Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) 32–45

DOI: 10.2478/ausm-2018-0003

On generalized Laguerre matrix polynomials

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Abstract. The main object of the present paper is to introduce and study the generalized Laguerre matrix polynomials for a matrix that satisfies an appropriate spectral property. We prove that these matrix polynomials are characterized by the generalized hypergeometric matrix function. An explicit representation, integral expression and some recurrence relations in particular the three terms recurrence relation are obtained here. Moreover, these matrix polynomials appear as solution of a differential equation.

1 Introduction

Laguerre, Hermite, Gegenbauer and Chebyshev matrix polynomials sequences have appeared in connection with the study of matrix differential equations [8, 7, ?, 4]. In [13], the Laguerre and Hermite matrix polynomials were introduced as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. The Laguerre matrix polynomials were introduced and studied in [11, ?, ?, ?]. In [?], it is shown that these matrix polynomials are orthogonal with respect to a non-diagonal

2010 Mathematics Subject Classification: 15A15, 33C45, 42C05
Key words and phrases: Laguerre matrix polynomials, three terms recurrence relation, generalized hypergeometric matrix function and Gamma matrix function

Sobolev-Laguerre matrix polynomials matrix moment functional. Recently, the numerical inversion of Laplace transforms using Laguerre matrix polynomials has been given in [?]. A generalized form of the Gegenbauer matrix polynomials is presented in [2]. Moreover, two generalizations of the Hermite matrix polynomials have been given in [1, ?].

The main aim of this paper is to consider a new generalization of the Laguerre matrix polynomials. The structure of this paper is the following. After a section introducing the notation and preliminary results, we characterize, in Section 3, the definition of the generalized Laguerre matrix polynomials and an explicit representation and integral expression are given. Finally, Section 4 deals with some recurrence relations in particular the three terms recurrence relation for these matrix polynomials. Furthermore, we prove that the generalized Laguerre matrix polynomials satisfy a matrix differential equation.

2 Preliminaries

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. We say that a matrix A is a positive stable if $\mathrm{Re}(\mu) > 0$ for every eigenvalue $\mu \in \sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [5, p. 558], it follows that $f(A)g(A) = g(A)f(A)$. The reciprocal gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable z. Then, for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if

$$A + nI \text{ is invertible for every integer } n \geq 0, \tag{1}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and it follows that [6, p. 253]

$$(A)_n = A(A+I)...(A+(n-1)I);\ n \geq 1, \tag{2}$$

with $(A)_0 = I$.

For any non-negative integers m and n, from (2), one easily obtains

$$(A)_{n+m} = (A)_n (A+nI)_m, \tag{3}$$

and

$$(A)_{mn} = m^{mn} \prod_{s=1}^{m} \left(\frac{1}{m}(A + (s-1)I) \right)_n. \tag{4}$$

Let P and Q be commuting matrices in $\mathbb{C}^{N\times N}$ such that for all integer $n \geq 0$ one satisfies the condition

$$P + nI, \quad Q + nI, \quad \text{and} \quad P + Q + nI \quad \text{are invertible.} \tag{5}$$

Then by [10, Theorem 2] one gets

$$B(P, Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P + Q), \tag{6}$$

where the gamma matrix function, $\Gamma(A)$, and the beta matrix function, $B(P, Q)$, are defined respectively [9] by

$$\Gamma(A) = \int_0^\infty \exp(-t)t^{A-I}dt, \tag{7}$$

and

$$B(P, Q) = \int_0^1 t^{P-I}(1 - t)^{Q-I}dt. \tag{8}$$

In view of (7), we have [10, p. 206]

$$(A)_n = \Gamma(A + nI)\Gamma^{-1}(A); \quad n \geq 0. \tag{9}$$

If λ is a complex number with $\mathrm{Re}(\lambda) > 0$ and A is a matrix in $\mathbb{C}^{N\times N}$ with $A + nI$ invertible for every integer $n \geq 1$, then the n-th Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ is defined by [8, p. 58]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k\lambda^k}{k!(n-k)!}(A + I)_n[(A + I)_k]^{-1}x^k, \tag{10}$$

and the generating function of these matrix polynomials is given [8] by

$$G(x, t, \lambda, A) = (1 - t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1 - t}\right) = \sum_{n\geq 0} L_n^{(A,\lambda)}(x)t^n. \tag{11}$$

