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Alternative proofs of some formulas for two tridiagonal determinants

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Abstract. In the paper, the authors provide five alternative proofs of two formulas for a tridiagonal determinant, supply a detailed proof of the inverse of the corresponding tridiagonal matrix, and provide a proof for a formula of another tridiagonal determinant. This is a companion of the paper [F. Qi, V. Čerňanová, and Y. S. Semenov, *Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81** (2019), in press.

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1 Introduction

For $c \in \mathbb{C}$ and $k \in \mathbb{N}$, define the $k \times k$ tridiagonal matrix $M_k(c)$ by

$$M_k(c) = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{pmatrix}_{k \times k}$$

and denote the determinant $|M_k(c)|$ of the $k \times k$ tridiagonal matrix $M_k(c)$ by $D_k(c)$. In [7, Remark 4.4], the explicit expression

$$D_k(-6) = \frac{1}{6^k} \sum_{\ell=0}^{k} (-1)^\ell 6^{2\ell} \binom{\ell}{k-\ell}$$

was derived from some results in [7, Theorem 1.2] for the Cauchy products of central Delannoy numbers, where $\binom{p}{q} = 0$ for $q > p \geq 0$. For information on central Delannoy numbers, please refer to the papers [6, 7] and plenty of references cited therein. In [7, Remar 4.4], the authors guessed that the explicit formula

$$D_k(c) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell c^{2\ell-k} \binom{\ell}{k-\ell} = \sum_{m=0}^{k} (-1)^m c^{k-2m} \binom{k-m}{m} \qquad (1)$$

should be valid for all $c \in \mathbb{C}$ and $k \in \mathbb{N}$ and claimed that the equality (1) can be verified by induction on $k \in \mathbb{N}$ straightforwardly.

In the paper [6], the authors discovered a generating function of the sequence $D_k(c)$, provided an analytic proof of the explicit formula (1), established a simple formula for computing the tridiagonal determinant $D_k(c)$, found a determinantal expression for $D_k(c)$, presented the inverse of the symmetric tridiagonal matrix $M_k(c)$, connected $D_k(c)$ with the Chebyshev polynomials [6, 9, 11] and the Fibonacci numbers and polynomials [1, 6, 8], reviewed computation of general diagonal determinants, supplied two new formulas for computing general diagonal determinants, generalized central Delannoy numbers [6, 7], and represented the Cauchy product of the generalized central Delannoy numbers [6] in terms of $D_k(c)$.

In this paper, we pay our attention on the following four conclusions.

Theorem 1 ([6, Theorem 2.2]) *For $k \ge 0$ and $c \in \mathbb{C}$, the formula (1) is valid.*

Theorem 2 ([6, Theorem 3.1]) *For $c \in \mathbb{C}$, $\alpha = \frac{1}{\beta} = \frac{c+\sqrt{c^2-4}}{2}$, and $k \ge 0$, the tridiagonal determinant $D_k(c)$ can be computed by*

$$D_k(c) = \begin{cases} \dfrac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, & c \neq \pm 2; \\ k+1, & c = 2; \\ (-1)^k(k+1), & c = -2. \end{cases} \tag{2}$$

Theorem 3 ([6, Theorem 5.1]) *For $k \in \mathbb{N}$, the inverse of the symmetric tridiagonal matrix $M_k(c)$ can be computed by $M_k^{-1}(c) = \left(R_{ij}\right)_{k\times k}$, where*

$$R_{ij} = \begin{cases} -\dfrac{\left(\lambda^i - \mu^i\right)\left(\lambda^{k-j+1} - \mu^{k-j+1}\right)}{(\lambda - \mu)(\lambda^{k+1} - \mu^{k+1})}, & c \neq \pm 2 \\ (-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2 \\ -\dfrac{i(k-j+1)}{k+1}, & c = -2 \end{cases}$$

for $i < j$, $R_{ij} = R_{ji}$ for $i > j$, and λ and μ are defined by

$$\lambda = \frac{1}{\mu} = \frac{2}{\sqrt{c^2-4} - c} = -\alpha = -\frac{1}{\beta}.$$

