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Computing metric dimension of compressed zero divisor graphs associated to rings

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Abstract. For a commutative ring R with $1 \neq 0$, a compressed zero-divisor graph of a ring R is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0],[1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$. In this paper, we study the metric dimension of the compressed zero divisor graph $\Gamma_E(R)$, the relationship of metric dimension between $\Gamma_E(R)$ and $\Gamma(R)$, classify the rings with same or different metric dimension and obtain the bounds for the metric dimension of $\Gamma_E(R)$. We provide a formula for the number of vertices of the family of graphs given by $\Gamma_E(R \times \mathbb{F})$. Further, we discuss the relationship between metric dimension, girth and diameter of $\Gamma_E(R)$.

1 Introduction

Beck [7] first introduced the notion of a zero divisor graph of a ring R and his interest was mainly in coloring of zero divisor graphs. Anderson and Livingston [3] studied zero divisor graph of non-zero zero divisors of a commutative ring

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R. For a commutative ring R with $1 \neq 0$, let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero divisors of R. A *zero divisor graph* $\Gamma(R)$ is the undirected graph with vertex set $Z^*(R)$ and the two vertices x and y are adjacent if and only if $xy = 0$. This zero divisor graph has been studied extensively and even more the idea has been extended to the ideal based zero divisor graphs in [15, 23] and modules in [20]. Inspired by ideas from Mulay [16], we study the zero divisor graph of equivalence classes of zero divisors of a ring R. Anderson and LaGrange [4] studied this under the term compressed zero divisor graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$, constructed by taking the vertices to be equivalence classes $[x] = \{y \in R \mid \mathfrak{ann}(x) = \mathfrak{ann}(y)\}$, for every $x \in R \setminus ([0] \cup [1])$ and each pair of distinct classes $[x]$ and $[y]$ is joined by an edge if and only if $[x][y] = 0$, that is, if and only if $xy = 0$. If x and y are distinct adjacent vertices in $\Gamma(R)$, we note that $[x]$ and $[y]$ are adjacent in $\Gamma_E(R)$ if and only if $[x] \neq [y]$. It is clear that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$ and that $[x] \subseteq Z(R) \setminus \{0\}$, for each $x \in R \setminus ([0] \cup [1])$. Some results on the compressed zero divisor graph can be seen in [5].

For example, consider $R = \mathbb{Z}_{12}$. Here, $Z^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ is the vertex set of $\Gamma(R)$, see Fig 1(a). For the vertex set of $\Gamma_E(R)$, we have $\mathfrak{ann}(2) = \{6\}$, $\mathfrak{ann}(3) = \{4, 8\}$, $\mathfrak{ann}(4) = \{3, 6, 9\}$, $\mathfrak{ann}(6) = \{2, 4, 6, 8, 10\}$, $\mathfrak{ann}(8) = \{3, 6, 9\}$, $\mathfrak{ann}(9) = \{4, 8\}$, $\mathfrak{ann}(10) = \{6\}$.

So, $Z(R_E) = \{[2], [3], [4], [6]\}$ is the vertex set of $\Gamma_E(R)$, see Fig 1(b).

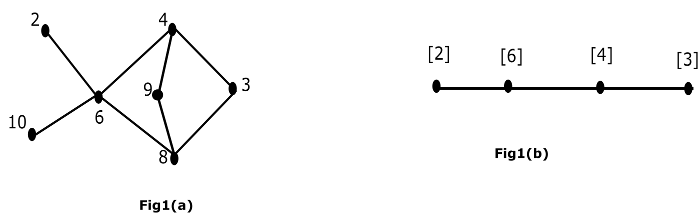

Figure 1: $\Gamma(\mathbb{Z}_{12})$ and $\Gamma_E(\mathbb{Z}_{12})$

We note that the vertices of the graph $\Gamma_E(R)$ correspond to *annihilator* ideals in the ring and hence *prime* ideals if R is a Noetherian ring in which case $Z(R_E)$ is called as the *spectrum* of a ring. Clearly $\Gamma_E(R)$ is connected and $\mathfrak{diam}(\Gamma_E(R)) \leq 3$. Also $\mathfrak{diam}(\Gamma_E(R)) \leq \mathfrak{diam}(\Gamma(R))$. Anderson and LaGrange [5] showed that $\mathfrak{gr}(\Gamma_E(R)) \leq 3$ if $\Gamma_E(R)$ contains a cycle and determined the structure of $\Gamma_E(R)$ when it is acyclic and the monoids R_E when $\Gamma_E(R)$ is a star graph. In [4], they also show that $\Gamma_E(R) \cong \Gamma_E(S)$ for a Noetherian or finite commutative ring S.

The compressed zero-divisor graph has some advantages over the earlier studied zero divisor graph $\Gamma(R)$ as seen in [1, 2, 3] or subsequent zero divisor graph determined by ideal of R as seen in [15, 23]. For example, Spiroff and Wickham [[27], Proposition 1.10] showed that there are no finite regular graphs $\Gamma_E(R)$ for any ring R with more than two vertices. Further, they showed that R is a local ring (a ring R is said to be a local ring if it has a unique maximal ideal) if $\Gamma_E(R)$ is a star graph with at least four vertices.

Another important aspect of studying graphs of equivalence classes is the connection to associated primes of the ring. In general, all the associated primes of a ring R correspond to distinct vertices in $\Gamma_E(R)$. Through out, R will denote a commutative ring with unity, $U(R)$ its set of units. We will denote a finite field on q elements by $\mathbb{F}_q$, ring of integers modulo n by $\mathbb{Z}_n$ and all graphs are simple graphs in the sense that there are no loops. For basic definitions from graph theory we refer to [11, 17], and for commutative ring theory we refer to [6, 13].

A graph G is *connected* if there exists a path between every pair of vertices in G. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest $u - v$ path in G. If such a path does not exist, we define $d(u, v)$ to be infinite. The *diameter* of a graph is the maximum distance between any two vertices of G. The diameter is 0 if the graph consists of a single vertex. Also, the *girth* of a graph G, denoted by $gr(G)$, is the length of a smallest cycle in G. Slater [25] introduced the concept of a *resolving set* for a connected graph G under the term locating set. He referred to a minimum resolving set as a reference set for G and called the cardinality of a minimum resolving set (reference set) the location number of G. Independently, Harary and Melter [12] discovered these concepts as well but used the term metric dimension, rather than location number. The concept of metric dimension has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [8, 9], robot navigation [14], combinatorial optimization [24], sonar and coast guard Loran [26]. We adopt the terminology of Harary and Melter.