According to [8], Laguerre matrix polynomials satisfy the three-term recurrence relation

$$\begin{aligned}(n + 1)L_{n+1}^{(A,\lambda)}(x) + \big[\lambda xI - (A + (2n + 1)I)\big]L_n^{(A,\lambda)}(x)&\\ + (A + nI)L_{n-1}^{(A,\lambda)}(x) = \theta; \quad n \geq 0&,\end{aligned} \tag{12}$$

with $L_{-1}^{(A,\lambda)}(x) = \theta$ and $L_0^{(A,\lambda)}(x) = I$ where θ is the zero matrix in $\mathbb{C}^{N\times N}$.

Definition 1 [2] *Let* p *and* q *be two non-negative integers. The generalized hypergeometric matrix function is defined in the form:*

$$
\begin{aligned}
&{}_pF_q(A_1,\dots,A_p;B_1\dots,B_q;z)\\
&\qquad = \sum_{n\geq 0}(A_1)_n\dots(A_p)_n[(B_1)_n]^{-1}\dots[(B_q)_n]^{-1}\frac{z^n}{n!},
\end{aligned} \tag{13}
$$

where A_i *and* B_j *are matrices in* $\mathbb{C}^{N\times N}$ *such that the matrices* $B_j; 1 \leq j \leq q$ *satisfy the condition* (1)*.*

With $p = 1$ and $q = 0$ in (13), one gets the following relation due to [11, p. 213]

$$
(1-z)^{-A} = \sum_{n\geq 0}\frac{1}{n!}(A)_n z^n, \quad |z| < 1. \tag{14}
$$

The following lemma provides results about double matrix series. The proof are analogous to the corresponding for the scalar case c.f [?, p. 56] and [?, p. 101].

Lemma 1 [2, 3, ?] *If* $A(k,n)$ *and* $B(k,n)$ *are matrices in* $\mathbb{C}^{N\times N}$ *for* $n \geq 0$ *and* $k \geq 0$*, then it follows that:*

$$
\sum_{n\geq 0}\sum_{k\geq 0}A(k,n) = \sum_{n\geq 0}\sum_{k=0}^{n}A(k,n-k), \tag{15}
$$

$$
\sum_{n\geq 0}\sum_{k=0}^{\lfloor n/m\rfloor}A(k,n) = \sum_{n\geq 0}\sum_{k\geq 0}A(k,n+mk), \tag{16}
$$

and

$$
\sum_{n\geq 0}\sum_{k\geq 0}B(k,n) = \sum_{n\geq 0}\sum_{k=0}^{\lfloor n/m\rfloor}B(k,n-mk) \quad ; n > m, \tag{17}
$$

where $\lfloor a\rfloor$ is the standard floor function which maps a real number a to its next smallest integer.

It is obviously desirable, by (2), to have the following:

$$
\begin{aligned}
\frac{1}{(n-mk)!}I &= \frac{(-1)^{mk}}{n!}(-nI)_{mk}\\
&= \frac{(-1)^{mk}}{n!}m^{mk}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}I\right)_k ; \quad 0 \leq mk \leq n.
\end{aligned} \tag{18}
$$

3 Definition of generalized Laguerre matrix polynomials

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (1) and let λ be a complex number with $\mathrm{Re}(\lambda) > 0$. For a positive integer m, we can define the generalized Laguerre matrix polynomials [**GLMPs**] by

$$F(x,t,\lambda,A) = (1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = \sum_{n=0}^{\infty} L_{n,m}^{(A,\lambda)}(x)t^n. \tag{19}$$

By (14) one gets

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{k!n!}x^{mk}(A+I+mkI)_n t^{n+mk} = \sum_{n=0}^{\infty} L_{n,m}^{(A,\lambda)}(x)t^n,$$

which by using (17) and (3) and equating the coefficients of t^n, yields an explicit representation for the **GLMPs** in the form:

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}(A+I)_n[(A+I)_{mk}]^{-1}x^{mk}. \tag{20}$$

It should be observed that when $m = n$, the explicit representation (20) becomes

$$L_{n,n}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!} - \lambda x^n I.$$

If $m > n$, then from (20) one gets

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}.$$

Moreover, it is evident that

$$L_{n,m}^{(A,\lambda)}(0) = \frac{(A+I)_n}{n!} \quad \text{and} \quad L_{n,m}^{(A,\lambda)}(x) = L_{n,m}^{(A,1)}(\lambda^{\frac{1}{m}}x).$$

Note that the expression (20) coincides with (10) for the case $m = 1$.

In view of (4) and (18), we can rewrite the formula (20) in the form

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^{(m+1)k}\lambda^k}{k!}x^{mk}\prod_{p=1}^{m}\Big(\frac{p-n-1}{m}I\Big)_k \tag{21}$$

$$\times \left[\prod_{s=1}^{m} \left(\frac{1}{m}(A + sI \right)_k \right]^{-1}.$$

Therefore, in view of (13), the hypergeometric matrix representation of **GLMPs** is given in the form:

$$L_{n,m}^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}\, {}_mF_m\left(\frac{-n}{m}I, \cdots, \frac{(-n+m-1)}{m}I; \frac{A+I}{m}, \cdots, \frac{A+mI}{m}; (-1)^{m+1}\lambda x^m \right). \tag{22}$$

We give a generating matrix function of **GLMPs**. This result is contained in the following.

Theorem 1 *Let* A *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying* (1) *and let* λ *be a complex number with* $\mathrm{Re}(\lambda) > 0$. *Then*

$$\sum_{n\geq 0} [(A+I)_n]^{-1} L_{n,m}^{(A,\lambda)}(x) t^n = e^t\, {}_0F_m\left(-; \frac{A+I}{m}, \cdots, \frac{A+mI}{m}; -\lambda \left(\frac{xt}{m}\right)^m \right). \tag{23}$$

Proof. By virtue of (20) and applying (16), we have

$$\sum_{n\geq 0} [(A+I)_n]^{-1} L_{n,m}^{(A,\lambda)}(x) t^n = \left(\sum_{n\geq 0} \frac{t^n}{n!} \right) \left(\sum_{k\geq 0} \frac{(-1)^k \lambda^k}{k!} [(A+I)_{mk}]^{-1} x^{mk} t^{mk} \right),$$

which, by using (4) and (13), reduces to (23). □

It is clear that

$$e^t\, {}_0F_m\left(-; \frac{A+I}{m}, \cdots, \frac{A+mI}{m}; -\lambda \left(\frac{xyt}{m}\right)^m \right) = e^{(1-y)t} e^{ty}\, {}_0F_m\left(-; \frac{A+I}{m}, \cdots, \frac{A+mI}{m}; -\lambda \left(\frac{xyt}{m}\right)^m \right).$$

Thus, by using (23) and applying (15), it follows that

$$\sum_{n\geq 0} [(A+I)_n]^{-1} L_{n,m}^{(A,\lambda)}(xy) t^n = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{(1-y)^{n-k} y^k}{(n-k)!} [(A+I)_k]^{-1} L_{k,m}^{(A,\lambda)}(x) t^n.$$

By equating the coefficients of t^n, in the last series, one gets

$$L_{n,m}^{(A,\lambda)}(xy) = (A+I)_n \sum_{k=0}^{n} \frac{(1-y)^{n-k} y^k}{(n-k)!} [(A+I)_k]^{-1} L_{k,m}^{(A,\lambda)}(x).$$

Let B be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1). From (3), (4), (14) and (16) and taking into account (20) we have

$$\begin{aligned}&\sum_{n\geq 0}(B)_n[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n\\&\qquad=(1-t)^{-B}\sum_{n\geq 0}\frac{(-\lambda)^n}{n!}(B)_{mn}[(A+I)_{mn}]^{-1}\left(\frac{xt}{1-t}\right)^{mn}.\end{aligned}\tag{24}$$

