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Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers

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Abstract. This paper concerns the analysis of a Markovian queueing system with Bernoulli feedback, single vacation, waiting server and impatient customers. We suppose that whenever the system is empty the sever waits for a random amount of time before he leaves for a vacation. Moreover, the customer's impatience timer depends on the states of the server. If the customer's service has not been completed before the impatience timer expires, the customer leaves the system, and via certain mechanism, impatient customer may be retained in the system. We obtain explicit expressions for the steady-state probabilities of the queueing model, using the probability generating function (PGF). Further, we obtain some important performance measures of the system and formulate a cost model. Finally, an extensive numerical study is illustrated.

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1 Introduction

Queueing models with vacations have a great impact in many real life situations, such models occur naturally in different fields such as computer and communication systems, flexible manufacturing systems, telephone services, production line systems, machine operating systems, post offices, etc. Over the past few decades, vacation queueing systems have paid attention of many researchers, excellent surveys on queueing systems with vacations can be found in Doshi [9] and Takagi [16] and in the monographs of Tian [17] and Ke [11]. In recent years, there has been gowning interest in the study of queueing systems with impatient customers (balking and reneging). For related literature, interested readers may refer to Shin and Choo [15], El-Paoumy and Nabwey [10], Kumar et al. [12], Kumar and Sharma [13], Bouchentouf et al. [7], Baek et al. [6], Bouchentouf and Messabihi [8] and references therein.

The studies of queueing models with impatient customers were ranked depending on the causes of the impatience behavior. In queueing literature, models where customers may be impatient because of server vacations have been extensively analyzed. Yue [20] presented the optimal performance analysis of an $M/M/1/N$ queueing system with balking, reneging and server vacation. Altman and Yechiali [2] gave the analysis of some queueing models such as $M/M/1$, $M/G/1$ and $M/M/c$ queues with server vacations and customer impatience, both single and multiple vacation cases were studied. Further, Altman and Yechiali [3] investigated the infinite server queue with vacations and impatient customers. They obtained the probability generating function of the number of customers in the model and derived the performance measures of the system. Queueing systems with vacations and synchronized reneging have been done by Adan et al. [1]. Wu and Ke [19] presented computational algorithm and parameter optimization for a multi-server system with unreliable servers and impatient customers. Later, the model given in Altman and Yechiali [2] were extended by Yue et al. [21] by considering a variant of the multiple vacation policy which includes both single vacation and multiple vacations. In Padmavathy et al. [14], authors studied the steady state behavior of the vacation queues with impatient customers and a waiting server. Further, the transient solution of a $M/M/1$ multiple vacation queueing model with impatient customers has been investigated by Ammar [4]. Then, a study of single server Markovian queueing system with vacations and impatience timers which depend of the state of the server was presented in Yue et al. [22]. Recently, in Ammar [5], author established the transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers.

The main objective of this article is to study an M/M/1 vacation queueing system with Bernoulli feedback, waiting server, reneging, and retention of reneged customers. It is supposed that whenever the busy period ended the server waits a random duration of time before beginning on a vacation. Moreover, we assume that the impatience timers of customers depend on the server's states. We obtain the steady-state solution of the queueing model, using the probability generating function (PGF). Further, we give explicit expressions of useful measures of effectiveness and formulate a cost model. Then, we present a sensitive numerical experiments to illuminate the interests of our theoretical results and to show the impact of the diverse parameters on the behavior of the system. Finally, an appropriate economic analysis is carried out numerically.

The model analyzed in this paper has a number of applications in practice. In most studies cited earlier, authors considered that the server leaves the system once the system is empty, but in many practical life situations the server waits a certain period of time before he leaves the system even if there is no customers, especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc.

Further, our study has another great scope, in most studies mentioned in the above literature, the basis of the research is the supposition that customers may be impatient because of server vacations. However, there are many situations where the customer can become impatient due to the long wait in the queue even if the server is present in the system, another example when the customer may leave the system during busy period is when he cannot see the server state, these situations can be found in telecommunication systems, call centers and production inventory systems.

The rest of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we present the stationary analysis for the queueing model. In Section 4, we obtain different performance measures and formulate a cost model. Section 5 presents numerical results in the form of Tables and Figures. Finally, in Section 6 we conclude the paper.

2 System model

Consider a M/M/1 vacation queueing model with Bernoulli feedback, waiting server, reneging and retention of reneged customers. The model studied in this paper is based on following assumptions:

$*$ Customers arrive into the system according to a Poisson process with arrival rate λ, the service time is assumed to be exponentially distributed with parameter μ. The service discipline is FCFS and there is infinite space for customers to wait.

$*$ When the busy period is finished the server waits a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with parameter η.

$*$ If the server comes back from a vacation to an empty system he waits passively the first arrival, then he begins service. Otherwise, if there are customers waiting in the queue at the end of a vacation, the server starts immediately a busy period. That is single vacation policy. The period of vacation has an exponential distribution with parameter γ.

$*$ Whenever a customer arrives at the system and finds the server on vacation (respectively. busy), he activates an impatience timer T_0 (respectively. T_1), which is exponentially distributed with parameter ξ_0 (respectively. ξ_1). If the customer's service has not been completed before the impatience timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

$*$ Each reneged customer may leave the system without getting service with probability α and may be retained in the system with probability $\alpha' = (1-\alpha)$.

$*$ After completion of each service, the customer can either leave the system definitively with probability β or return to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$.

3 Stationary analysis

In this section, we use the probability generating function (PGF) to obtain the steady-state solution of the queueing system.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 1, & \text{when the server is in a busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the process $\{(L(t); J(t)); t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j, n) : j = 0, 1, \ \ n = 0, 1, ...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\}, \ j = 0, 1, n = 0, 1, ..., \ (j, n) \in \Omega$, denote the system state probabilities.

