Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) 5–17

DOI: 10.2478/ausm-2018-0001

Fredholm type integral equation with special functions

Frédéric Ayant
Teacher in High School, France
email: fredericayant@gmail.com

Dinesh Kumar*
College of Agriculture, Sumerpur-Pali,
Agriculture University of Jodhpur, India
Department of Mathematics & Statistics,
Jai Narain Vyas University, India
email: dinesh_dino03@yahoo.com

Abstract. Recently Chaurasia and Gill [7], Chaurasia and Kumar [8] have solved the one-dimensional integral equation of Fredholm type involving the product of special functions. We solve an integral equation involving the product of a class of multivariable polynomials, the multivariable H-function defined by Srivastava and Panda [29, 30] and the multivariable I-function defined by Prasad [21] by the application of fractional calculus theory. The results obtained here are general in nature and capable of yielding a large number of results (known and new) scattered in the literature.

1 Introduction and preliminaries

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of any arbitrary real or complex order. The widely investigated subject of fractional calculus has gained importance and popularity during

2010 Mathematics Subject Classification: 33C60, 26A33
Key words and phrases: multivariable H-function, multivariable I-function, class of multivariable polynomials, Fredholm type integral equation, Riemann-Liouville fractional integral, Weyl fractional integral

*Correspondig author

the past four decades or so, chiefly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics (see, for details, [6, 5, 15, 26, 31] and for recent works, see also [18, 19]). Under various fractional calculus operators, the computations of image formulas for special functions of one or more variables are important from the point of view the solution of differential and integral equations (see, [1, 2, 3, 13, 12, 15, 22, 23, 24, 25]). In this paper, we use the fractional calculus, more precisely the Weyl fractional operator to resolve the one-dimensional integral equation of Fredholm type involving the product of special functions. The integral equations occur in many fields of physics, mechanics and applied mathematics. In the last several years a large number of Fredholm type integral equations involving various polynomials or special functions as kernels have been studied by many authors notably Chaurasia et al. [7, 8], Buchman [4], Higgins [10], Love [16, 17], Prabhakar and Kashyap [20] and others. In the present paper, we obtain the solutions of following Fredholm integral equation.

$$\int_0^\infty y^{-\alpha}\, S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right]\times H^{0,\lambda':(\alpha',\beta');\cdots;(\alpha^{(r)},\beta^{(r)})}_{A',C':(M',N');\cdots;(M^{(r)},N^{(r)})}$$
$$\left[\begin{matrix} u_1(y)^p \\ \cdot \\ \cdot \\ u_r(y)^p \end{matrix}\;\middle|\; \begin{matrix}[(g_j);\gamma',\cdots,\gamma^{(r)}]_{1,A'}:(q',\eta')_{1,M'};\cdots;(q^{(r)},\eta^{(r)})_{1,M^{(r)}} \\ [(f_j);\xi',\cdots,\xi^{(r)}]_{1,C'}:(p',\epsilon')_{1,N'};\cdots;(p^{(r)},\epsilon^{(r)})_{1,N^{(r)}}\end{matrix}\right]$$
$$\times\; I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \tag{1}$$
$$\left[\begin{matrix} z_1\left(\frac{x}{y}\right)^q \\ \cdot \\ \cdot \\ z_r\left(\frac{x}{y}\right)^q \end{matrix}\;\middle|\; \begin{matrix}(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots;(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}: \\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots;(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:\end{matrix}\right.$$
$$\left.\begin{matrix}(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ (b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right] f(y)\,dy = g(x)\quad (0<x<\infty)$$

The multivariable I-function defined by Prasad [21] is an extension of the multivariable H-function defined by Srivastava and Panda [29, 30]. It is defined in term of multiple Mellin-Barnes type integral:

$$
\begin{aligned}
&I(z_1,\cdots,z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \\
&\left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{l} \left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\cdots; \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\cdots; \end{array}\right. \\
&\left.\begin{array}{l} \left(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj}\right)_{1,p_r}:\left(a^{(1)}_j,\alpha^{(1)}_j\right)_{1,p^{(1)}};\cdots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \left(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj}\right)_{1,q_r}:\left(b^{(1)}_j,\beta^{(1)}_j\right)_{1,q^{(1)}};\cdots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array}\right] \\
&= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(t_i)\, z_i^{t_i}\, dt_1\cdots dt_r,
\end{aligned} \tag{2}
$$

the existence and convergence conditions for the defined integral (2), see Prasad [21].