By using (4) and (13), the equation (24) gives the following generating function of **GLMPs**:

$$\begin{aligned}&\sum_{n\geq 0}(B)_n[(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)t^n=(1-t)^{-B}\\&{}_mF_m\left(\frac{B}{m},\cdots,\frac{B+(m-1)I}{m};\frac{A+I}{m},\cdots,\frac{A+mI}{m};-\lambda\left(\frac{xt}{1-t}\right)^m\right).\end{aligned}\tag{25}$$

Clearly, (25) reduces to (19) when $B=A+I$.

We now proceed to give an integral expression of **GLMPs**. For this purpose, we state the following result.

Theorem 2 *Let A and B be positive stable matrices in* $\mathbb{C}^{N\times N}$ *such that* $AB=BA$*. Then*

$$\begin{aligned}L_{n,m}^{(A+B,\lambda)}(x)&=\Gamma(A+B+(n+1)I)\Gamma^{-1}(B)\Gamma^{-1}(A+(n+1)I)\\&\quad\times\int_0^1 t^A(1-t)^{B-I}L_{n,m}^{(A,\lambda)}(xt)dt.\end{aligned}\tag{26}$$

Proof. According to (8) and (20), we can write

$$\begin{aligned}\Psi&=\int_0^1 t^A(1-t)^{B-I}L_{n,m}^{(A,\lambda)}(xt)dt\\&=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}(A+I)_n[(A+I)_{mk}]^{-1}x^{mk}B(A+(mk+1)I,B),\end{aligned}\tag{27}$$

and since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. Hence by (6) and (9) it follows that

$$
\begin{aligned}
\Psi &= \Gamma\big(A+(n+1)I\big)\Gamma(B)\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k x^{mk}}{k!(n-mk)!}\Gamma^{-1}\big(A+B+(mk+1)I\big)\\
&= \Gamma\big(A+(n+1)I\big)\Gamma(B)\Gamma^{-1}\big(A+B+(n+1)I\big)\\
&\qquad \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k x^{mk}}{k!(n-mk)!}(A+B+I)_n[(A+B+I)_{mk}]^{-1}.
\end{aligned}
\tag{28}
$$

From (20), (27) and (28), the expression (26) holds. □

We conclude this section giving an integral form of **GLMPs**.

Theorem 3 *For* **GLMPs** *the following holds*

$$
\begin{aligned}
\int_0^\infty x^A L_{n,m}^{(A,\lambda)}(x)e^{-x}dx &= \frac{\Gamma(A+(n+1)I)}{n!}\\
&\quad {}_mF_0\left(\frac{-n}{m},\cdots,\frac{-n+m-1}{m};-;(-1)^{m+1}\lambda m^m\right).
\end{aligned}
\tag{29}
$$

Proof. From (7), (9) and (20), it follows that

$$
\begin{aligned}
\int_0^\infty x^A L_{n,m}^{(A,\lambda)}(x)e^{-x}dx &= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}\\
&\qquad (A+I)_n[(A+I)_{mk}]^{-1}\Gamma(A+(mk+1)I)\\
&= \Gamma(A+(n+1)I)\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\lambda^k}{k!(n-mk)!}.
\end{aligned}
$$

Using (18) and taking into account (13) we arrive at (29). □

4 Recurrence relations

In addition to the three terms recurrence relation, some differential recurrence relations of **GLMPs** are obtained here.