Then, the steady-state balance equations of our model are given as follows:

$$(\lambda+\gamma)P_{0,0}=\alpha\xi_0 P_{0,1}+\eta P_{1,0}, \tag{1}$$

$$(\lambda+\gamma+n\alpha\xi_0)P_{0,n}=\lambda P_{0,n-1}+(n+1)\alpha\xi_0 P_{0,n+1},\ \ n\geq 1, \tag{2}$$

$$(\lambda+\eta)P_{1,0}=\gamma P_{0,0}+(\beta\mu+\alpha\xi_1)P_{1,1}, \tag{3}$$

$$(\lambda+\beta\mu+n\alpha\xi_1)\,P_{1,n}=\lambda P_{1,n-1}+\gamma P_{0,n}+(\beta\mu+(n+1)\alpha\xi_1)P_{1,n+1},\quad n\geq 1, \tag{4}$$

Theorem 1 *If we have a single server Bernoulli feedback queueing system with single vacation, waiting server, server's states-dependent reneging and retention of reneged customers, then*

1. *The steady-state probability* $P_{0,.}$ *is given by*

$$P_{0,.}=\left(\frac{\gamma\alpha\xi_0+\delta_1 K_0(1)(1-\gamma)}{\gamma K_0(1)}\right)P_{0,0}. \tag{5}$$

2. *The steady-state probability* $P_{1,.}$ *is given by*

$$\begin{aligned}P_{1,.}=e^{\frac{\lambda}{\alpha\xi_1}}\Bigg(&\frac{\gamma}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)-\frac{\gamma}{\alpha\xi_1}K_3(1)\\&+\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\Bigg)P_{0,0},\end{aligned} \tag{6}$$

where

$$\begin{aligned}P_{0,0}=\Bigg\{&\frac{\delta_1\delta_2K_0(1)+\delta_2(\alpha\xi_0-\delta_1K_0(1))}{\gamma\delta_2K_0(1)}+e^{\frac{\lambda}{\alpha\xi_1}}\Bigg[\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)\\&\left(\frac{\gamma}{\lambda+\eta}+\left(\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\right)\right)-\tfrac{\gamma}{\alpha\xi_1}K_3(1)\Bigg]\Bigg\}^{-1},\end{aligned} \tag{7}$$

$$K_0(z)=\int_0^z(1-s)^{\frac{\gamma}{\alpha\xi_0}-1}e^{-\frac{\lambda}{\alpha\xi_0}s}ds,$$

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \quad K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds,$$

and

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)}\right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0}-\frac{\lambda}{\alpha\xi_1}\right)s} ds.$$

Proof. Let

$$G_j(z) = \sum_{n=0}^{\infty} P_{j,n} z^n, \quad j = 0, 1.$$

Then, multiplying Equation (2) by z^n, using Equations (1) and (3) and summing all possible values of n, we get

$$\alpha\xi_0(1-z)G_0'(z) - (\lambda(1-z)+\gamma)G_0(z) = -\{\delta_1 P_{00} + \delta_2 P_{11}\}, \tag{8}$$

with

$$\delta_1 = \left(\frac{\gamma\eta}{\lambda+\eta}\right) \text{ and } \delta_2 = \left(\frac{\eta(\beta\mu+\alpha\xi_1)}{\lambda+\eta}\right),$$

where $G_0'(z) = \frac{d}{dz}G_0(z)$.

In the same manner, from Equations (3) and (4) we obtain

$$\alpha\xi_1 z(1-z)G_1'(z) - (\lambda z - \beta\mu)(1-z)G_1(z) = -\gamma z G_0(z) + (\beta\mu(1-z)+\eta z)P_{1,0}. \tag{9}$$

Next, let $\Gamma = \delta_1 P_{00} + \delta_2 P_{11}$. Then, for $z \neq 1$, Equation (8) can be rewritten as follows

$$G_0'(z) - \left(\frac{\lambda}{\alpha\xi_0} + \frac{\gamma}{\alpha\xi_0(1-z)}\right)G_0(z) = -\frac{\Gamma}{\alpha\xi_0(1-z)}. \tag{10}$$

Multiplying both sides of Equation (10) by $e^{\frac{-\lambda}{\alpha\xi_0}}(1-z)^{\frac{\gamma}{\alpha\xi_0}}$, then integrating from 0 to z, we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{G_0(0) - \frac{\Gamma}{\alpha\xi_0}K_0(z)\right\}, \tag{11}$$

with

$$K_0(z) = \int_0^z (1-s)^{\frac{\gamma}{\alpha\xi_0}-1} e^{-\frac{\lambda}{\alpha\xi_0}s} ds. \tag{12}$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of the denominator of the right hand side of Equation (11), so $z = 1$ must be the root of the numerator of the right hand side of Equation (11).

