the same time, clearly,*

$$\Re e \left(\frac{z f'''(z)}{f''(z)} \right) = 1 - \Re e \left(\frac{1}{1 + 12 a_3 z} \right) < \frac{1 - 2|a_2|}{1 + 4|a_2|^2} = \frac{2}{5} \, .$$

In that case, as a result of Corollary 2, it is clear that $f(z) \in \mathcal{S}^*$. We also indicate that, since $\left| f''(z) \right| = \left| \frac{1}{2} + 6a_3 z \right| < 1$, Lemma 1 immediately implies that the function $f(z)$ is starlike in $\mathbb{U}$.

Proposition 2 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < \frac{1}{2}$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e \left(\frac{z f'''(z)}{\delta - 2a_2 + f''(z)} \right) < \Phi(\alpha, \delta, a_2) \quad \Rightarrow \quad f(z) \in \mathcal{K}(\alpha) \ ,$$

where

$$\Phi(\alpha, \delta, a_2) := \frac{[(1-\alpha) - 2(2-\alpha)|a_2|]\,[(1-\alpha) - (2-\alpha)(2|a_2| + \delta)]}{(2-\alpha)^2 \delta^2 + [(2-\alpha)(\delta - 2|a_2|) + (1-\alpha)]^2} \ .$$

Proof. If we put

$$M + 2|a_2| := \frac{1-\alpha}{2-\alpha} \quad (0 \leq \alpha < 1)$$

in Theorem 1 and just then use Lemma 2, we easily arrive at the desired result in Proposition 2. □

By putting $\alpha = 0$ in Proposition 2, we then get the following result.

Corollary 3 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < \frac{1}{2}$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e \left(\frac{z f'''(z)}{\delta - 2a_2 + f''(z)} \right) < \frac{(1 - 4|a_2|)(1 - 4|a_2| - 2\delta)}{(1 - 4|a_2| + 2\delta)^2 + 4\delta^2} \quad \Rightarrow \quad f(z) \in \mathcal{K} \ .$$

By setting $\delta := 2|a_2|$ in Corollary 3, we also get the following corollary.

Corollary 4 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < 2|a_2| < \frac{1}{2}$, *and let* $f''(z) \neq 0$. *Then,*

$$\Re e \left(\frac{z f'''(z)}{f''(z)} \right) < \frac{1 - 4|a_2|}{1 + 4|a_2|^2} \quad \Rightarrow \quad f(z) \in \mathcal{K} \ .$$

The following can be also given to exemplify the result given above.

Example 2 *Take* $f(z) = z + \frac{1}{8} z^2 + a_3 z^3$ *and let* $|a_3| < \frac{1}{24\sqrt{2}}$. *Since*

$$|f''(z)| = \left| \frac{1}{4} + 6a_3 \right| \geq \frac{1}{4} - 6|a_3| > \frac{1}{4} - \frac{1}{4\sqrt{2}} > 0 \ ,$$

we obtain $f''(z) \neq 0$. *Furthermore, it is clear that* $|a_2| = \frac{1}{4} < \frac{1}{2}$. *At the same time, obviously,*

$$\Re e \left(\frac{z f'''(z)}{f''(z)} \right) = 1 - \Re e \left(\frac{1}{1 + 24 a_3 z} \right) < \frac{1 - 4|a_2|}{1 + 4|a_2|^2} = \frac{8}{17} \ .$$

In this case, as a result of Corollary 4, it is clear that $f(z) \in \mathcal{K}$. *Then, since* $|f''(z)| = \left|\frac{1}{4} + 6a_3\right| < 1$, *Lemma2 immediately implies that function* $f(z)$ *is convex in* $\mathbb{U}$.

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