In this paper, we study the notion of metric dimension of $\Gamma_E(R)$. We explore the relationship between metric dimension of $\Gamma_E(R)$ and $\Gamma_E(R)$. We obtain the metric dimension of $\Gamma_E(R)$ whenever it exists. We also classify the rings having the same or different metric dimension and obtain bounds for the metric dimension of $\Gamma_E(R)$. We also provide relationship between the metric dimension, girth and diameter of $\Gamma_E(R)$.

2 Metric dimension of some graphs $\Gamma_E(R)$

Let G be a connected graph with $n \geq 2$ vertices. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector as the metric representation (locating code) of v with respect to W as

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

The set W is a resolving set of G if distinct vertices have distinct metric representations (codes) and a resolving set containing the minimum number of vertices is called a *metric basis* for G and the *metric dimension*, denoted by $\dim(G)$, of G is the cardinality of a metric basis. If W is a finite metric basis, we say that $r(v|W)$ are the metric coordinates of vertex v with respect to W. The only vertex of G whose metric coordinate with respect to W has 0 in its i^{th} coordinate of $r(v|W)$ is $\{w_i\}$. So the vertices of W necessarily have distinct metric representations. Since only those vertices of G that are not in W have coordinates all of which are positive, it is only these vertices that need to be examined to determine if their representations are distinct. This implies that the metric dimension of G is at most $n-1$. In fact for every connected graph G of order $n \geq 2$, we have $1 \leq \dim(G) \leq n-1$.

For example, consider the graph G given in Figure 2. Take $W_1 = \{v_1, v_3\}$. So, $r(v_1|W_1) = (0, 1)$, $r(v_2|W_1) = (1, 1)$, $r(v_3|W_1) = (1, 0)$, $r(v_4|W_1) = (1, 1)$, $r(v_5|W_1) = (2, 1)$. Notice, $r(v_2|W_1) = (1, 1) = r(v_4|W_1)$, therefore W_1 is not a resolving set. However, if we take $W_2 = \{v_1, v_2\}$, then $r(v_1|W_2) = (0, 1)$, $r(v_2|W_2) = (1, 0)$, $r(v_3|W_2) = (1, 1)$, $r(v_4|W_2) = (1, 2)$, $r(v_5|W_2) = (2, 1)$. Since distinct vertices have distinct metric representations, W_2 is a minimum resolving set and thus this graph has metric dimension 2.

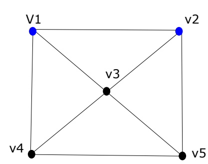

Figure 2: $\dim(G) = 2$

Now, we have the following observation.

Lemma 1 *A connected graph G of order n has metric dimension 1 if and only if $G \cong P_n$, where P_n denotes a path on n vertices of length $n-1$.*

Proof. Suppose $G \cong P_n$. Let $x_1 - x_2 - \cdots - x_n$ be a path on n vertices of G. Since $d(x_i, x_1) = i-1$ for $1 \leq i \leq n$, it follows $\{x_1\}$ is a minimum resolving set and therefore metric basis for $\Gamma_E(R)$. So $dim(P_n) = 1$.

Conversely, let G be not a path. Then either G is a cycle or it contains a vertex v whose degree is at least 3. But, G can not be a cycle as $dim(G) = 2$, see ([18], Lemma 2.3). Let u_1, u_2, $\ldots$, u_k be the vertices adjacent to v. Since $dim(G) = 1$ and if $W = \{w\}$ is a metric basis for G, then the metric representation of every vertex has a single coordinate. If d is the length of the shortest path from v to w, the coordinates of each u_i with respect to W is one of $\{d-1, d, d+1\}$, but $d(u_i, w) = d$ can not occur for all i $(1 \leq i \leq k)$. Therefore, it follows that at least two adjacent vertices of v have the same metric coordinates, which is a contradiction. Hence G is a path. $\square$

A graph $G(V, E)$ in which each pair of distinct vertices is joined by an edge is called a complete graph. A *complete* graph of n vertices is denoted by K_n. A graph G is said to be *bipartite* if its vertex set V can be partitioned into two sets V_1 and V_2 such that every edge of G has one end in V_1 and another in V_2. A bipartite graph is complete if each vertex of one partite set is joined to every vertex of the other partite set. We denote the complete bipartite graph with partite sets of order m and n by $K_{m,n}$. More generally, a graph is complete r-partite if the vertices can be partitioned into r distinct subsets, but no two elements of the same subset are adjacent. Based on the above definitions, we have the following observations.

Proposition 1 *The metric dimension of the compressed zero divisor graph $\Gamma_E(R)$ is 0 if and only if the zero divisor graph $\Gamma(R)$ of R ($R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$) is a complete graph.*

Proof. If $\Gamma(R) \cong K_n$, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z^*(R)$. Let $v_1, v_2, \ldots, v_n$ be the zero divisors of $\Gamma(R)$, then $[v_1] = [v_2], \cdots = [v_n]$ implies that all the vertices of $\Gamma(R)$ would collapse to a single vertex in $\Gamma_E(R)$ and we know the metric dimension of a single vertex graph is 0.

Conversely, assume that $\Gamma(R)$ is not isomorphic to K_n. Then $\Gamma(R)$ contains at least one vertex not adjacent to all the other vertices. Thus $|\Gamma_E(R)| \geq 2$, so that $dim(\Gamma_E(R)) \geq 1$. $\square$

We can also obtain the converse part by letting $dim(\Gamma_E(R)) = 0$. Then $\Gamma_E(R) = \{[a]\}$ for some $a \in Z^*(R)$, that is, $\Gamma_E(R)$ is a graph on a single vertex, which then implies $\Gamma(R)$ is either isomorphic to a single vertex or a complete graph K_n, for all $n \geq 1$. If G is a connected graph of order $n \geq 2$, we say two distinct vertices u and v are distance similar, if $d(u, a) = d(v, a)$ for all

$a \in V(G) - \{u, v\}$. It can be seen that the distance similar relation ($\sim$) is an equivalence relation on $V(G)$ and two distinct vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$, or $uv \in E(G)$ and $N[u] = N[v]$. Further we can find several results on metric dimension for zero divisor graphs of rings in [18, 19, 21].