Thus, we get

$$P_{0,0} = G_0(0) = \frac{\Gamma}{\alpha\xi_0} K_0(1). \tag{13}$$

This implies

$$P_{0,0} = \frac{\delta_2 K_0(1)}{\alpha\xi_0 - \delta_1 K_0(1)} P_{1,1}. \tag{14}$$

Consequently

$$P_{1,1} = \frac{\alpha\xi_0 - \delta_1 K_0(1)}{\delta_2 K_0(1)} P_{0,0}. \tag{15}$$

Next, substituting Equation (13) into (11), we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0} z} (1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{ 1 - \frac{K_0(z)}{K_0(1)} \right\} P_{0,0}. \tag{16}$$

For $z \neq 1$ and $z \neq 0$, Equation (9) can be rewritten as follows

$$\begin{aligned} G_1'(z) &- \left(\frac{\lambda}{\alpha\xi_1} - \frac{\beta\mu}{\alpha\xi_1 z} \right) G_1(z) \\ &= \left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)} \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z). \end{aligned} \tag{17}$$

Then, we multiply both sides of Equation (17) by $e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}}$, we get

$$\begin{aligned} &\frac{d}{dz} \left(e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}} G_1(z) \right) \\ &= \left\{ \left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)} \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z) \right\} e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}}. \end{aligned} \tag{18}$$

Integrating from 0 to z, we have

$$\begin{aligned} G_1(z) = e^{\frac{\lambda}{\alpha\xi_1} z} z^{-\frac{\beta\mu}{\alpha\xi_1}} &\left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} \right. \\ &\left. - \frac{\gamma}{\alpha\xi_1} \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} G_0(s) ds \right\}, \end{aligned} \tag{19}$$

where

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \quad K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds. \tag{20}$$

Using Equation (14) and substituting Equation (16) into (19), we get

$$G_1(z) = e^{\frac{\lambda z}{\alpha\xi_1}} z^{-\frac{\beta\mu}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(z) P_{0,0} \right\}, \tag{21}$$

with

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)} \right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0} - \frac{\lambda}{\alpha\xi_1}\right)s} ds. \tag{22}$$

Next, putting $z = 1$ in Equation (8), we get the probability that the server is on vacation, $\left(P_{0,.} = G_0(1) = \sum_{n=0}^{\infty} P_{0,n} \right)$,

$$P_{0,.} = \left(\frac{\delta_1 P_{0,0} + \delta_2 P_{1,1}}{\gamma} \right). \tag{23}$$

And, putting $z = 1$ in Equation (21), we find the probability that the server is in busy period, $\left(P_{1,.} = G_1(1) = \sum_{n=0}^{\infty} P_{1,n} \right)$,

$$P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(1) P_{0,0} \right\}. \tag{24}$$

From Equation (3), it yields

$$P_{1,0} = \left(\frac{\gamma}{\lambda+\eta} \right) P_{0,0} + \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1}. \tag{25}$$

Substituting Equation (25) into (24), we have

$$\begin{aligned} P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \Bigg\{ & \left(\frac{\gamma}{\lambda+\eta} \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) - \frac{\gamma}{\alpha\xi_1} K_3(1) \right) P_{0,0} \\ & + \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1} \Bigg\}. \end{aligned} \tag{26}$$

Next, substituting Equation (15) into (23), we get (5). Then, substituting Equation (15) into (26), we obtain (6).

Finally, using the normalizing condition

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1,$$

which is equivalent to

$$P_{0,.} + P_{1,.} = 1. \tag{27}$$

And substituting Equations (15), (23) and (26) into (27), we find (7) □

4 Performance measures and cost model

4.1 Performance measures

In this subpart useful performance measures are presented.

∗ The probability that the server is in a busy period (P_B).

$$\mathbb{P}(\text{Busy period}) = P_B = P_{1,.}.$$

∗ The probability that the server is on vacation (P_V).

$$\mathbb{P}(\text{Vacation period}) = P_V = 1 - \mathbb{P}(\text{Busy period}).$$

∗ The probability that the server is idle during busy period (P_I).

$$P_I = P_{1,0}.$$

∗ The average number of customers in the system when the server is taking vacation ($\mathbb{E}(L_0)$).

From Equation (8), using L'Hopital rule, we have

$$\mathbb{E}(L_0) = \lim_{z \to 1} G_0'(z) = \frac{-\lambda P_{0,.} + \gamma \mathbb{E}(L_0)}{-\alpha \xi_0}.$$

This implies

$$\mathbb{E}(L_0) = \left(\frac{\lambda}{\gamma + \alpha \xi_0} \right) P_{0,.}.$$

∗ The average number of customers in the system when the server is in busy period ($\mathbb{E}(L_1)$).

From Equation (9), using L'Hopital rule, we get

$$\mathbb{E}(L_1) = \lim_{z\to 1} G_1'(z) = \left(\frac{\lambda-\beta\mu}{\alpha\xi_1}\right) P_{1,.} + \frac{\gamma}{\alpha\xi_1}\mathbb{E}(L_0) + \frac{\beta\mu}{\alpha\xi_1(\lambda+\eta)}\left(\gamma + \frac{(\beta\mu+\alpha\xi_1)(\alpha\xi_0-\delta_1 K_0(1))}{\delta_2 K_0(1)}\right) P_{0,0}.$$

$*$ The average number of customers in the system ($\mathbb{E}(L)$).

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1).$$

$*$ The average number of customers in the queue ($\mathbb{E}(L_q)$).

$$\mathbb{E}(L_q) = \sum_{n=0}^{+\infty} nP_{0n} + \sum_{n=1}^{+\infty}(n-1)P_{1n}$$

$$= \mathbb{E}(L) - (P_{1,.} - P_{1,0}).$$

$*$ The mean waiting time of a customer in the system (W_s).

$$W_s = \frac{\mathbb{E}(L_0)+\mathbb{E}(L_1)}{\lambda} = \frac{\mathbb{E}(L)}{\lambda}.$$

$*$ The expected number of customers served per unit of time (E_{cs}).

$$E_{cs} = \beta\mu(P_{1,.} - P_{1,0}).$$

$*$ The average rates of reneging and retention of impatient customers during vacation period.

$$R_{ren_0} = \alpha\xi_0\mathbb{E}(L_0), \quad R_{ret_0} = (1-\alpha)\xi_0\mathbb{E}(L_0).$$