Theorem 4 *The generalized Laguerre matrix polynomials satisfy the following relations:*

$$\begin{aligned}\sum_{r=0}^{2m}\binom{2m}{r}(-1)^r(n+1-r)L_{n+1-r,m}^{(A,\lambda)}(x) &= (A+I)\sum_{r=0}^{2m-1}\binom{2m-1}{r}(-1)^rL_{n-r,m}^{(A,\lambda)}(x)\\ -m\lambda x^m\sum_{r=0}^{m}\binom{m}{r}(-1)^rL_{n-r-m+1,m}^{(A,\lambda)}(x) &- m\lambda x^m\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^rL_{n-r-m,m}^{(A,\lambda)}(x),\end{aligned}\tag{30}$$

and

$$\sum_{r=0}^{m}\binom{m}{r}(-1)^rDL_{n-r,m}^{(A,\lambda)}(x) = -\lambda m x^{m-1}L_{n-m,m}^{(A,\lambda)}(x).\tag{31}$$

Proof. Differentiating (19) with respect to t yields

$$\begin{aligned}(1-t)^{2m}\sum_{n\geq 1}nL_{n,m}^{(A,\lambda)}(x)t^{n-1} &= (A+I)(1-t)^{2m-1}\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n\\ -\lambda m x^m t^{m-1}(1-t)^m\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n &- \lambda x^m t^m(1-t)^{m-1}\sum_{n\geq 0}L_{n,m}^{(A,\lambda)}(x)t^n.\end{aligned}$$

With the help of the binomial theorem, it follows that

$$\begin{aligned}&\sum_{n\geq 0}\sum_{r=0}^{2m}\binom{2m}{r}(-1)^r(n+1)L_{n+1,m}^{(A,\lambda)}(x)t^{n+r}\\ &= (A+I)\sum_{n\geq 0}\sum_{r=0}^{2m-1}\binom{2m-1}{r}(-1)^rL_{n,m}^{(A,\lambda)}(x)t^{n+r}\\ &\quad -\lambda x^m\Bigg[m\sum_{n\geq m-1}\sum_{r=0}^{m}\binom{m}{r}(-1)^rL_{n-m+1,m}^{(A,\lambda)}(x)t^{n+r}\\ &\quad +\sum_{n\geq m}\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^rL_{n-m,m}^{(A,\lambda)}(x)t^{n+r}\Bigg].\end{aligned}$$

Hence, by equating the coefficients of t^n, equation (30) holds.

Now, by differentiating (19) with respect to x one gets

$$\sum_{n\geq 0}DL_{n,m}^{(A,\lambda)}(x)t^n = \frac{-\lambda m x^{m-1}t^m}{(1-t)^m}(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right).$$

Thus, it follows that

$$\sum_{n\geq m}\sum_{r=0}^{m}\binom{m}{r}(-1)^r DL_{n-r,m}^{(A,\lambda)}(x)t^n = -\lambda m x^{m-1}\sum_{n\geq m} L_{n-m,m}^{(A,\lambda)}(x)t^n,$$

which, by equating the coefficients of t^n, gives (31). □

It is worthy to mention that (30) reduces to (12) for $m = 1$. Also, for the case $m = 1$, the expression (31) gives the result for the Laguerre matrix polynomials in the form

$$DL_n^{(A,\lambda)}(x) = DL_{n-1}^{(A,\lambda)}(x) - \lambda L_{n-1}^{(A,\lambda)}(x).$$

Differentiating (19) with respect to x again we obtains

$$\begin{aligned}\sum_{n\geq 0} DL_{n,m}^{(A,\lambda)}(x)t^n &= -\lambda m x^{m-1}t^m(1-t)^{-(A+(m+1)I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right)\\ &= -\lambda m x^{m-1}\sum_{n\geq m} L_{n-m,m}^{(A+mI,\lambda)}(x)t^n.\end{aligned}$$

Hence, by equating the coefficients of t^n, we readily obtain

$$DL_{n,m}^{(A,\lambda)}(x) = -\lambda m x^{m-1} L_{n-m,m}^{(A+mI,\lambda)}(x). \tag{32}$$

It may be noted that the formula (32) reduces to the result of [12, p. 16] for Laguerre matrix polynomials, when $m = 1$, in the form

$$DL_n^{(A,\lambda)}(x) = -\lambda L_{n-1}^{(A+I,\lambda)}(x).$$

Using the fact that

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = (1-t)^m(1-t)^{-(A+(m+1)I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right),$$

and (19), one gets

$$\sum_{n\geq 0} L_{n,m}^{(A,\lambda)}(x)t^n = \sum_{n\geq r}\sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-r,m}^{(A+mI,\lambda)}(x)t^n.$$