Proposition 2 *The metric dimension of $\Gamma_E(R)$ is 1 if $\Gamma(R)$ is isomorphic to a complete bipartite graph $K_{m,n}$, with m or $n \geq 2$.*

Proof. Let $\Gamma(R)$ be isomorphic to a complete bipartite graph $K_{m,n}$ with two distance similar classes V_1 and V_2. Let $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$ such that $u_i v_j = 0$ for all $i \neq j$. Clearly, each of V_1 and V_2 is an independent set. We see that $[u_1] = [u_2] = \cdots = [u_m]$ and $[v_1] = [v_2] = \cdots = [v_n]$, so that V_1 and V_2 each represents a single vertex in $\Gamma_E(R)$. Since the graph is connected, $\Gamma_E(R)$ is isomorphic to $K_{1,1}$, a path on two vertices. Therefore by Lemma 1, we have $\dim(\Gamma_E(R)) = 1$. □

Remark 1 *Note that the converse of this result need not be true, the graph illustrated in Fig.1 being a counter example. However, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma_E(R) \cong K_{1,1}$ with metric dimension 1 and $\Gamma(R) \cong \Gamma_E(R)$.*

One of the important differences between $\Gamma(R)$ and $\Gamma_E(R)$ is that the later can not be complete with at least three vertices, as seen in ([27], Proposition 1.5). However, if $\Gamma_E(R)$ is complete r-partite, then $r = 2$ and $\Gamma_E(R) \cong K_{n,1}$, for some $n \geq 1$, see ([27], Proposition 1.7). A second look at the above result allows us to deduce some facts about star graphs. A complete bipartite graph of the form $K_{n,1}, n \in \mathbb{N} \cup \{\infty\}$ is called a *star graph*. If $n = \infty$, we say the graph is an infinite star graph.

Corollary 1 *If R is a ring such that $\Gamma_E(R)$ is a star graph $K_{n,1}$ with $n \geq 2$, then $\dim(\Gamma_E(R)) = n - 1$.*

Proof. First we identify a centre vertex of $K_{n,1}$ adjacent to n vertices. Then partition the vertex set V of order $n+1$ into two distance similar classes, with centre vertex in one class V_1 and the remaining n vertices in another class V_2 which is clearly an independent set. Choose a subset of vertices W of V and $u \sim v$. Then $r(u|W) = r(v|W)$ whenever both $u, v \notin W$. Hence the metric basis contains all except at most two vertices one from each class V_i, $1 \leq i \leq 2$. Therefore, $\dim(K_{n,1}) = |V(\Gamma_E(R))| - 2 = n + 1 - 2 = n - 1$. □

For example the metric dimension of $K_{1,3}$ is 2, see Figure 3.

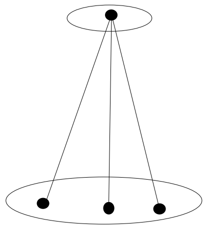

Figure 3: $\dim(K_{1,3} = 2)$

Corollary 2 *If* R *is a commutative ring such that* $\Gamma_E(R)$ *has at least* $n \geq 3$ *vertices, then* $\dim(\Gamma_E(R)) \neq n-1$.

Proof. Suppose $\dim(\Gamma_E(R)) = n-1$, $(n \geq 3)$. Then, by [18, Lemma 2.2], $\Gamma_E(R)$ is a complete graph on n vertices which is a contradiction to the argument prior to Corollary 1. Therefore $\dim(\Gamma_E(R)) \neq n-1$. □

Remark 2 *It is not known whether for each positive integer* n, *the star graph* $K_{n,1}$ *can be realized as* $\Gamma_E(R)$ *for some ring* R. *However, there is a ring* $R = \mathbb{Z}_2[x,y,z]/(x^2,y^2)$ *whose* $\Gamma_E(R)$ *is a star graph with infinitely many ends, that is,* $\Gamma_E(R)$ *is an infinite star graph. This ring also shows that the Noetherian condition is not enough to force* $\Gamma_E(R)$ *to be finite, see* [27]. *For* $n = 3$, *if the local ring* R *is isomorphic to* $\mathbb{Z}_4[x]/(x^2)$ *or* $\mathbb{Z}_2[x,y]/(x^2,y^2)$ *or* $\mathbb{Z}_4[x,y]/(x^2,y^2,xy-2,2x,2y)$, *then* $\Gamma_E(R) \cong K_{1,3}$ *and therefore* $\dim(\Gamma_E(R)) = 2$. *For* $n = 4$, *if the local ring* R *is isomorphic to* $\mathbb{Z}_8[x,y]/(x^2,y^2,4x,4y,2xy)$, *then* $\Gamma_E(R) \cong K_{1,4}$ *and therefore* $\dim(\Gamma_E(R)) = 3$. *For* $n = 5$, *if* $R \cong \mathbb{Z}_2[x,y,z]/(x^2,y^2,z^2,xy)$, *then* $\Gamma_E(R) \cong K_{5,1}$ *and therefore* $\dim(\Gamma_E(R)) = 4$. *This star graph* $K_{1,5}$ *is the smallest star graph that can be realized as* $\Gamma_E(R)$, *but not as a zero divisor graph.*

By definition of the compressed zero divisor graph $\Gamma_E(R)$ of a ring R, it is clear that each vertex in $\Gamma_E(R)$ is a representative of a distinct class of zero divisor activity in R. Thus, $\dim(\Gamma_E(R)) \leq \dim(\Gamma(R))$. However, the strict inequality holds if $\Gamma_E(R)$ has at least 3 vertices.

Example 1 *In the rings* $R = \frac{\mathbb{Z}_2[x,y]}{(x^2,xy,2x)}$, $R = \frac{\mathbb{Z}_4[x,]}{(x^2)}$, $R = \mathbb{Z}_{16}$, $R = \frac{\mathbb{Z}_8[x]}{(2x,x^2)}$, *it is easy to find that* $\dim(\Gamma_E(R)) < \dim(\Gamma(R))$.

It will be interesting to see the family of rings in which the equality $\dim(\Gamma_E(R) = \dim(\Gamma(R)$ occurs.

A ring R is called a *Boolean ring* if $a^2 = a$ for every $a \in R$. Clearly a Boolean ring R is commutative with char(R) = 2, where char(R) denotes the characteristic of a ring R. More generally, a commutative ring is *von Neumann regular* ring if for every $a \in R$, there exists $b \in R$ such that $a = a^2 b$, or equivalently, R is a reduced zero dimensional ring, see [13, Theorem 3.1]. A Boolean ring is clearly a von Neumann regular, but not conversely. For example, let $\{F_i\}_{i\in I}$ be a family of fields, then $\prod\limits_{i\in I} F_i$ is always von Neumann regular, but it is Boolean if and only if $F_i \cong \mathbb{Z}_2$ for all $i \in I$. Also the set $B(R) = \{a \in R \mid a^2 = a\}$ of idempotents of a commutative ring R becomes a Boolean ring with multiplication defined in the same way as in R, and addition defined by the mapping $(a, b) \mapsto a + b - 2ab$. In [13, Lemma 3.1], if $r, s \in \Gamma(R)$, the conditions $N(r) = N(s)$ and $[r] = [s]$ are equivalent if R is a reduced ring, and these are equivalent to the condition $rR = sR$ if R is a von Neumann regular ring. Furthermore, if R is a von Neumann regular ring and $B(R)$ is the set of idempotent elements of R, the mapping defined by $e \mapsto [e]$ is isomorphism from the subgraph of $\Gamma(R)$ induced by $B(R) \setminus \{0, 1\}$ onto $\Gamma_E(R)$ [13, Proposition 4.5]. In particular, if R is a Boolean ring (i.e., R = B(R)), then $\Gamma_E(R) \cong \Gamma(R)$. From this discussion, we have the following characterization.