$*$ The average rates of reneging and retention of impatient customers during busy period.

$$R_{ren_1} = \alpha\xi_1\mathbb{E}(L_1), \quad R_{ret_1} = (1-\alpha)\xi_1\mathbb{E}(L_1).$$

Thus,

$*$ The average rate of abandonment of a customer due to impatience (R_{ren}).

$$R_{ren} = R_{ren_0} + R_{ren_1}.$$

$*$ The average rate of retention of impatient customers (R_{ret}).

$$R_{ret} = R_{ret_0} + R_{ret_1}.$$

4.2 Cost model

This subpart is devoted to develop a model for the costs incurred in the queueing system using the following symbols:

- C_1 : Cost per unit time when the server is working during busy period.
- C_2 : Cost per unit time when the server is idle during busy period.
- C_3 : Cost per unit time when the server is on vacation.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per service per unit time.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Cost per unit time when a customer is retained.
- C_8 : Cost per unit time when a customer returns to the system as a feedback customer.

Let
* R be the revenue earned by providing service to a customer.
* Γ be the total expected cost per unit time of the system.

$$\Gamma = C_1 P_B + C_2 P_I + C_3 P_V + C_4 \mathbb{E}(L_q) + C_6 R_{ren} + C_7 R_{ret} + \mu(C_5 + \beta' C_8).$$

* Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu(1 - P_V - P_{1,0}).$$

* Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

5 Numerical analysis

5.1 Impact of system parameters on performance measures

Different performance measures of interest computed under different scenarios are given. These measures are obtained by using a MATLAB program coded by the authors. To illustrate the system numerically, the values for default parameters are considered using the following cases

- Table 1: $\lambda = 1.00 : 0.05 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 2: $\lambda = 1.50$, $\mu = 2.00 : 0.40 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 3: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50 : 0.05 : 0.95$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85 : 0.05 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 5: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10 : 0.05 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 6: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10 : 0.05 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 7: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$.
- Table 8: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

5.2 General comments

$*$ From Table 1 it is clearly seen that with the increases of the arrival rate λ, $P_{0,0}$ and P_V decrease, while P_B increases. Thus, the mean number of customers in the system during the busy period $\mathbb{E}(L_1)$ increases significatively, which leads to an increase in the number of customers served E_{cs}. Moreover, $\mathbb{E}(L_0)$ is not monotone with λ, while W_s increases as the arrival rate increases, this implies an increases in the average reneging and retention rates R_{ren} and R_{ret}.

$*$ According to Table 2 we see that along the increases of the service rate μ, $P_{0,0}$, P_V, $\mathbb{E}(L_0)$ and E_{cs} increase, whereas P_B and $\mathbb{E}(L_1)$ both decrease, as it should be expected. Moreover, with the increase in μ, the mean waiting time of a customer in the system W_s deceases, this leads to a decrease in R_{ren} and R_{ret}. Obviously, the higher the service rate, the smaller the average rate of abandonment and the larger the number of customers served.

$*$ From Table 3 we remark that when the reneging rate during vacation period ξ_0 increases, P_B, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease, while $P_{0,0}$, P_V, R_{ren} and R_{ret} increase. Consequently, E_{cs} decreases. As intuitively expected, the bigger the rate of reneging, the smaller the number of customers served.

Table 1: Performance measures vs. λ

λ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
1.00	0.0272	0.7720	0.2280	0.6840	0.7883	1.4022	0.5060	0.5060	0.5440
1.05	0.0248	0.7795	0.2205	0.6931	0.8654	1.4169	0.5411	0.5411	0.5589
1.10	0.0227	0.7869	0.2131	0.7002	0.9439	1.4296	0.5762	0.5762	0.5738
1.15	0.0208	0.7943	0.2057	0.7052	1.0237	1.4407	0.6114	0.6114	0.5886
1.20	0.0191	0.8017	0.1983	0.7083	1.1049	1.4505	0.6466	0.6466	0.6034
1.25	0.0176	0.8090	0.1910	0.7094	1.1874	1.4591	0.6820	0.6820	0.6180
1.30	0.0161	0.8163	0.1837	0.7087	1.2713	1.4667	0.7175	0.7175	0.6325
1.35	0.0148	0.8234	0.1766	0.7063	1.3566	1.4735	0.7531	0.7531	0.6469
1.40	0.0137	0.8305	0.1695	0.7021	1.4434	1.4797	0.7890	0.7890	0.6610
1.45	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750

Table 2: Performance measures vs. μ

μ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
2.00	0.0144	0.8143	0.1857	0.7959	1.2864	1.3882	0.7457	0.7457	0.7543
2.40	0.0160	0.7938	0.2062	0.8839	1.0741	1.3053	0.6775	0.6775	0.8225
2.80	0.0174	0.7757	0.2243	0.9614	0.8883	1.2331	0.6179	0.6179	0.8821
3.20	0.0186	0.7597	0.2403	1.0300	0.7240	1.1694	0.5652	0.5652	0.9348
3.60	0.0197	0.7455	0.2545	1.0909	0.5775	1.1123	0.5182	0.5182	0.9818
4.00	0.0207	0.7328	0.2672	1.1453	0.4459	1.0607	0.4758	0.4758	1.0242
4.40	0.0216	0.7214	0.2786	1.1941	0.3268	1.0140	0.4374	0.4374	1.0626
4.80	0.0224	0.7111	0.2889	1.2383	0.2187	0.9713	0.4025	0.4025	1.0975
5.20	0.0231	0.7017	0.2983	1.2786	0.1201	0.9325	0.3707	0.3707	1.1293
5.60	0.0238	0.6931	0.3069	1.3154	0.0300	0.8970	0.3416	0.3416	1.1584

$*$ According to Table 4, we observe that along the increases of the reneging rate during busy period ξ_1, P_B, $\mathbb{E}(L_1)$ and W_s decrease, this leads to a decrease in E_{cs}. Further, as expected, the increasing of ξ_1 implies an increase in $P_{0,0}$, P_V, $\mathbb{E}(L_0)$, R_{ren} and R_{ret}.