The condition for absolute convergence of multiple Mellin-Barnes type contour (1) can be obtained by extension of the corresponding conditions for multivariable H-function:

$$|\arg z_i| < \frac{1}{2}\Omega_i\,\pi$$

where

$$
\begin{aligned}
\Omega_i = &\sum_{k=1}^{n^{(i)}}\alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}}\beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta^{(i)}_k \\
&+\left(\sum_{k=1}^{n_2}\alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2}\alpha^{(i)}_{2k}\right)+\cdots+\left(\sum_{k=1}^{n_s}\alpha^{(i)}_{sk} - \sum_{k=n_s+1}^{p_s}\alpha^{(i)}_{sk}\right) \\
&-\left(\sum_{k=1}^{q_2}\beta^{(i)}_{2k}+\sum_{k=1}^{q_3}\beta^{(i)}_{3k}+\cdots+\sum_{k=1}^{q_s}\beta^{(i)}_{sk}\right),
\end{aligned} \tag{3}
$$

where $i = 1,\cdots,r$. The complex numbers z_i are not zero. We establish the asymptotic expansion in the following convenient form:

$$
\begin{aligned}
I(z_1,\cdots,z_r) &= O\quad \left(|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\right),\ \max\left(|z_1|,\cdots,|z_r|\right)\to 0 \\
I(z_1,\cdots,z_r) &= O\quad \left(|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\right),\ \min\left(|z_1|,\cdots,|z_r|\right)\to \infty
\end{aligned}
$$

where $k = 1, \cdots, r$; $\alpha'_k = \min\left[\Re\left(\frac{b_j^{(k)}}{\beta_j^{(k)}}\right)\right]$, $j = 1, \cdots, m^{(k)}$;
and $\beta'_k = \max\left[\Re\left(\frac{a_j^{(k)}-1}{\alpha_j^{(k)}}\right)\right]$, $j = 1, \cdots, n^{(k)}$.
The generalized class of multivariable polynomials defined by Srivastava [28], is given as

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[y_1,\cdots,y_s\right] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!} \\ A\left[N_1, K_1; \cdots; N_s, K_s\right] y_1^{K_1} \cdots y_s^{K_s}, \tag{4}$$

where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex.

The generalized polynomials of one variable defined by Srivastava [27], is given in the following manner:

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A\left[N, K\right] x^K, \tag{5}$$

where the coefficients$A\left[N, K\right]$ are arbitrary constants real or complex.
The series representation of the multivariable H-function defined by Srivastava and Panda [29, 30] is given by Chaurasia and Olka [9] as

$$H\left[u_1,\cdots,u_r\right] = H_{A',C':(M',N');\cdots;\left(M^{(r)},N^{(r)}\right)}^{0,\lambda':(\alpha',\beta');\cdots;\left(\alpha^{(r)},\beta^{(r)}\right)} \\ \left[\begin{array}{c} u_1 \\ \cdot \\ \cdot \\ u_r \end{array}\middle| \begin{array}{c} \left[(g_j);\gamma',\cdots,\gamma^{(r)}\right]_{1,A'} : \left(q^{(1)},\eta^{(1)}\right)_{1,M^{(1)}};\cdots;\left(q^{(r)},\eta^{(r)}\right)_{1,M^{(r)}} \\ \left[(f_j);\xi',\cdots,\xi^{(r)}\right]_{1,C'} : \left(p^{(1)},\epsilon^{(1)}\right)_{1,N^{(1)}};\cdots;\left(p^{(r)},\epsilon^{(r)}\right)_{1,N^{(r)}} \end{array}\right] \tag{6} \\ = \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n'_i=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i\, u_i^{U_i} (-)^{\sum_{i=1}^r n'_i}}{\prod_{i=1}^r \epsilon_{m_i}^i\, n'_i!},$$