Proposition 3 *Let R be a reduced commutative ring with unity. Then, metric dimension of the zero divisor graph $\Gamma(R)$ equals to metric dimension of its corresponding compressed zero divisor graph if R is a Boolean ring.*

Note that the converse of this result is not true in general. For example, the graphs in Figure 4 being a counter example, where $\dim(\Gamma(\mathbb{Z}_6)) = \dim(\Gamma_E(\mathbb{Z}_6))$, but R is not a Boolean ring.

Figure 4: $\dim(\Gamma(\mathbb{Z}_6)) = \dim(\Gamma_E(\mathbb{Z}_6)) = 1$

Corollary 3 *Let R and S be commutative reduced rings with unity 1. If $\Gamma(R) \cong \Gamma(S)$, then $\dim(\Gamma_E(R)) = \dim(\Gamma_E(S))$.*

Remark 3 *As seen in* [21, Theorem 2], *for the graph $\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)$ of a finite Boolean ring*

$$\dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n, \quad \dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n-1$$

for $n = 2, 3, 4$ and $\dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) = n$ for $n = 5$. This is also true for $\Gamma_E(R)$, follows by Proposition 3. The case $n > 5$ is still open.

3 Bounds for the metric dimension of $\Gamma_E(R)$

In this section, we investigate the role of metric dimension in the study of the structure of the graph $\Gamma_E(R)$. We also obtain metric dimension of some special type of rings that exhibit $\Gamma_E(R)$. Pirzada et al [18] characterized those graphs $\Gamma(R)$ for which the metric dimension is finite and for which the metric dimension is undefined [18, Theorem 3.1]. The analogous of this result is as follows.

Theorem 1 *Let R be a commutative ring. Then*
(i) $\dim(\Gamma_E(R))$ *is finite if and only if R is finite.*
(ii) $\dim(\Gamma_E(R))$ *is undefined if and only if R is an integral domain.*

However, $\dim(\Gamma_E(R))$ may be finite if R is infinite. For example, $R = \mathbb{Z}[x,y]/(x^3, xy)$ has $\Gamma_E(R)) \cong K_{1,3} + e$ (or paw graph), see Figure 5, and therefore has $\dim = 2$.

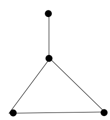

Figure 5:

The following lemma will be used to find the metric dimension of finite local rings.

Lemma 2 *If R is a finite local ring, then $|R| = p^n$, for some prime p and some positive integer n.*

Now, we have the following results.

Proposition 4 *If R is a local ring with $|R| = p^2$ and $p = 2, 3, 5$, then $\dim(\Gamma_E(R))$ is either 0 or undefined.*

Proof. Consider all local rings of order p^2 with p a prime. According to [10, p. 687] local rings of order p^2 are precisely $\mathbb{F}_{p^2}$, $\frac{\mathbb{F}_p[x]}{(x^2)}$, and $\mathbb{Z}_{p^2}$. If R is a field of order p^2, i.e., $R \cong \mathbb{F}_{p^2}$, then $\Gamma_E(R)$ is an empty graph, which implies $\dim(\Gamma_E(R))$ is undefined. If R is not a field and $|R| = p^2$, i.e., $R \cong \frac{\mathbb{F}_p[x]}{(x^2)}$, or

$\mathbb{Z}_{p^2}$ then $\Gamma_E(R)$ is a single vertex, when $p = 2,\ 3$ or 5 which then immediately gives that $\dim(\Gamma_E(R)) = 0$. □

From the above result, we also observe that $\dim(\Gamma(R)) = \dim(\Gamma_E(R))$, if

$$R \cong \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proposition 5 *If* R *is a local ring (not a field) of order*

(i) p^3 *with* $p = 2$ *or* 3, *then* $\dim(\Gamma_E(R))$ *is* 0, *and* $\dim(\Gamma_E(R)) = 1$ *only if* $R \cong \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 6)$ *or* $\mathbb{Z}_{27}$

(ii) p^4 *with* $p = 2$, *then* $\dim(\Gamma_E(R))$ *is* 0, 1 *or* 2.

Proof. (i) The following is the list of all the local rings of order p^3.

$$\mathbb{F}_{p^3}, \frac{\mathbb{F}_p[x, y]}{(x, y)^2}, \frac{\mathbb{F}_p[x]}{(x^3)}, \frac{\mathbb{Z}_p{}^2[x]}{(px, x^2)}, \frac{\mathbb{Z}_p{}^2[x]}{(px, x^2 - p)}$$

Case(a). When $p = 2$, the equivalence classes of the zero divisors in the local rings $\mathbb{Z}_2[x, y]/(x, y)^2$ and $\mathbb{Z}_4[x]/(2x, x^2)$ are same and is given by $[a] = \{x, y, x + y\}$ for any zero divisor a of the first ring and $[b] = \{2, x, x + 2\}$ for any zero divisor b of the second ring, that is, they get collapsed to a single vertex. Therefore $\dim(\Gamma_E(R)) = 0$. However, $\Gamma_E(R)$ of the rings $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$ and $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ is isomorphic to the graph $K_{1,1}$, which then, by Lemma 1, gives $\dim_E(R) = 1$.

Case(b). When $p = 3$ in the above list of local rings, we find that the compressed zero divisor graph structure of the rings $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 6)$ and $\mathbb{Z}_{27}$ is same and is isomorphic to $K_{1,1}$. Then, by Lemma 1, we have $\dim_E(R) = 1$. Also, in the rings $\dfrac{\mathbb{Z}_3{}^2[x]}{(3x, x^2)}$ and $\dfrac{\mathbb{Z}_3[x, y]}{(x, y)^2}$, the equivalence classes of all the zero divisors is same and is given by $[a] = \{3, 6, x, 2x, x + 3, x + 6, 2x + 3, 2x + 6\}$ for any non-zero zero divisor a of the first ring and $[b] = \{x, 2x, y, 2y, x + y, 2x + y, x + 2y, 2x + 2y\}$ for any non-zero zero divisor b of the later ring. Thus, $\Gamma_E(R)$ for both rings is a graph on a single vertex and follows that $\dim(\Gamma_E(R) = 0$.