$*$ Table 5 illustrates that P_B increases with increasing values of the vacation rate γ, while $P_{0,0}$ is not monotonic with γ. Further, P_V, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease with the increase of γ, this implies an increase in E_{cs}. On the other hand, R_{ren} and R_{ret} decrease significantly as the vacation rate increases, which agrees with the intuitive expectation; the higher the rate of vacation, the bigger the probability of busy period and the greater the number of customers served.

Table 3: Performance measures vs. ξ_0

ξ_0	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.50	0.0130	0.8374	0.1626	0.6506	1.5209	1.4477	0.8253	0.8253	0.6747
0.55	0.0134	0.8372	0.1628	0.6106	1.5117	1.4148	0.8256	0.8256	0.6744
0.60	0.0139	0.8370	0.1630	0.5752	1.5036	1.3859	0.8260	0.8260	0.6740
0.65	0.0143	0.8369	0.1631	0.5438	1.4964	1.3601	0.8263	0.8263	0.6737
0.70	0.0148	0.8367	0.1633	0.5157	1.4899	1.3371	0.8266	0.8266	0.6734
0.75	0.0153	0.8365	0.1635	0.4904	1.4842	1.3164	0.8269	0.8269	0.6731
0.80	0.0158	0.8364	0.1636	0.4675	1.4790	1.2976	0.8272	0.8272	0.6728
0.85	0.0163	0.8362	0.1638	0.4466	1.4742	1.2806	0.8275	0.8275	0.6725
0.90	0.0167	0.8361	0.1639	0.4276	1.4699	1.2650	0.8278	0.8278	0.6722
0.95	0.0172	0.8360	0.1640	0.4101	1.4659	1.2507	0.8281	0.8281	0.6719

Table 4: Performance measures vs. ξ_1

ξ_1	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.85	0.0131	0.8310	0.1690	0.7242	1.4598	1.4560	0.8380	0.8380	0.6620
0.90	0.0136	0.8248	0.1752	0.7508	1.3951	1.4306	0.8504	0.8504	0.6496
0.95	0.0140	0.8189	0.1811	0.7763	1.3364	1.4084	0.8623	0.8623	0.6377
1.00	0.0145	0.8132	0.1868	0.8007	1.2828	1.3890	0.8737	0.8737	0.6263
1.05	0.0149	0.8077	0.1923	0.8241	1.2338	1.3719	0.8846	0.8846	0.6154
1.10	0.0153	0.8024	0.1976	0.8467	1.1886	1.3568	0.8951	0.8951	0.6049
1.15	0.0157	0.7974	0.2026	0.8683	1.1469	1.3435	0.9052	0.9052	0.5948
1.20	0.0161	0.7925	0.2075	0.8892	1.1083	1.3317	0.9150	0.9150	0.5850
1.25	0.0164	0.7878	0.2122	0.9093	1.0723	1.3211	0.9244	0.9244	0.5756
1.30	0.0168	0.7833	0.2167	0.9288	1.0389	1.3117	0.9334	0.9334	0.5666

$*$ According to Table 6, it is clearly observed that with the increase in the waiting server rate η, the probability of busy period P_B decreases which leads to a decrease in the mean number of customers served E_{cs}; this is because W_s, P_V and $\mathbb{E}(L_0)$ increase with η, which implies an increase in R_{ren}, R_{ret} and $P_{0,0}$. On the other hand the number of customers in the system during busy period $\mathbb{E}(L_1)$ increases; the reason is that the size of the system during vacation period becomes large with η.

Table 5: Performance measures vs. γ

γ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0284	0.7420	0.2580	0.9674	2.1932	2.1071	1.1740	1.1740	0.6646
0.15	0.0290	0.7933	0.2067	0.6890	1.9385	1.7517	0.9961	0.9961	0.7106
0.20	0.0290	0.8276	0.1724	0.5172	1.7849	1.5348	0.8879	0.8879	0.7414
0.25	0.0286	0.8522	0.1478	0.4032	1.6856	1.3925	0.8172	0.8172	0.7635
0.30	0.0281	0.8706	0.1294	0.3234	1.6179	1.2942	0.7685	0.7685	0.7801
0.35	0.0276	0.8850	0.1150	0.2654	1.5699	1.2235	0.7336	0.7336	0.7930
0.40	0.0270	0.8965	0.1035	0.2218	1.5348	1.1711	0.7077	0.7077	0.8033
0.45	0.0264	0.9059	0.0941	0.1882	1.5085	1.1311	0.6882	0.6882	0.8118
0.50	0.0258	0.9138	0.0862	0.1617	1.4884	1.1001	0.6730	0.6730	0.8189
0.55	0.0252	0.9204	0.0796	0.1404	1.4728	1.0755	0.6610	0.6610	0.8249