where

$$\phi = \frac{\prod_{j=1}^{\lambda'} \Gamma\left(1 - g_j + \sum_{i=1}^r \gamma_j^{(i)} U_i\right)}{\prod_{j=\lambda'+1}^{A} \Gamma\left(g_j - \sum_{i=1}^r \gamma_j^{(i)} U_i\right) \prod_{j=1}^{C'} \Gamma\left(1 - f_j + \sum_{i=1}^r \xi_j^{(i)} U_i\right)}, \tag{7}$$

and

$$\phi_i = \frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}} \Gamma\left(p_j^{(i)} - \epsilon_j^{(i)} U_i\right) \prod_{j=1}^{\beta^{(i)}} \Gamma\left(1 - q_j^{(i)} + \eta_j^{(i)} U_i\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}} \Gamma\left(1 - p_j^{(i)} + \epsilon_j^{(i)} U_i\right) \prod_{j=\beta^{(i)}+1}^{M^{(i)}} \Gamma\left(q_j^{(i)} - \eta_j^{(i)} U_i\right)}, \tag{8}$$

for $i = 1, \cdots, r$ and $U_i = \frac{p_{m_i}^{(i)} + n_i'}{\epsilon_{m_i}^{(i)}}$, $\quad i = 1, \cdots, r$, which is valid under the following conditions:

$$\epsilon_{m_i}^{(i)} \left[p_j^{(i)} + p_i\right] \neq \epsilon_j^{(i)} \left[p_{m_i} + n_i\right],$$

for $j = m_i,\ m_i = 1, \cdots, \alpha^{(i)};\ \ p_i, n_i' = 0, 1, 2, \cdots; u_i \neq 0$

$$\Sigma_i = \sum_{j=1}^{A'} \gamma_j^{(i)} - \sum_{j=1}^{C'} \xi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{D^{(i)}} \epsilon_j^{(i)} < 0, \ \ \forall i \in \{1, \cdots, r\}. \tag{9}$$

Let $\mathfrak{I}$ denote the space of all functions f which are defined on $\mathbb{R}^+$ and satisfy
(i) $f \in C^\infty(\mathbb{R}^+)$
(ii) $\lim_{x\to\infty} [x^\gamma f^r(x)] = 0$ for all non-negative integersγ and r.
(iii) $f(x) = O(1)$ as $x \to 0$.
For correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$.

The Riemann-Liouville fractional integral (of order μ) is defined by

$$\begin{aligned} D^{-\mu}\{f(x)\} &= {}_0D_x^{-\mu}\{f(x)\} \\ &= \frac{1}{\Gamma(\mu)} \int_0^x (x-\omega)^{\mu-1}\ f(\omega)\, d\omega \qquad (\Re(\mu) > 0, f \in \mathfrak{I}), \end{aligned} \tag{10}$$

where $D^\mu\{f(x)\} = \phi(x)$ is understood to mean that ϕ is a locally integrable solution of $f(x) = D^{-\mu}\{\phi(x)\}$, implying that D^μ is the inverse of the fractional operator $D^{-\mu}$.

The Weyl fractional (of order h) is defined by Laurent [14] as following:

$$W^{-h}\{f(x)\} = {}_xD_\infty^{-h}\{f(x)\} = \frac{1}{\Gamma(h)} \int_x^\infty (\xi - x)^{h-1}\ f(\xi) d\xi, \tag{11}$$

where $\Re(h) > 0$ and $f \in \mathfrak{I}$.

2 Solution of the Fredholm integral equation (1)

For convenience, we shall use these following notations:

$$U_r = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; \quad V_r = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}, \tag{12}$$

$$W_r = \left(p^{(1)}, q^{(1)}\right); \cdots ; \left(p^{(r)}, q^{(r)}\right); \quad X_r = \left(m^{(1)}, n^{(1)}\right); \cdots ; \left(m^{(r)}, n^{(r)}\right), \tag{13}$$

$$A = \left(a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}\right)_{1,p_2}; \cdots ; \left(a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots , \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}}, \tag{14}$$

$$\mathbb{A} = \left(a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots , \alpha_{rk}^{(r)}\right)_{1,p_r} : \left(a_k^{(1)}, \alpha_k^{(1)}\right)_{1,p^{(1)}}; \cdots ; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}, \tag{15}$$

$$B = \left(b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}\right)_{1,q_2}; \cdots ; \left(b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots , \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}}, \tag{16}$$