(ii) Consider the local rings of order p^4, when $p = 2$. Corbas and Williams [10] conclude that there are 21 non-isomorphic commutative local rings with identity of order 16. The rings with $\dim(\Gamma_E(R)) = 0$ are $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y, z]/(x, y, z)^2$ and $\mathbb{Z}_4[x]/(x^2 + x + 1)$. The rings with $\dim(\Gamma_E(R)) = 1$ are $Z_2[x]/(x^4)$, $Z_2[x, y]/(x^3, xy, y^2)$, $Z_4[x]/(2x, x^3 - 2)$, $\mathbb{Z}_4[x]/(x^2 - 2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_4[x]$

$/(x^2-2x-2)$, $\mathbb{Z}_8[x]/(2x, x^2-2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_2[x]/(x^4)$ and $\mathbb{Z}_2[x]/(x^4)$. Further the rings with $\dim(\Gamma_E(R)) = 2$ are $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y]/(x^2, y^2)$ and $\mathbb{Z}_2[x, y]/(x^2-y^2, xy)$. □

Now, we find the metric dimension of $\Gamma_E(\mathbb{Z}_n)$.

Proposition 6 *Let* p *be a prime number.*
(i) *If* $n = 2p$ *and* $p > 2$*, then* $\dim(\Gamma_E(\mathbb{Z}_n)) = 1$*.*
(ii) *If* $n = p^2$*, then* $\dim(\Gamma_E(\mathbb{Z}_n)) = 0$*.*

Proof. (i) If $p = 2$, since $\Gamma_E(\mathbb{Z}_4)$ is a graph with single vertex. So, $\dim(\Gamma_E(\mathbb{Z}_4)$ $= 0$.

If $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{2, 2.2, 2.3, \ldots, 2.(p-1), p\}$. Since, $char(\mathbb{Z}_n) = 2p$, it follows that p is adjacent to all other vertices. Thus the equivalence classes of these zero divisors are given by
$[p] = \{2, 2.2, 2.3, \ldots, 2.(p-1)\}$, $[2] = [2.2] = \cdots = [2.(p-1)] = \{p\}$.

So, the vertex set of $\Gamma_E(\mathbb{Z}_n)$ is $Z(R_E) = \{[p], [2x]\}$ for any positive integer $x = 1, 2, \ldots, p-1$. Thus $\Gamma_E(\mathbb{Z}_n)$ is a path P_2 which then, by Lemma 1, gives $\dim(\Gamma_E(\mathbb{Z}_n) = 1$.
(ii) If $n = p^2$ and $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{p, p.2, p.3, \ldots, p(p-1)\}$. Since $char(\mathbb{Z}_n) = p^2$, it follows that the equivalence class of all these zero divisors is same and is $\{p, p.2, p.3, \ldots, p.(p-1)\}$. Thus, $\Gamma_E(R)$ in this case is a graph on a single vertex and therefore $\dim(\Gamma_E(R) = 0$. □

From the above result, we have the following observations.

Corollary 4 *Let* p *be a prime number*
(i) *If* $n = 2p$ *and* $p > 2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = 2$*.*
(ii) *If* $n = p^2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = 1$*.*
(iii) *If* $n = p^k$*,* $k > 3$ *and* $p > 2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = k-1$*.*

Proof. (i) and (ii) follow from Proposition 6.
(iii) When $n = p^k$, $k > 3$ and $p \geq 2$, the zero divisors of $\mathbb{Z}_n$ are
$Z(\mathbb{Z}_n) = \{up^i | u \in U(\mathbb{Z}_n)\}$, for $i = 1, 2, \ldots, k-1$. Now the equivalence classes of zero divisors are $[up] = \{up^{k-1}\}$, $[up^2] = \{up^{k-1}, up^{k-2}\}$, $\ldots$, $[up^{k-1}] = \{up^{k-1}, up^{k-2}, \ldots, up^2, up\}$.
In this way, we get $k-1$ distinct equivalence classes. Thus, $|\Gamma_E(\mathbb{Z}_n)| = k-1$. □

Corollary 5 $\dim(\Gamma_E(\mathbb{Z}_n)) \leq 2k-2$*, where* $n = p^k$*, for any prime* $p > 2$ *and* $k > 3$*.*

Proof. By [18, Theorem 2.1]. If G is a connected graph with G partitioned into m distance similar classes that consist of a single vertex, then $dim(G) \leq |V(G)| + m$.

Using part (iii) of Corollary 4, the result follows. □

The following important lemma, which is used later in the proof of several results, provides a combinatorial formula for the number of vertices of the compressed zero divisor graph $\Gamma_E(R \times \mathbb{F}_q)$.

Lemma 3 *Let R be a finite commutative local ring with unity 1 and $|R| = p^k$ and let $\mathbb{F}_q$ be a finite prime field. Then $|Z^*((R \times F_q)_E)| = 2k$ or $2(1 + |Z^*(R_E)|$.*

Proof. Let R be a finite commutative local ring with unity and $|R| = p^k$, $k \geq 1$. We consider the following three cases.

Case 1. $R \cong \mathbb{F}_p$, for some prime p. Then the zero divisor set of $Z^*(\mathbb{F}_p \times \mathbb{F}_q) = \{\{(a, 0)\}, \{(0, x)\}\}$, for every $a \in U(R)$ and $0 \neq x \in \mathbb{F}_q$. Now, to find the equivalence classes of these zero divisors, the set $\{(a, 0)\}$ and $\{(0, x)\}$ respectively correspond to vertices $[(a, 0)]$ and $[(0, x)]$ in $\Gamma_E(R \times \mathbb{F}_q)$, for any $a \in U(R)$ and for any $x \in \mathbb{F}_q$. Therefore, $|Z^*(R \times \mathbb{F}_q)_E| = 2k$, where $k = 1$.