Theorem 4 ([6, Section 8]) *For $n \in \mathbb{N}$ and $a, b, c \in \mathbb{C}$, we have*

$$D_n = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n}$$

$$= \begin{cases} \dfrac{\left(a + \sqrt{a^2-4bc}\,\right)^{n+1} - \left(a - \sqrt{a^2-4bc}\,\right)^{n+1}}{2^{n+1}\sqrt{a^2-4bc}}, & a^2 \neq 4bc; \\ (n+1)\left(\dfrac{a}{2}\right)^n, & a^2 = 4bc. \end{cases} \tag{3}$$

In Section 2 of this paper, we will supply two alternative proofs of Theorem 1. In Section 3, we will provide three alternative proofs of Theorem 2. In Section 4, we will present a detailed proof of Theorem 3. In Section 5, we will provide a proof of Theorem 4. In the last section of this paper, we will list several remarks.

2 Two alternative proofs of Theorem 1

Now we are in a position to supply two alternative proofs of Theorem 1.

Proof. [First alternative proof of Theorem 1] Let $D_0(c) = 1$. Theorem 2.1 in [6] states that the sequence $D_k(c)$ for $k \ge 0$ can be generated by

$$F_c(t) = \frac{1}{t^2 - ct + 1} = \sum_{k=0}^{\infty} D_k(c) t^k. \tag{4}$$

By the formula for the sum of a geometric progression, the generating function $F_c(t)$ can be expanded as

$$F_c(t) = \sum_{\ell=0}^{\infty} (-1)^\ell \left(t^2 - ct\right)^\ell = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} t^{\ell+m} \tag{5}$$

for $\left|t^2 - ct\right| < 1$. Hence, it follows for $k \ge 0$ that

$$\begin{aligned} [F_c(t)]^{(k)} &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} \left(t^{\ell+m}\right)^{(k)} \\ &\to \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} \lim_{t\to 0} \left(t^{\ell+m}\right)^{(k)} \\ &= (-1)^k k! \sum_{\ell=0}^{k} (-1)^\ell \binom{\ell}{k-\ell} c^{2\ell-k} \end{aligned}$$

for $\left|t^2 - ct\right| < 1$ and as $t \to 0$. The formula (1) is thus proved. □

Proof. [Second alternative proof of Theorem 1] Taking $k = \ell + m$ in (5) leads to

$$F_c(t) = \sum_{k=0}^{\infty} \left[\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{\ell}{k-\ell} c^{2\ell-k}\right] t^k = \sum_{k=0}^{\infty} D_k(c) t^k$$

for $\left|t^2 - ct\right| < 1$. The formula (1) is proved again. The proof of Theorem 1 is complete. □

3 Three alternative proofs of Theorem 2

We now start out to provide three alternative proofs of Theorem 2.

Proof. [First alternative proof of Theorem 2] It is clear that the generating function $F_c(t)$ in (4) can be rewritten as $F_c(t)=\frac{1}{t-\alpha}\frac{1}{t-\beta}$. By virtue of the Leibniz theorem for the product of two functions, we have

$$[F_c(t)]^{(k)}=\left(\frac{1}{t-\alpha}\frac{1}{t-\beta}\right)^{(k)}=\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{t-\alpha}\right)^{(\ell)}\left(\frac{1}{t-\beta}\right)^{(k-\ell)}$$
$$=\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{\ell}\ell!}{(t-\alpha)^{\ell+1}}\frac{(-1)^{k-\ell}(k-\ell)!}{(t-\beta)^{k-\ell+1}}\to\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{\ell}\ell!}{(-\alpha)^{\ell+1}}\frac{(-1)^{k-\ell}(k-\ell)!}{(-\beta)^{k-\ell+1}}$$
$$=k!\sum_{\ell=0}^{k}\frac{1}{\alpha^{\ell+1}}\frac{1}{\beta^{k-\ell+1}}=\frac{k!}{\beta^k}\sum_{\ell=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\ell}=\frac{k!}{\beta^k}\frac{1-(\beta/\alpha)^{k+1}}{1-\beta/\alpha}=k!\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}$$

as $t\to0$. The formula (2) is thus proved. □

Proof. [Second alternative proof of Theorem 2] The generating function $F_c(t)$ can also be rewritten as

$$F_c(t)=\frac{1}{\alpha-\beta}\left(\frac{1}{t-\alpha}-\frac{1}{t-\beta}\right). \tag{6}$$