Table 6: Performance measures vs. η

η	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0161	0.7919	0.2081	0.8919	1.5729	1.6432	0.8914	0.8914	0.6532
0.15	0.0187	0.7579	0.2421	1.0375	1.6647	1.8015	0.9669	0.9669	0.6369
0.20	0.0208	0.7316	0.2684	1.1502	1.7899	1.9601	1.0483	1.0483	0.6243
0.25	0.0224	0.7107	0.2893	1.2400	1.9383	2.1189	1.1338	1.1338	0.6142
0.30	0.0237	0.6936	0.3064	1.3132	2.1034	2.2778	1.2223	1.2223	0.6060
0.35	0.0248	0.6794	0.3206	1.3741	2.2811	2.4368	1.3130	1.3130	0.5992
0.40	0.0258	0.6674	0.3326	1.4254	2.4684	2.5959	1.4054	1.4054	0.5935
0.45	0.0266	0.6571	0.3429	1.4694	2.6632	2.7550	1.4992	1.4992	0.5886
0.50	0.0272	0.6483	0.3517	1.5074	2.8639	2.9142	1.5940	1.5940	0.5843
0.55	0.0278	0.6405	0.3595	1.5407	3.0696	3.0735	1.6897	1.6897	0.5806

$*$ The effect of non-feedback probability β is presented in Table 7, we see that P_B and W_s both decrease with increasing values of β. Further, as expected, $P_{0,0}$, P_V and $\mathbb{E}(L_0)$ increase as β increases, whereas $\mathbb{E}(L_1)$ decreases with increasing values of β; this is because the mean system size during vacation period increases with β. Further, it is well shown that R_{ren} and R_{ret} both decrease along the increasing of non-feedback probability β, which results in the increase of E_{cs}.

Table 7: Performance measures vs. β

β	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0020	0.9741	0.0259	0.1109	4.3719	2.9885	1.1207	1.1207	0.3793
0.20	0.0038	0.9503	0.0497	0.2128	3.6255	2.5589	0.9596	0.9596	0.5404
0.30	0.0060	0.9221	0.0779	0.3336	2.9646	2.1988	0.8246	0.8246	0.6754
0.40	0.0083	0.8932	0.1068	0.4578	2.3968	1.9031	0.7137	0.7137	0.7863
0.50	0.0104	0.8658	0.1342	0.5752	1.9133	1.6590	0.6221	0.6221	0.8779
0.60	0.0123	0.8410	0.1590	0.6815	1.4995	1.4541	0.5453	0.5453	0.9547
0.70	0.0140	0.8189	0.1811	0.7760	1.1416	1.2784	0.4794	0.4794	1.0206
0.80	0.0155	0.7995	0.2005	0.8594	0.8282	1.1251	0.4219	0.4219	1.0781
0.90	0.0169	0.7822	0.2178	0.9333	0.5510	0.9895	0.3711	0.3711	1.1289
1.00	0.0181	0.7669	0.2331	0.9991	0.3038	0.8686	0.3257	0.3257	1.1743

Table 8: Performance measures vs. α

α	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0019	0.9710	0.0290	0.2900	6.3941	4.4560	0.5580	5.0220	0.9420
0.20	0.0049	0.9273	0.0727	0.5454	3.4759	2.6808	0.6454	2.5817	0.8546
0.30	0.0076	0.8916	0.1084	0.6502	2.4282	2.0523	0.7167	1.6724	0.7833
0.40	0.0101	0.8623	0.1377	0.6886	1.8756	1.7095	0.7754	1.1631	0.7246
0.50	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750
0.60	0.0151	0.8162	0.1838	0.6892	1.2957	1.3233	0.8676	0.5784	0.6324
0.70	0.0178	0.7976	0.2024	0.6746	1.1238	1.1989	0.9048	0.3878	0.5952
0.80	0.0205	0.7813	0.2187	0.6562	0.9926	1.0992	0.9375	0.2344	0.5625
0.90	0.0231	0.7667	0.2333	0.6362	0.8892	1.0170	0.9666	0.1074	0.5334
1.00	0.0258	0.7537	0.2463	0.6157	0.8056	0.9475	0.9926	0.0000	0.5074

$*$ The impact of non-retention probability α is shown in Table 8. As intuitively expected, along the increase of α, P_B and $\mathbb{E}(L_1)$ decrease, while P_V increases as α increases. Further, $\mathbb{E}(L_0)$ is not monotonic with the probability of non-retention. Moreover, W_s and R_{ret} both decrease with increasing of α whereas R_{ren} increases with the probability α, this leads to a decrease of E_{cs}. This is quite reasonable; the smaller the probability of retaining impatient customers, the larger the average rate of reneged customers and the smaller the number of customers served.

5.3 Economic analysis

In this subpart, a sensitive economic analysis of the model is performed numerically and the results are discussed appropriately.

We present the variation in total expected cost, total expected revenue and total expected profit with the change in different parameters of the system. For the whole numerical study we fix the costs at $C_1 = 5$, $C_2 = 3$, $C_3 = 5$, $C_4 = 3$, $C_5 = 4$, $C_6 = 3$, $C_7 = 2$, $C_8 = 2$, and $R = 50$.

Impact of arrival rate λ

We examine the impact of λ by keeping all other variables fixed, to this end we take $\lambda = 1.00 : 0.05 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. Results of the analysis are summarized in Table 9 and Figure 1.