$$\mathbb{B} = \left(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots , \beta_{rk}^{(r)}\right)_{1,q_r} : \left(b_k^{(1)}, \beta_k^{(1)}\right)_{1,q^{(1)}}; \cdots ; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}, \tag{17}$$

$$a_s = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!} A\left[N_1, K_1; \cdots ; N_s, K_s\right], \tag{18}$$

$$U = (\alpha', \beta'); \cdots ; \left(\alpha^{(r)}, \beta^{(r)}\right); \; V = (M', N'); \cdots ; \left(M^{(r)}, N^{(r)}\right), \tag{19}$$

$$C = \left[(g_j); \gamma', \cdots , \gamma^{(r)}\right]_{1,A'} : \left(q^{(1)}, \eta^{(1)}\right)_{1,M^{(1)}}; \cdots ; \left(q^{(r)}, \eta^{(r)}\right)_{1,M^{(r)}}, \tag{20}$$

$$D = \left[(f_j); \xi', \cdots , \xi^{(r)}\right]_{1,C'} : \left(p^{(1)}, \epsilon^{(1)}\right)_{1,N^{(1)}}; \cdots ; \left(p^{(r)}, \epsilon^{(r)}\right)_{1,N^{(r)}}. \tag{21}$$

We have the following formula:

Lemma 1

$$W^{\beta-\alpha}\left[y^{-\alpha} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t, \cdots , u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';U}\left(\begin{array}{c|c} v_1\left(\frac{x}{y}\right)^p & C \\ \vdots & \vdots \\ v_r\left(\frac{x}{y}\right)^p & D \end{array}\right)\right.$$

$$\left.\times\; I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c|c} z_1\left(\frac{x}{y}\right)^q & A; \mathbb{A} \\ \vdots & \vdots \\ z_r\left(\frac{x}{y}\right)^q & B; \mathbb{B} \end{array}\right)\right]$$

$$= y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^{r} \phi_i v_i^{U_i} (-)^{\sum_{i=1}^{r} n_i'}}{\prod_{i=1}^{r} \epsilon_{m_i}^{i} n_i'!}$$

$$a_s u_1^{K_1} \cdots u_s^{K_s} \left(\frac{x}{y}\right)^{t\sum_{i=1}^{s} K_i + p\sum_{i=1}^{r} U_i} \times I_{U_r:p_r+1,q_r+1;W_r}^{V_r;0,n_r+1;X_r} \tag{22}$$

$$\left(\begin{array}{c|c} z_1 \left(\frac{x}{y}\right)^{q} & A; \left(1-\beta - t\sum_{i=1}^{s} K_i - p\sum_{i=1}^{r} U_i; q, \cdots, q\right), \mathbb{A} \\ \vdots & \\ z_r \left(\frac{x}{y}\right)^{q} & B; \left(1-\alpha - t\sum_{i=1}^{s} K_i - p\sum_{i=1}^{r} U_i; q, \cdots, q\right), \mathbb{B} \end{array} \right),$$

where ϕ_1, ϕ_i *and* a_s *are defined respectively by* (7), (8) *and* (18). *Also, provided that*

(a) $\Re(\alpha) > \Re(\beta)$,

(b) $\Re\left[\beta + p\sum_{i=1}^{r} \min_{1\le j\le M^{(i)}} \left(\frac{p_j^{(i)}}{\epsilon_j^{(i)}}\right) + q\sum_{i=1}^{r} \min_{1\le j\le m^{(i)}} \left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right] > 0$,

(c) $|\arg z_i| < \frac{1}{2}\Omega_i \pi$, *where* Ω_i *is defined by* (4),

(d) *which is valid under the following conditions:* $\epsilon_{m_i}^{(i)}[p_j^{(i)} + p_i] \neq \epsilon_j^{(i)}[p_{m_i} + n_i]$ *for* $j = m_i, m_i = 1 \cdots, \alpha^{(i)};\ p_i, n_i' = 0, 1, 2, \cdots; v_i \neq 0$,

$$\Sigma_i = \sum_{j=1}^{A'} \gamma_j^{(i)} - \sum_{j=1}^{C'} \xi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{D^{(i)}} \epsilon_j^{(i)} < 0, \quad \forall i \in \{1, \cdots, r\}$$