Case 2. $R \cong \mathbb{Z}_p^k$, $(k \geq 2)$. The equivalence class of each element $(a, 0)$, for every $a \in U(R)$ is same, since $[(a, 0)] = \{(0, x)\}$, for all $x \in \mathbb{F}_q$. In this way, we get one vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Also, the equivalence classes of each element $(0, x)$, for every $0 \neq x \in \mathbb{F}_q$ is same, since $[(0, x)] = \{(a, 0)\}$. So, this gives another vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Moreover, for any unit u in R, we get two zero divisor sets of equivalence classes given by

$$Z_1 = \{[(up, 0)], [(up^2, 0)], \dots, [(up^{k-1}, 0)]\}$$
$$Z_2 = \{[(up, 1)], [(up^2, 1)], \dots, [(up^{k-1}, 1)]\}.$$

We note that there is no other possible equivalence class. Claim $[(up^{k-1}, 1)] = [(up^{k-1}, x_i]$, for all $1 \leq i \leq q - 2$. If $[(up^{k-1}, 1)] \neq [(up^{k-1}, x_i]$, there exists some zero divisor in $R \times \mathbb{F}_q$, say $(a_1, 0)$ adjacent to $(up^{k-1}, 1)$ but not adjacent to (up^{k-1}, x_i), which is a contradiction.

The total number of zero divisors is $|Z^*((R \times \mathbb{F})_E)| = 2 + |Z_1| + |Z_2| = 2 + k - 1 + k - 1 = 2k$ or $2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|)$.

Case 3. R is a local ring other than $\mathbb{F}_p$ and $\mathbb{Z}_{p^k}$. So, we consider all local rings R with $|R| = p^k$, especially $k = 2$, 3 or 5 and the rings of order p^2, p^3 or p^4 are mentioned in proof of Proposition 4 and 5. Then the set of zero divisors of equivalence classes include

$$[(a, 0)],\ a \in U(R)$$

$$[(0, x_i)], \text{ for any } i,\ 1 \le i \le q-2$$
$$Z_1 = \{[(a_1, 0)], [(a_2, 0)], \dots, [(a_r, 0)]\}$$
$$Z_2 = \{[(a_1, 1)], [(a_2, 1)], \dots, [(a_r, 1)]\}.$$
where $a_1, a_2, \dots, a_r$ are the non-zero zero divisors of the set $Z(R_E)$.

There is no other possible equivalence class as a zero divisor. Claim $[(a_i, 1)] = [(a_i, x_j)]$, $1 \le i \le r$ and $1 \le j \le q-2$. For if, $[(a_i, 1)] \ne [(a_i, x_j)]$, there exists some zero divisor $(a_k, 0)$ adjacent to one of $[(a_i, 1)]$ or $[(a_i, x_j)]$, but not to the other, which is a contradiction.

Thus, $|Z^*((R \times \mathbb{F}_q)_E)| = 2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|$. □

Example 2 *Consider the ring* $\mathbb{Z}_8 \times \mathbb{Z}_3$*, Here,* $R = \mathbb{Z}_{2^3}$*,* $k = 3$*, and* $U(R) = \{1, 3, 5, 7\}$*. For the zero divisors of equivalence classes, we have*
$[(1, 0)] = \{(0, 1), (0, 2)\}$*,* $[(3, 0)] = \{(0, 1), (0, 2)\}$*,* $[(5, 0)] = \{(0, 1), (0, 2)\}$*,*
$[(7, 0)] = \{(0, 1), (0, 2)\}$*.*
Also, $[(2, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$*,*
$[(4, 0)] = \{(0, 1), (0, 2), (2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$*,*
$[(6, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$*.*
Moreover, $[(0, 1)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$*,*
$[(0, 2)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$*,* $[(2, 1)] = \{(4, 0)\}$*,*
$[(4, 2)] = \{(2, 0), (4, 0), (6, 0)\}$*,* $[(4, 1)] = \{(2, 0), (4, 0), (6, 0)\}$*,* $[(2, 2)] = \{(4, 0)\}$*,*
$[(6, 1)] = \{(4, 0)\}$*,* $[(6, 2)] = \{(4, 0)\}$
Thus, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = \{[(0, 1)], [(1, 0)], [(2, 0)], [(4, 0)], [(2, 1)], [(4, 1)]\}$*.*
Using Lemma 3, we can directly have, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = 2 \times 3 = 6$*.*

Remark 4 *Lemma 3 holds if we replace* $\mathbb{F}_q$ *by any finite field* $\mathbb{F}$*. More generally, let* R *be any finite commutative ring with unity* 1*. We know* $R \cong R_1 \times R_2$*, where each* R_i*,* $1 \le i \le 2$*, is a local ring. If either* R_1 *or* R_2 *is a field, the number of vertices is always given by the formula* $2(1+|Z^*(R_{1E})|$ *or* $2(1+|Z^*(R_{2E})|$*, since the equivalence classes of zero divisors of* $\Gamma_E(R_1 \times R_2)$ *are always of the form* $\{[(0, 1)], [(1, 0)], [(a, 0)], [(0, b)], [(a, 1)], [(1, b)], [(a, b)]$*, where* a *and* b *are the non-zero zero divisors and* $Z^*(R_{1E})$*,* $Z^*(R_{2E})$ *denote the number of zero divisor equivalence classes of* R_1 *and* R_2 *respectively. The result holds trivially if both* R_1 *and* R_2 *are fields.*

Theorem 2 *Let* R *be a finite commutative local ring with unity* 1 *and finite field* $\mathbb{F}_q$*. Then,* $\dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$ *or at most* $4k$ *or* $4t$ *where* $k \ge 2$ *and* t *are integers,* $t = 1 + |Z^*(R_E)|$*.*

Proof. Let R be a finite commutative local ring with unity 1. We consider the following three cases.

Case 1. R is a field. Then, by Case 1 of Lemma 3, $\Gamma_E(R \times \mathbb{F}_q)$ is a path on two vertices. Therefore, by Lemma 2.1, $\dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$.
Case 2. $R \cong \mathbb{Z}_{p^k}$, $k \geq 2$. In this case, we partition the vertices into distance similar classes in $\Gamma_E(R)$ given by

$$\begin{aligned}
V_1 &= \{[(a, 0)]\}, \text{ for any } a \in U(R) \\
V_2 &= \{[(0, x)]\}, \text{ for any } x \in \mathbb{F}_q \\
Z_1 &= \{[(up, 0)]\}, Z_2 = \{[(up^2, 0)]\}, \ldots, Z_{k-1} = \{[(up^{k-1}, 0)]\} \\
W_1 &= \{[(up, 1)]\}, W_2 = \{[(up^2, 1)]\}, \ldots, W_{k-1} = \{[(up^{k-1}, 1)]\}
\end{aligned}$$

Then, $\dim(\Gamma_E(R \times \mathbb{F}_q) \leq |Z^*((R \times \mathbb{F}_q)_E) + m$ where m is the number of distance similar classes that consist of a single vertex. Hence by case 2 of Lemma 3, we have

$$\dim(\Gamma_E(R \times \mathbb{F}_q) \leq 2k + 2(k-1) + 2 = 4k.$$

Case 3. R is a local ring other than $\mathbb{Z}_p^k$ and $\mathbb{F}_p^k (k \geq 1)$. Then, by Case 3 of Lemma 3, $\dim(\Gamma_E(R \times \mathbb{F}_q)) \leq 2(1 + |Z^*(R_E)| + 2|Z^*(R_E)| + 2 = 4(1 + |Z^*(R_E)|) = 4t$ where t is any integer given by $t = 1 + |Z^*(R_E)|$. □

We say that a graph G has a bounded degree if there exists a positive integer M such that the degree of every vertex is at most M. In the next theorems, we obtain an upper bound for the number of zero divisors in a finite commutative ring R with unity 1 with finite metric dimension. The analogous of these results holds in case of $\Gamma_E(R)$.