Table 9: Γ, Δ and Θ for different values of λ

λ	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45
Γ	22.63	23.04	23.45	23.86	24.26	24.67	25.07	25.48	25.88	26.29
Δ	54.39	55.89	57.38	58.86	60.33	61.80	63.25	64.68	66.10	67.50
Θ	31.76	32.84	33.92	35.00	36.06	37.12	38.17	39.20	40.21	41.20

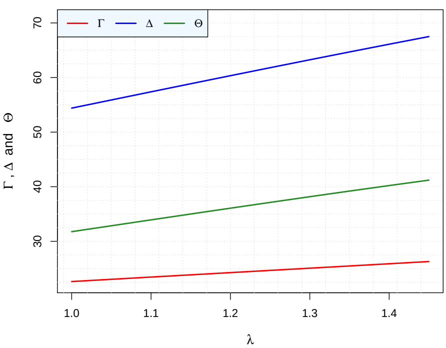

Figure 1: Γ, Δ and Θ vs. λ

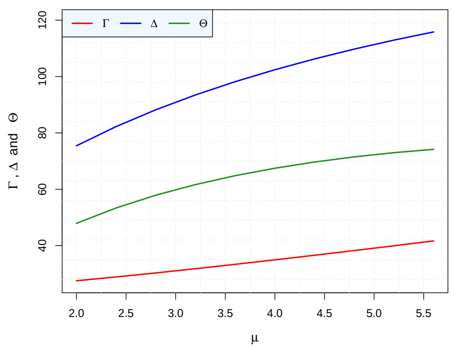

Figure 2: Γ, Δ and Θ vs. μ

Following the obtained results we observe that Γ, Δ, and Θ all increase with the increasing of the arrival rate λ. This result agrees with our intuition; the number of the customers in the system increases with the increasing of

λ, therefore a large number of customers is served. Consequently, the total expected profit increases.

Impact of service rate μ

To check the impact of service rate μ, the values of the parameters are chosen as follows: $\lambda = 1.50$, $\mu = 2.00 : 0.40 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 10: Γ, Δ and Θ for different values of μ

μ	2.00	2.40	2.80	3.20	3.60	4.00	4.40	4.80	5.20	5.60
Γ	27.53	28.88	30.31	31.80	33.35	34.95	36.58	38.25	39.94	41.66
Δ	75.43	82.25	88.21	93.48	98.18	102.4	106.2	109.7	112.9	115.8
Θ	47.89	53.37	57.90	61.67	64.82	67.46	69.67	71.49	72.98	74.17

According to Table 10 and Figure 2 we see that Γ and Δ increase with increasing values of μ, this generates an increase in Θ. This result makes perfect sense, the higher the service rate, the greater the total expected profit of the system.

Impact of reneging rates ξ_0 and ξ_1

Let's study the effect of reneging rates in vacation and busy periods ξ_0 and ξ_1, to this end we consider the following cases

- Table 11: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 1.00$, $\xi_0 = 2.00 : 0.50 : 6.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 12: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85 : 0.05 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 11: Γ, Δ and Θ for different values of ξ_0

ξ_0	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50
Γ	23.97	23.99	24.01	24.02	24.03	24.04	24.05	24.05	24.05	24.06
Δ	78.59	78.47	78.37	78.28	78.21	78.15	78.10	78.06	78.03	78.01
Θ	54.62	54.48	54.36	54.26	54.18	54.11	54.06	54.01	53.97	53.95

From Tables 11 and 12 and Figures 3 and 4 we observe that

Table 12: Γ, Δ and Θ for different values of ξ_1

ξ_1	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30
Γ	26.24	26.21	26.19	26.17	26.17	26.17	26.18	26.19	26.20	26.22
Δ	66.20	64.96	63.77	62.63	61.54	60.48	59.47	58.50	57.56	56.65
Θ	39.95	38.74	37.58	36.45	35.36	34.31	33.29	32.31	31.36	30.43

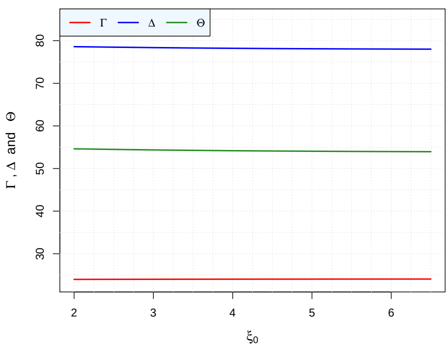

Figure 3: Γ, Δ and Θ vs. ξ_0

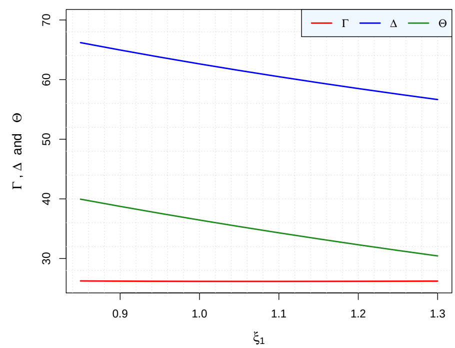

Figure 4: Γ, Δ and Θ vs. ξ_1

$*$ As expected, along the increasing of ξ_0, Γ increases while Θ and Δ decrease with ξ_0, this is because the average rate of reneged customers increases with ξ_0. Therefore the number of customers served decreases, which results in the decrease of the total expected profit.

$*$ With the increase of ξ_1, Δ decreases, while Γ is not monotonic with the parameter ξ_1. Further, Θ decreases with the increasing values of the impatience rate, this is because the number of customers in the system decreases with ξ_1, this implies a decrease in P_B which results in the decrease of E_{cs}.

Impact of vacation rate γ

To examine the impact of the vacation rate γ on the total expected profit, we take $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10 : 0.05 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

From Table 13 and Figure 5 it is easily seen that the increases of the vacation rate γ implies a decrease in Γ and a considerable increase in Δ and Θ. This is quite explicable; as γ increases the vacation duration decreases and the server switches to busy period during which customers are served. This leads to a significant increase in the total expected profit.