Then a straightforward computation reveals

$$[F_c(t)]^{(k)}=\frac{1}{\alpha-\beta}\left[\frac{(-1)^k k!}{(t-\alpha)^{k+1}}-\frac{(-1)^k k!}{(t-\beta)^{k+1}}\right]$$
$$\to -k!\frac{1}{\alpha-\beta}\left(\frac{1}{\alpha^{k+1}}-\frac{1}{\beta^{k+1}}\right)=k!\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}$$

as $t\to0$. The proof of Theorem 2 is complete. □

Proof. [Third alternative proof of Theorem 2] The formula for the sum of a geometric progression yields

$$\frac{1}{t-\alpha}=-\sum_{k=0}^{\infty}\frac{t^k}{\alpha^{k+1}}\quad\text{and}\quad\frac{1}{t-\beta}=-\sum_{k=0}^{\infty}\frac{t^k}{\beta^{k+1}}$$

for $|t|<\min\{|\alpha|,|\beta|\}$. Thus, in view of $\alpha\beta=1$ and (6), we obtain

$$F_c(t)=\frac{1}{\alpha-\beta}\sum_{k=0}^{\infty}\left(\frac{1}{\beta^{k+1}}-\frac{1}{\alpha^{k+1}}\right)t^k=\sum_{k=0}^{\infty}\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}t^k=\sum_{k=0}^{\infty}D_k(c)t^k$$

for $|t|<\min\{|\alpha|,|\beta|\}$. The formula (2) is thus proved. The proof of Theorem 2 is complete. □

4 A detailed proof of Theorem 3

We now present a detailed proof of Theorem 3.

In the paper [2], the inverse of the symmetric tridiagonal matrix $M_k(c)$ was discussed. We denote the inverse matrix of $M_k(c)$ by $M_k^{-1}(c) = (R_{ij})_{k\times k}$. Then, basing on discussions in [2, Eq. (9)], one can see without difficulty that the elements R_{ij} can be represented as

$$R_{ij} = (-1)^{i+j}\frac{D_{i-1}(c)D_{k-j}(c)}{D_k(c)}, \quad 1 \le i < j \le k$$

and $R_{ij} = R_{ji}$ for $1 \le j < i \le k$. Making use of the formula (2) yields

$$\begin{aligned}
R_{ij} &= \begin{cases}
(-1)^{i+j}\dfrac{\dfrac{\alpha^{i-1+1}-\beta^{i-1+1}}{\alpha-\beta}\dfrac{\alpha^{k-j+1}-\beta^{k-j+1}}{\alpha-\beta}}{\dfrac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}}, & c \ne \pm 2\\[2ex]
(-1)^{i+j}\dfrac{(\pm1)^{i-1}(i-1+1)(\pm1)^{k-j}(k-j+1)}{(\pm1)^k(k+1)}, & c = \pm 2
\end{cases}\\
&= \begin{cases}
(-1)^{i+j}\dfrac{\left(\alpha^i-\beta^i\right)\left(\alpha^{k-j+1}-\beta^{k-j+1}\right)}{(\alpha-\beta)(\alpha^{k+1}-\beta^{k+1})}, & c \ne \pm 2\\[2ex]
(-1)^{i+j}(\pm1)^{i-j-1}\dfrac{i(k-j+1)}{k+1}, & c = \pm 2
\end{cases}\\
&= \begin{cases}
-\dfrac{\left[(-\alpha)^i-(-\beta)^i\right]\left[(-\alpha)^{k-j+1}-(-\beta)^{k-j+1}\right]}{[(-\alpha)-(-\beta)][(-\alpha)^{k+1}-(-\beta)^{k+1}]}, & c \ne \pm 2\\[2ex]
(-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2\\[2ex]
-\dfrac{i(k-j+1)}{k+1}, & c = -2
\end{cases}\\
&= \begin{cases}
-\dfrac{\left(\lambda^i-\mu^i\right)\left(\lambda^{k-j+1}-\mu^{k-j+1}\right)}{(\lambda-\mu)(\lambda^{k+1}-\mu^{k+1})}, & c \ne \pm 2\\[2ex]
(-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2\\[2ex]
-\dfrac{i(k-j+1)}{k+1}, & c = -2
\end{cases}
\end{aligned}$$

for $1 \le i < j \le k$. The proof of Theorem 3 is complete.