Proposition 7 *If $\Gamma(R)$ is a zero divisor graph with finite metric dimension k, then $|Z^*(R)| \leq 3^k + k$.*

Proof. Let $\Gamma(R)$ be a zero divisor graph with metric dimension k. We choose two vertices, say w_1 and w_2, from the metric basis W. Since the diameter of $\Gamma(R)$ is at most 3, each coordinate of metric representation is an integer between 0 and 3 and only the vertices of a metric basis have one coordinate 0. The remaining vertices must get a unique code from one of the 3^k possibilities. Therefore, $|Z^*(R)| \leq 3^k + k$. □

Proposition 8 *Let R be a commutative ring and $\Gamma_E(R)$ be a corresponding compressed zero divisor graph with $|Z^*(R)| \geq 2$. Then $\dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d$, where d is the diameter of $\Gamma_E(R)$.*

Proof. By [21, Theorem 5.2], if R is a commutative ring and $\Gamma(R)$ is the corresponding zero divisor graph of R such that $|Z^*(R)| \geq 2$, then $\dim(\Gamma(R)) \leq |Z^*(R)| - d'$ where d' is the diameter of $\Gamma(R)$. Since

$$\dim(\Gamma_E(R)) \leq \dim(\Gamma(R)) \quad \text{and} \quad |Z^*(R_E)| \leq |Z^*(R)|,$$

therefore

$$\dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d,$$

where d is the diameter of $\Gamma_E(R)$. □

Proposition 9 *If $\Gamma(R)$ is a finite graph with metric dimension k, then every vertex of this graph has degree at most $3^k - 1$.*

Proof. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a metric basis of $\Gamma(R)$ with cardinality k. Consider a vertex v with metric representation

$$(d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

If u is adjacent to v, then $r(v|W) \neq r(u|W)$ and $|d(v, w_i) - d(u, w_i)| \leq 1$ for all $w_i \in W$, $1 \leq i \leq k$. If d is distance from v to w_i, then the distance of u from w_i is one of the numbers $\{d, d-1, d+1\}$. Thus, there are three possible numbers for each of the k coordinates of $r(u|W)$, but $d(u, w_i) \neq d(v, w_i)$ for all $1 \leq i \leq k$. This implies that there are at most $3^k - 1$ different possibilities for $r(u|W)$. Since all vertices must have distinct metric coordinates, the degree of v is at most $3^k - 1$. □

A graph G is *realizable* as $\Gamma_E(R)$ if $G \cong \Gamma_E(R)$ for some ring R. There are many results which imply that most graphs are not realizable as $\Gamma_E(R)$, like $\Gamma_E(R)$ is not a cycle graph, nor a complete graph with at least three vertices.

Proposition 10 *The metric dimension of realizable graphs $\Gamma_E(R)$ with 3 vertices is 1.*

Proof. Spiroff et al. proved that the only one realizable graph $\Gamma_E(R)$ with exactly three vertices as a graph of equivalence classes of zero divisors for some ring R is P_3, see Figure 6. Clearly, its metric dimension is 1. □

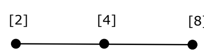

Figure 6: $\mathbb{Z}_{16}$

Proposition 11 *The metric dimension of realizable graphs $\Gamma_E(R)$ with 4 vertices is either 1 or 2.*

Proof. All the realizable graphs $\Gamma_E(R)$ on 4 vertices are shown in Figure 7. It is easy to see their metric dimension is either 1 or 2. $\square$

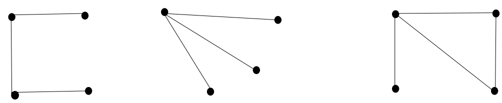

Figure 7: $(\mathbb{Z}_4 \times \mathbb{F}_4)$ $\quad \mathbb{Z}_4[x]/(x^2)$ $\quad \frac{\mathbb{Z}[x,y]}{(x^3,xy)}$

Proposition 12 *The metric dimension of realizable graphs $\Gamma_E(R)$ with 5 vertices is either 2 or 3.*

Proof. The only realizable graphs of equivalence classes of zero divisors of a ring R with 5 vertices are shown in Figure 8. It is easy to see the metric dimension of the first three graphs is 2 and for the star graph is 3 (by Corollary 1). $\square$

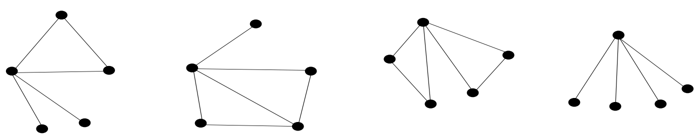

Figure 8: $(\mathbb{Z}_9[x]/(x^2)$, $\mathbb{Z}_{64}$, $Z_3[x,y]/(xy, x^3, y^3, x^2 - y^2)$, $\mathbb{Z}_8[x,y]/(x^2, y^2, 4x, 4y, 2xy)$

4 Relationship between metric dimension, girth and diameter of $\Gamma_E(R)$

In this section, we examine the relationship between girth, diameter and metric dimension of $\Gamma_E(R)$. Since $gr(\Gamma_E(R)) \in \{3, \infty\}$, it is worth to mention that, for a reduced commutative ring R with $1 \neq 0$, $gr(\Gamma_E(R)) = 3$ if and only if $gr(\Gamma(R)) = 3$ and that $gr(\Gamma_E(R)) = \infty$ if and only if $gr(\Gamma(R)) \in \{4, \infty\}$. However, if R is not reduced, then we may have $gr(\Gamma(R)) = 3$ and either $gr(\Gamma_E(R)) = 3$ or ∞. The following result gives the metric dimension of $\Gamma_E(R)$ in terms of the girth of $\Gamma_E(R)$ of a ring R.

Theorem 3 *Let* R *be a finite commutative ring with* $gr(\Gamma_E(R)) = \infty$.

(i) *If* R *is a reduced ring, then* $dim(\Gamma_E(R)) = 1$.