Table 13: Γ, Δ and Θ for different values of γ

γ	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	30.58	28.11	26.61	25.61	24.93	24.45	24.09	23.81	23.60	23.43
Δ	66.46	71.06	74.13	76.34	78.00	79.29	80.33	81.18	81.89	82.49
Θ	35.87	42.94	47.53	50.72	53.06	54.85	56.24	57.37	58.29	59.06

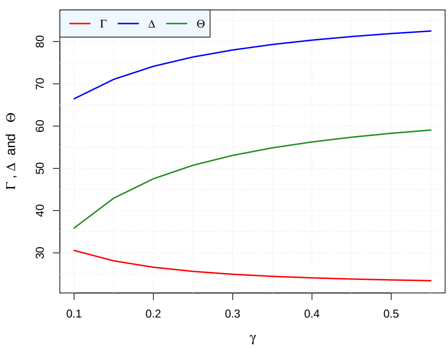

Figure 5: Γ, Δ and Θ vs. γ

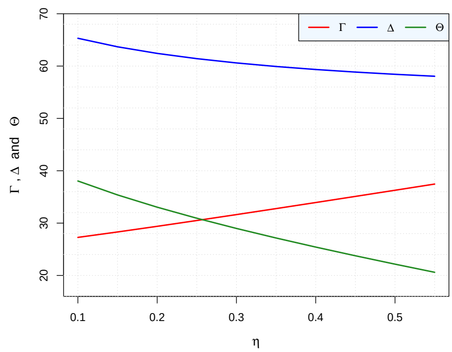

Figure 6: Γ, Δ and Θ vs. η

Impact of waiting rate of a server η

Here, we examine the sensitivity of the total expected profit versus the waiting server rate η. For this case, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10 : 0.05 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. The numerical results are presented in Table 14 and Figure 6.

From the obtained results we remark that with the increase in η, total expected cost Γ increases, while Δ and Θ monotonically decease with the parameter η. This is due to the fact that the probability of busy period during which service is provided decreases with the parameter η. Therefore, the total expected profit decreases considerably.

Table 14: Γ, Δ and Θ for different values of η

η	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	27.26	28.30	29.38	30.49	31.62	32.77	33.93	35.09	36.27	37.45
Δ	65.31	63.68	62.42	61.42	60.60	59.92	59.34	58.85	58.43	58.06
Θ	38.04	35.38	33.04	30.92	28.98	27.15	25.42	23.75	22.15	20.60

Impact of non-retention probability α

To study the impact of α on the total expect profit, we choose the parameters values as follows: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

Table 15: Γ, Δ and Θ for different values of α

α	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	46.85	34.38	30.05	27.75	26.29	25.26	24.49	23.88	23.39	22.98
Δ	94.20	85.45	78.32	72.45	67.50	63.24	59.52	56.25	53.34	50.74
Θ	47.34	51.07	48.27	44.69	41.20	37.97	35.03	32.36	29.95	27.76

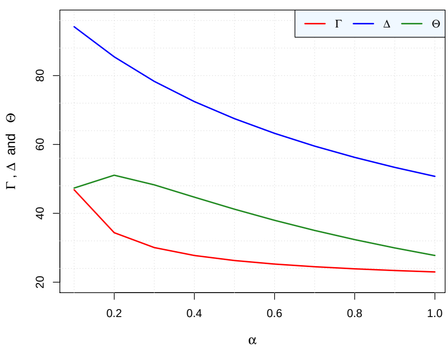

Figure 7: Γ, Δ and Θ vs. α

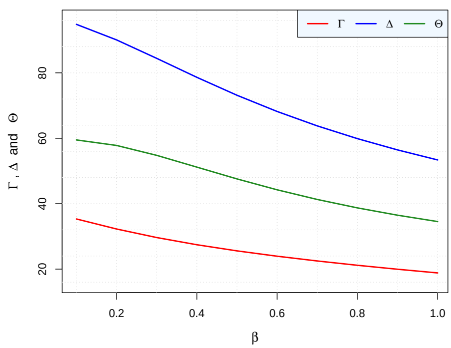

Figure 8: Γ, Δ and Θ vs. β

According to Table 15 and Figure 7 we observe that the increases of non-retention probability α implies a decrease in Γ, Δ and Θ. A slight increase is observed in Θ when the parameter α is below a certain value, ($\alpha = 0.2$). Therefore, we can see that the probability of retaining reneged customers α' has a noticeable effect on the total expected profit of the system. This is because the number of customers served increases with the parameter α'. Thus, it is quite clear that the probability of retention has a positive impact in the economy.

Impact of non-feedback probability β

Here, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$. The numerical results obtained for this situation are given in Table 16 and Figure 8.

Table 16: Γ, Δ and Θ for different values of β

β	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	35.32	32.26	29.65	27.45	25.57	23.94	22.49	21.17	19.96	18.83
Δ	94.81	90.07	84.42	78.63	73.15	68.19	63.78	59.89	56.44	53.37
Θ	59.49	57.80	54.77	51.18	47.57	44.24	41.29	38.72	36.48	34.54

From the obtained results, it is clearly shown that Γ, Δ and Θ monotonically decrease as non-feedback probability β increases. The reason is that the number of the customers in the system decreases with the increasing of β, which leads to a decrease in the total expected profit.

6 Conclusion

In this paper we studied an $M/M/1$ Bernoulli feedback queueing system with single exponential vacation, waiting server, reneging and retention of reneged customers, wherein the impatience timers of customers depend on the states of the server. The explicit expressions of the steady-state probabilities are obtained, using probability generating functions (PGFs).

Useful measures of effectiveness of the queueing system are presented and a cost model is developed. Finally, an extensive numerical study is presented. Our system can be considered as a generalized version of the existing queueing models given by Yue et al.[22] and Ammar [5] associated with several practical situations.

The model considered in this paper can be extended to multiserver queueing system with delayed state-dependent service times, breakdowns and repairs.

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