Proof. To prove the lemma first we use the definition of Weyl fractional integral given by (11), express the class of multivariable polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$ in series with the help of (4), the multivariable H-function in series with the help of (6) and the multivariable I-function defined by Prasad [21] in Mellin-Barnes type contour integral. Now we interchange the order of summation and integrations (which is permissible under the conditions stated), we evaluate the t-integral and reinterpreting the resulting Mellin-Barnes contour integral as terms of the multivariable I-function, we obtain the desired result. □

Theorem 1

$$\int_0^{\infty} y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^{r} \phi_i v_i^{U_i} (-)^{\sum_{i=1}^{r} n_i'}}{\prod_{i=1}^{r} \epsilon_{m_i}^{i} n_i'!}$$

$$a_s \left(\frac{x}{y}\right)^{t\sum_{i=1}^{s} K_i + p\sum_{i=1}^{r} U_i} u_1^{K_1} \cdots u_s^{K_s} \times I_{U_r:p_r+1,q_r+1;W_r}^{V_r;0,n_r+1;X_r}$$

$$\left(\begin{array}{c} z_1\left(\frac{x}{y}\right)^q \\ \vdots \\ z_r\left(\frac{x}{y}\right)^q \end{array}\middle| \begin{array}{c} \mathrm{A};(1-\beta-t\sum_{i=1}^s K_i - p\sum_{i=1}^r U_i;q,\cdots,q),\mathbb{A} \\ \mathrm{B};(1-\alpha-t\sum_{i=1}^s K_i - p\sum_{i=1}^r U_i;q,\cdots,q),\mathbb{B} \end{array}\right) f(y)\,dy$$

$$= \int_0^\infty y^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';U}\left(\begin{array}{c} v_1\left(\frac{x}{y}\right)^p \\ \vdots \\ v_r\left(\frac{x}{y}\right)^p \end{array}\middle| \begin{array}{c} C \\ \vdots \\ D \end{array}\right)$$

$$\times I_{U_r:p_r,q_r;W_r}^{V_r;0,n_r;X_r}\left(\begin{array}{c} z_1\left(\frac{x}{y}\right)^q \\ \vdots \\ z_r\left(\frac{x}{y}\right)^q \end{array}\middle| \begin{array}{c} A,\mathbb{A} \\ \vdots \\ B,\mathbb{B} \end{array}\right) D^{\beta-\alpha}\left[f(x)\right]\,dy, \tag{23}$$

under the same conditions and notations that (22)*.*

Proof. Let E denote the first member of the equation (23). Then using the 1 and applying (11), we have

$$E = \int_0^\infty \frac{f(y)}{\Gamma(\alpha-\beta)}\,(\xi-y)^{\alpha-\beta-1}\,\xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right]$$
$$\times H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right] I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right] d\xi\,dy$$
$$= \int_0^\infty \xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right] H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right]$$
$$\times I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right]\left\{\int_0^\xi \frac{(\xi-y)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(y)\,dy\right\} d\xi. \tag{24}$$

□

The change the order of integration is assumed to be permissible just as in the proof of the lemma 1. Now by applying to definition (11) and (24), it gives

$$E = \int_0^\infty \xi^{-\alpha}\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{\xi}\right)^t,\cdots,u_s\left(\frac{x}{\xi}\right)^t\right] H\left[v_1\left(\frac{x}{\xi}\right)^p,\cdots,v_r\left(\frac{x}{\xi}\right)^p\right]$$
$$\times I\left[z_1\left(\frac{x}{\xi}\right)^q,\cdots,z_r\left(\frac{x}{\xi}\right)^q\right]\left\{D^{\beta-\alpha}\{f(y)\}\,dy\right\} d\xi. \tag{25}$$

We obtain the desired result where $f \in \mathfrak{I}$ and$x > 0$, under the same conditions that (22).