(ii) *If* $R \cong \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3)$ *or* $\mathbb{Z}_4[x]/(2x, x^2-2)$, *then* $dim(\Gamma_E(R)) = |Z^*(R_E)| - 1$.

(iii) *If* $R \cong \mathbb{Z}_4,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2)$, *then* $dim(\Gamma_E(R)) = 0$.

(iv) $\dim(\Gamma_E(R)) = 0$ *or* 1 *if and only if* $gr(\Gamma(R)) \in \{4, \infty\}$.

Proof. If R is a reduced ring and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we know $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Therefore, by Remark 4, R has two equivalence classes of zero divisors $[(0, 1)]$ and $[(1, 0)]$, adjacent to each other. Hence, $dim(\Gamma_E(R)) = 1$. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then R being a Boolean ring, implies $\Gamma(R) \cong \Gamma_E(R)$. Therefore, by Case 1 of Lemma 3, the result follows. In part (ii), these rings are non reduced and $\Gamma_E(R)$ are isomorphic to $K_{1,1}$. Rings listed in part (iii) represents $\Gamma_E(R)$ on a single vertex, part (iv) follows from the above comments. □

We can also prove the Part (i) by using the fact that if R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Thus $\Gamma(R)$ is a complete bipartite and the result follows from Proposition 2. Now, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R) \cong K_{1,1}$, whose metric dimension is 1. Since, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean ring, therefore by Proposition 3, we have $dim(\Gamma_E(R)) = 1$.

If R is a reduced ring with non-trivial zero divisor graph, then $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$ for some integer $k \geq 2$ and for finite fields $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$. If R is not a reduced ring, then either R is local or $R \cong R_1 \times R_2 \times \cdots \times R_t$, for some integer $t \geq 2$ and local rings $R_1, R_2, \ldots, R_t$, where at least one R_i is not a field. Now, we have the following observations for the finite commutative rings whose zero divisor graphs can be seen in [22].

Corollary 6 *If* R *is a finite commutative ring with unity* 1 *and* $gr(\Gamma_E(R)) = \infty$, *then the compressed zero divisor graph of the reduced rings* $R \times \mathbb{F}$ *where* $\mathbb{F}$ *is a finite field, is isomorphic to the compressed zero divisor graph of the following local rings with metric dimension* 1, R *being any local ring.*
$\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x, y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_9[x]/(3x, x^2-6)$, $\mathbb{Z}_9[x]/(3x, x^2-3)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_{27}$.

Proof. The reduced rings $R \times \mathbb{F}$ with $gr(\Gamma_E(R)) = \infty$, all have compressed zero divisor graph isomorphic to $K_{1,1}$, by Case 2 of Lemma 3. Also, the local rings listed above have the same compressed zero divisor graph isomorphic to $K_{1,1}$. □

Proposition 13 *Let* R *be a finite commutative ring with* 1 *and* $gr(\Gamma_E(R)) = \infty$. *The following are the non reduced rings with* $\dim(\Gamma_E(R)) = 1$
$\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_5 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_5 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2,x)^2$.

Proof. If R is not a local ring, we can write $R \cong R_1 \times R_2 \times \cdots \times R_k$, where $k \geq 2$ and each R_i is a local ring. In case of above rings $R \cong R_1 \times R_2$, where either R_1 or R_2 is a field. Therefore, using Remark 4, we have $|\Gamma_E(R)| = 4$ and it is easy to see that $\Gamma_E(R)$ isomorphic to a path on 3 vertices. Thus, $gr(\Gamma_E(R)) = \infty$ and $\dim(\Gamma_E(R)) = 1$. □

If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, then it is easy to see that the three vertices $[(1,0,0)]$, $[(0,1,0)]$ and $[(0,0,1)]$ are adjacent with ends $[(0,1,1)]$, $[(1,0,1)]$, and $[(1,1,0)]$ respectively and thus $|\Gamma_E(R)| = 6$.

Proposition 14 *Let* R *be a reduced commutative ring and* $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$. *Then,* $gr(\Gamma_E(R)) = 3$ *and* $\dim(\Gamma_E(R)) = 2$.

We now proceed to study the relationship between diameter and metric dimension of compressed zero divisor graphs. Since $diam(\Gamma_E(R)) \leq 3$, if $\Gamma_E(R)$ contains a cycle. We have the following results.

Theorem 4 *Let* R *be commutative ring and* $\Gamma_E(R)$ *be its corresponding compressed zero divisor graph.*

(i) $\dim(\Gamma_E(R)) = 0$ *if and only if* $diam(\Gamma_E(R)) = 0$.

(ii) $\dim(\Gamma_E(R)) = 0$ *if and only if* $diam(\Gamma(R)) = 0$ *or* 1, $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) $\dim(\Gamma_E(R)) = diam(\Gamma_E(R)) = 1$ *if* $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, *where* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *are fields.*

(iv) $\dim(\Gamma_E(R)) = 1$ *and* $diam(\Gamma_E(R)) = 3$, *if* R *is non reduced ring isomorphic to the rings given in Proposition 13.*

(v) $\dim(\Gamma_E(R)) = 0$ *if* $Z(R)^2 = 0$ *and* $|Z(R)| \geq 2$.

Proof.

(i) $\dim(\Gamma_E(R)) = 0$ if and only if $\Gamma_E(R)$ is a single vertex graph if and only if $diam(\Gamma_E(R)) = 0$.

(ii) Let $\mathrm{dim}(\Gamma_E(R)) = 0$. Then $\Gamma(R)$ is complete and thus $\mathrm{diam}(\Gamma(R)) = 0$ or 1. Conversely, let $\mathrm{diam}(\Gamma(R)) = 0$ or 1, then $\Gamma(R)$ is complete, thus $\mathrm{dim}(\Gamma_E(R)) = 0$ unless $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Let $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, then by Case 1 of Lemma 3, $|\Gamma_E(R)| = 2$, since the only equivalence classes of zero divisors are $[(0, 1)]$ and $[(1, 0)]$. So, $\Gamma_E(R) \cong K_{1,1}$. Thus, $\mathrm{dim}(\Gamma_E(R)) = \mathrm{diam}(\Gamma_E(R)) = 1$.

(iv) Rings listed in this case correspond to a path of length 3.

(v) Let $|Z(R)| \geq 2$ and $(Z(R))^2 = 0$. Hence $\mathrm{ann}(a) = \mathrm{ann}(b)$, for each $a, b \in Z(R)^*$, which implies that $\mathrm{diam}(\Gamma_E(R)) = 0$. Therefore, $\mathrm{dim}(\Gamma_E(R)) = 0$.

□

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