5 A proof of Theorem 4

The determinant D_n satisfies the recurrence relation $D_n = aD_{n-1} - bcD_{n-2}$. Solving the equation $x^2 - ax + bc = 0$ reaches to two roots $\alpha = \frac{a+\sqrt{a^2-4bc}}{2}$ and $\beta = \frac{a-\sqrt{a^2-4bc}}{2}$. These two roots satisfy $\alpha + \beta = a$ and $\alpha\beta = bc$. Then by the above recurrence relation one can write

$$\begin{aligned} D_n - \alpha D_{n-1} = \beta[D_{n-1} - \alpha D_{n-2}] = \beta^2[D_{n-2} - \alpha D_{n-3}] = \cdots \\ = \beta^{n-2}[D_2 - \alpha D_1] = \beta^{n-2}[(a^2 - bc) - \alpha a] = \beta^n. \end{aligned}$$

Similarly, one can deduce that $D_n - \beta D_{n-1} = \alpha^n$. Accordingly, when $\alpha \neq \beta$, that is, $a^2 \neq 4bc$, one finds $(\alpha - \beta)D_n = \alpha^{n+1} - \beta^{n+1}$, that is,

$$D_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\left(a + \sqrt{a^2 - 4bc}\,\right)^{n+1} - \left(a - \sqrt{a^2 - 4bc}\,\right)^{n+1}}{2^{n+1}\sqrt{a^2 - 4bc}}.$$

When $\alpha = \beta$, that is, $a^2 = 4bc$, we have

$$\begin{aligned} D_n &= \alpha^n + \alpha D_{n-1} = \alpha^n + \alpha(\alpha^{n-1} + \alpha D_{n-2}) = \cdots = (n-1)\alpha^n + \alpha^{n-1}D_1 \\ &= (n-1)\alpha^n + \alpha^{n-1}(2\alpha) = (n+1)\alpha^n = (n+1)\left(\frac{a}{2}\right)^n. \end{aligned}$$

The formula (3) is thus proved. The proof of Theorem 4 is complete.

6 Several remarks

Finally, we list several remarks on tridiagonal determinants.

Remark 1 *The identities*

$$\mathcal{D}_k(c) \triangleq \begin{vmatrix} -c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & -2c & 1 & \cdots & 0 & 0 & 0 \\ 0 & 6 & -3c & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(k-2)c & 1 & 0 \\ 0 & 0 & 0 & \cdots & (k-1)(k-2) & -(k-1)c & 1 \\ 0 & 0 & 0 & \cdots & 0 & k(k-1) & -kc \end{vmatrix}$$

$$= (-1)^k k! \begin{vmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{vmatrix}_{k\times k}$$

$$= \frac{k!}{c^k}\sum_{\ell=0}^{k}(-1)^\ell c^{2\ell}\binom{\ell}{k-\ell} = \begin{cases} k!\dfrac{\lambda^{k+1}-\mu^{k+1}}{\lambda-\mu}, & c \ne \pm 2 \\ (-1)^k (k+1)!, & c = 2 \\ (k+1)!, & c = -2 \end{cases}$$

are neither trivial nor obvious, where $\lambda = \frac{1}{\mu} = \frac{2}{\sqrt{c^2-4}-c} = -\alpha = -\frac{1}{\beta}$. *The determinant* $\mathcal{D}_k(c)$ *satisfies*

$$\mathcal{D}_0(c) = 1, \quad \mathcal{D}_1(c) = -c, \quad \mathcal{D}_2(c) = 2(c^2-1),$$

and

$$\mathcal{D}_k(c) = -kc\mathcal{D}_{k-1}(c) - k(k-1)\mathcal{D}_{k-2}(c), \quad k \ge 2. \tag{7}$$