Hence, we obtain that

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{r=0}^{m}\binom{m}{r}(-1)^r L_{n-r,m}^{(A+mI,\lambda)}(x).$$

Let B be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1) with $AB = BA$. Note that

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right) = (1-t)^{-(A-B)}(1-t)^{-(B+I)}\exp\left(\frac{-\lambda x^m t^m}{(1-t)^m}\right).$$

Using (14), (15) and (19) , it follows that

$$\sum_{n\geq 0} L_{n,m}^{(A,\lambda)}(x)t^n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(A-B)_k}{k!}L_{n-k,m}^{(B,\lambda)}(x)t^n.$$

Identifying the coefficients of t^n, in the last series, gives

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{n}\frac{(A-B)_k}{k!}L_{n-k,m}^{(B,\lambda)}(x). \tag{33}$$

By reversing the order of summation in (33), we obtain that

$$L_{n,m}^{(A,\lambda)}(x) = \sum_{k=0}^{n}\frac{(A-B)_{n-k}}{(n-k)!}L_{k,m}^{(B,\lambda)}(x). \tag{34}$$

And finally, we prove the following result.

Theorem 5 *The* **GLMPs** *is a solution of the following differential equation*

$$\begin{aligned}\Bigg[\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)+(-1)^m\lambda m x^m \\ \times\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)I\Bigg]L_{n,m}^{(A,\lambda)}(x)=\theta,\end{aligned} \tag{35}$$

where $\Theta = x\frac{d}{dx}$.

Proof. It is clear that $\frac{1}{m}\Theta x^{mk} = kx^{mk}$. According to (22) we can write

$$\begin{aligned} W &= {}_mF_m\left(-\frac{n}{m}I,\ldots,-\frac{n-m+1}{m}I;\frac{A+I}{m},\ldots,\frac{A+mI}{m};(-1)^{m+1}\lambda x^m\right)\\ &= \sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_k\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI\right)_k\right]^{-1}(-1)^{(m+1)k}\lambda^k\frac{x^{mk}}{k!}.\end{aligned}$$

It follows after replacing k by $k+1$ and using (3) that

$$\begin{aligned}
&\frac{1}{m}\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)W\\
&=\sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_{k+1}\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI)\right)_{k}\right]^{-1}(-1)^{(m+1)(k+1)}\lambda^{k+1}\frac{x^{m(k+1)}}{k!}\\
&=(-1)^{m+1}\lambda x^{m}\sum_{k=0}^{\lfloor n/m\rfloor}\prod_{p=1}^{m}\left(\frac{p-n-1}{m}\right)_{k+1}\left[\prod_{s=1}^{m}\left(\frac{1}{m}(A+sI)\right)_{k}\right]^{-1}(-1)^{(m+1)k}\lambda^{k}\frac{x^{mk}}{k!}\\
&=(-1)^{m+1}\lambda x^{m}\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)W.
\end{aligned}$$

Therefore, W is a solution of the following differential equation

$$\left[\frac{1}{m}\Theta\prod_{s=1}^{m}\left(\frac{1}{m}(\Theta-1)I+\frac{1}{m}(A+sI)\right)+(-1)^{m}\lambda x^{m}\prod_{p=1}^{m}\left(\frac{1}{m}\Theta+\frac{p-n-1}{m}\right)\right]W=\theta.$$

Since $W=n![(A+I)_n]^{-1}L_{n,m}^{(A,\lambda)}(x)$, then (35) follows immediately. $\square$

It is worth noticing that taking $m=1$ in (35) gives the following [8]

$$\left[xI\frac{d^2}{dx^2}+(A+(1-\lambda x)I)\frac{d}{dx}+\lambda nI\right]L_{n}^{(A,\lambda)}(x)=\theta.$$

Acknowledgments

The authors wish to express their gratitude to the unknown referee for several helpful suggestions.

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Received: January 5, 2015