3 Particular cases

We obtain the similar formula with the class of polynomials of one variable defined by Srivastava [27]. We have

Corollary 1

$$\int_0^\infty y^{-\beta} \sum_{K=0}^{[N/M]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i v_i^{U_i} (-)^{\sum_{i=1}^r n_i'}}{\prod_{i=1}^r \epsilon_{m_i}^i n_i'!} \frac{(-N)_{MK}}{K!}$$

$$A[N,K] \left(\frac{x}{y}\right)^{tK+p\sum_{i=1}^r U_i} u^K \times I^{V_r;0,n_r+1;X_r}_{U_r:p_r+1,q_r+1;W_r}$$

$$\left(\begin{array}{c} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{array} \middle| \begin{array}{c} A; (1-\beta-tK-p\sum_{i=1}^r U_i; q, \cdots, q), \mathbb{A} \\ B; (1-\alpha-tK-p\sum_{i=1}^r U_i; q, \cdots, q), \mathbb{B} \end{array} \right) f(y)\, dy \tag{26}$$

$$= \int_0^\infty y^{-\alpha} S_N^M \left(u \left(\frac{x}{y}\right)^t \right) H^{0,\lambda';u}_{A',C';V} \left(\begin{array}{c} v_1 \left(\frac{x}{y}\right)^p \\ \vdots \\ v_r \left(\frac{x}{y}\right)^p \end{array} \middle| \begin{array}{c} C \\ \vdots \\ D \end{array} \right)$$

$$I^{V_r;0,n_r;X_r}_{U_r:p_r,q_r;W_r} \left(\begin{array}{c} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{array} \middle| \begin{array}{c} A; \mathbb{A} \\ \vdots \\ B; \mathbb{B} \end{array} \right) \times D^{\beta-\alpha} [f(x)]\, dy,$$

under the same conditions and notations that (22).

If the multivariable I-function reduces to the multivariable H-function defined by Srivastava and Panda [29, 30], we obtain the following result:

Corollary 2

$$\int_0^\infty y^{-\beta} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{m_i=0}^{\alpha^{(i)}} \sum_{n_i'=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i v_i^{U_i} (-)^{\sum_{i=1}^r n_i'}}{\prod_{i=1}^r \epsilon_{m_i}^i n_i'!}$$

$$a_s \left(\frac{x}{y}\right)^{t\sum_{i=1}^s K_i + p\sum_{i=1}^r U_i} u_1^{K_1} \cdots u_s^{K_s} \times H^{0,n_r+1;X_r}_{p_r+1,q_r+1;W_r}$$

$$\left(\begin{array}{c} z_1 \left(\frac{x}{y}\right)^q \\ \vdots \\ z_r \left(\frac{x}{y}\right)^q \end{array} \middle| \begin{array}{c} (1-\beta-t\sum_{i=1}^s K_i - p\sum_{i=1}^r U_i; q, \cdots, q), \mathbb{A} \\ (1-\alpha-t\sum_{i=1}^s K_i - p\sum_{i=1}^r U_i; q, \cdots, q), \mathbb{B} \end{array} \right) f(y)\, dy$$

$$= \int_0^{\infty} y^{-\alpha} S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[u_1\left(\frac{x}{y}\right)^t,\cdots,u_s\left(\frac{x}{y}\right)^t\right] H_{A',C';V}^{0,\lambda';U}\left(\begin{array}{c|c} v_1\left(\frac{x}{y}\right)^p & C \\ \vdots & \vdots \\ v_r\left(\frac{x}{y}\right)^p & D \end{array}\right)$$

$$\times H_{p_r,q_r;W_r}^{0,n_r;X_r}\left(\begin{array}{c|c} z_1\left(\frac{x}{y}\right)^q & \mathbb{A} \\ \vdots & \vdots \\ z_r\left(\frac{x}{y}\right)^q & \mathbb{B} \end{array}\right) D^{\beta-\alpha}\left[f(x)\right]\, dy, \tag{27}$$

under the same conditions and notations that (22) *with* $U_r = V_r = A = B = 0$.

4 Conclusion

The equation (1) is of general character. By suitably specializing the various parameters of the multivariable I-function, the multivariable H-function and the class of multivariable polynomials, our results can be reduce to a large number of integral equations involving various polynomials or special functions of one and several variables occur in many fields of physics, mechanics and applied mathematics.

Acknowledgments

I am grateful to the National Board of Higher Mathematics (NBHM) India, for granting a Post-Doctoral Fellowship (sanction no.2/40(37)/2014/R&D-II/14131).

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Received: August 9, 2017