Then, if letting $\mathcal{F}_c(t) = \sum_{k=0}^{\infty}\mathcal{D}_k(c)t^k$, *we have*

$$\sum_{k=2}^{\infty}\mathcal{D}_k(c)t^k = -ct\sum_{k=2}^{\infty}k\mathcal{D}_{k-1}(c)t^{k-1} - t^2\sum_{k=2}^{\infty}k(k-1)\mathcal{D}_{k-2}(c)t^{k-2},$$

$$\sum_{k=0}^{\infty}\mathcal{D}_k(c)t^k - \mathcal{D}_0(c) - \mathcal{D}_1(c)t = -ct\sum_{k=1}^{\infty}(k+1)\mathcal{D}_k(c)t^k$$
$$- t^2\sum_{k=0}^{\infty}(k+2)(k+1)\mathcal{D}_k(c)t^k,$$

$$\mathcal{F}_c(t) - 1 + ct = -ct\frac{d}{dt}\left[\sum_{k=1}^{\infty}\mathcal{D}_k(c)t^{k+1}\right] - t^2\frac{d^2}{dt^2}\left[\sum_{k=0}^{\infty}\mathcal{D}_k(c)t^{k+2}\right],$$

$$\mathcal{F}_c(t) - 1 + ct = -ct\frac{d}{dt}\left[t\sum_{k=1}^{\infty}\mathcal{D}_k(c)t^k\right] - t^2\frac{d^2}{dt^2}\left[t^2\sum_{k=0}^{\infty}\mathcal{D}_k(c)t^k\right],$$

$$\mathcal{F}_c(t) - 1 + ct = -ct\frac{d}{dt}[t(\mathcal{F}_c(t)-1)] - t^2\frac{d^2}{dt^2}\big[t^2\mathcal{F}_c(t)\big],$$
$$t^4\mathcal{F}_c''(t) + t^2(4t+c)\mathcal{F}_c'(t) + \big(2t^2+ct+1\big)\mathcal{F}_c(t) - 1 = 0.$$

This means that the generating function of the sequence $\mathcal{D}_k(c) = (-1)^k k! D_k(c)$ *is the solution of the second order linear ordinary differential equation*

$$t^4 f''(t) + t^2(4t + c)f'(t) + \big(2t^2 + ct + 1\big)f(t) - 1 = 0$$

with initial values $f(0) = 1$ *and* $f'(0) = -c$*. This differential equation is solvable, but its solution is not elementary.*

Remark 2 *The method used in the proof of* [6, Theorem 3.1] *can not be applied to the sequence* $\mathcal{D}_k(c)$*, since its recurrence relation* (7) *is not a homogeneous linear recurrence relation with constant coefficients.*

Remark 3 *The central Delannoy numbers* $D(k)$ *were generalized in* [10] *as*

$$D_{a,b}(k) = \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t^{k+1}}\,\mathrm{d}\,t, \quad k \ge 0, \quad b > a > 0$$

and, by [7, Lemma 2.4]*, we find that* $D_{a,b}(k)$ *can be generated by*

$$\frac{1}{\sqrt{(x+a)(x+b)}} = \sum_{k=0}^{\infty} D_{a,b}(k)x^k.$$

By virtue of conclusions in [4, Section 2.4] *and* [3, Remark 4.1]*, the generalized central Delannoy numbers* $D_{a,b}(k)$ *for* $k \ge 0$ *can be computed by*

$$D_{a,b}(k) = \frac{1}{a^{k+1}}\, {}_2F_1\left(k+1, \frac{1}{2}; 1; 1 - \frac{b}{a}\right), \quad 2a > b > a > 0, \quad k \ge 0,$$

where ${}_2F_1$ *is the classical hypergeometric function which is a special case of the generalized hypergeometric series*

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n}\frac{z^n}{n!}$$

for complex numbers $a_i \in \mathbb{C}$ *and* $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$*, for positive integers* $p, q \in \mathbb{N}$*, and for*

$$(x)_\ell = \begin{cases} \prod_{k=0}^{\ell-1}(x+k), & \ell \ge 1 \\ 1, & \ell = 0 \end{cases}$$

which is called the rising factorial of $x \in \mathbb{R}$*.*

Remark 4 *This paper and* [6] *are extracted from different parts of the preprint* [5]*.*

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