Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) 189–197

DOI: 10.2478/ausm-2018-0016

Cyclic flats and corners of the linking polynomial

Koko K. Kayibi
Department of Physics and Mathematics,
University of Hull, UK
email: kokokayibi@hull.ac.uk

U. Samee
Department of Mathematics,
University of Kashmir, India
email: pzsamee@yahoo.co.in

S. Pirzada
Department of Mathematics,
University of Kashmir, India
email: pirzadasd@kashmiruniversity.ac.in

Abstract. Let $T(M; x, y) = \sum_{ij} T_{ij} x^i y^j$ denote the Tutte polynomial of the matroid M. If T_{ij} is a corner of $T(M; x, y)$, then T_{ij} counts the sets of corank i and nullity j and each such set is a cyclic flat of M. The main result of this article consists of extending the definition of cyclic flats to a pair of matroids and proving that the corners of the linking polynomial give the lower bound of the number of the cyclic flats of the matroid pair.

1 Introduction

Let A and B be two sets. We denote by $A \setminus B$ the set difference between A and B. We write $A \setminus e$ for $A \setminus \{e\}$. Similarly, we write $A \cup f$ instead of $A \cup \{f\}$. A *matroid* M defined on a finite nonempty set E consists of the set E and a collection $\mathcal{I}$ of subsets of E, satisfying the following axioms:

I1: $\emptyset \in \mathcal{I}$

I2: if $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$

2010 Mathematics Subject Classification: 52B40, 90C27, 52C40
Key words and phrases: matroid, cyclic flat, Tutte polynomial, linking polynomial

I3: if I_1 and $I_2 \in \mathcal{I}$ and, $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Elements of $\mathcal{I}$ are the *independent sets* of the matroid M. A *circuit* of M is a subset C that is not independent but $X \setminus e$ is independent for every $e \in X$. That is, a circuit is a minimal non-inependent set. A basis of M is a maximal independent set. The *dual* matroid of M, denoted by M^*, is the matroid whose bases are the complements of bases of M.

Let 2^E denotes the set of all the subsets of E and let $\mathcal{N}^+$ denotes the set of non-negative integers. The *rank function* of M, denoted by r, is a function from 2^E to $\mathcal{N}^+$, where, for $X \subseteq E$, $r(X)$ is the cardinality of the largest independent set I contained in X. The *dual* matroid of M, denoted by M^*, is the matroid whose bases are the complements of the bases of M.

For a matroid M defined on E, the Tutte polynomial of M, denoted by $T(M; x, y)$, is a two-variable polynomial defined as follows.

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where r is the rank function of M. This polynomial is much researched on since it encodes much information about the matroidal properties of combinatorial structures and is found useful in counting combinatorial invariants. For instance, let $G = (V, E)$ be a graph whose vertex set is V and edge set is E. Let M_G be the matroid defined on E, where the set of independent sets of M_G is the set of subsets $X \subseteq E$ that do not contain a closed path. It can be proved that for $A \subseteq E$, $r(A) = |V| - c(A)$, where $c(A)$ is the number of connected components of the graph (V, A). Thus, $T(M_G; 1-\lambda, 0)$ is proportional to the number of coloring of G using λ colors. See [3, 5, 6, 4, 17] for an extensive exposition to this topic.

Let M and N be two matroids defined on the set E with rank function r and s respectively. We call the pair (M, N) a *matroid pair*. The *dual matroid pair* is the pair (M^*, N^*) where M^* and N^* denote the dual matroids of M and N respectively. The *linking polynomial* of (M, N), denoted $Q(M, N; x, y, u, v)$ is defined in [24] as follows.

$$Q(M, N; x, y, u, v) = \sum_{X \subseteq E} x^{r(E)-r(X)} y^{|X|-r(X)} u^{s(E)-s(X)} v^{|X|-s(X)}.$$

The linking polynomial contains, as a specialisation the Tutte polynomial of a matroid and it also partially contains the Tutte invariant of *2-polymatroids* defined by Oxley and Whittle in [14, 15].

There is a *weak map* from a matroid M to another matroid N if every independent set in N is also independent in M, whereas a weak map is a *strong map* if every closed set of N is closed in M. A strong map is a *matroid perspective* if M and N are defined on the same set.

The linking polynomial is equivalent to the Tutte polynomial of a *matroid perspective*, the polynomial $T(P; x, y, z)$, defined and studied by the late Las Vergnas [9, 19, 20, 21, 22, 23] and in [10]. One of its most interesting evaluations is $T(P; 0, 0, 1)$. An *oriented matroid* is a matroid where an orientation is assigned to every element e. One of the simplest examples of oriented matroids is the cycle matroid of a graph G whose edges are oriented. If P is a perspective from an oriented matroid M to the oriented matroid N, then $T(P; 0, 0, 1)$ counts the number of subsets A such that A is acyclic in M and totally cyclic in N [22]. An obvious application is when there is a strong map from a cycle matroid of a graph G to a cycle matroid of a graph G' and one defines an orientation on the edges of G. This orientation is carried to the edges of G' in an obvious way. Then, $T(P; 0, 0, 1)$ counts the number of subsets A of edges, such that A is acyclic in G and totally cyclic in G'. This evaluation is paramount as it generalizes results on bounded regions of real hyperplane arrangements [25], non Radon partitions of real spaces [3]. More of such applications can be found in [19].

Moreover, the *bond matroid* of a graph G is the matroid whose independent sets are the subsets of edges of G that do not contain cutsets. Suppose that G and G^* are dually imbedded on a surface. Then there is a matroid perspective from the bond matroid of G^* to the cycle matroid of G. For the case of 4-valent graphs imbedded in the projective plane or a torus, Las Vergnas in [23] relates the Tutte polynomial of matroid perspective to Eulerian tours and cycles decompositions of G. This result sparks a renewed interest in the Tutte polynomial of matroid perspective because of its connection with the Bollobas-Riordan polynomial and Krushkal polynomial, which find many applications in the theory of graphs embedding on surfaces [8]. A generalization of matroid perspectives to a sequence of perspectives in [1] finds applications in electrical network theory. More applications of strong maps in engineering and in the theory of rigidity matroids can be seen in [2, 18]. Thanks to these many applications, the Tutte polynomial of matroid perspective deserves to be more studied algebraically and the Linking polynomial seems one of the best ways to investgate this algebraic structure. This paper looks at the corners of the linking polynomial and gives the lower bound of the number of the cyclic flats of the matroid pair.

For a matroid M defined on E, we write $M|X$ to denote the matroid M

restricted to the subset $X \subseteq E$. For a matroid M defined on E with rank function r, a subset X is a *flat* if for all $e \in E \setminus X$ we have $r(X \cup e) = r(X) + 1$.

A *cyclic flat* of M is a subset $X \subseteq E$ such that X is a flat of M and X is a union of circuits of M. In other words, X is a cyclic flat of M if X is a flat of M and there is no element $f \in X$ such that f is a coloop in $M|X$. A third equivalent definition is that X is a cyclic flat of a matroid M defined on E with rank function r if X is a flat of M and $r(X \setminus e) = r(X)$, for all $e \in X$.

2 Main results

We extend the above definitions to matroid pairs as follows. Let (M, N) be a matroid pair defined on E with rank funtions r and s for M and N respectively. We define a subset $X \subseteq E$ to be a *flat* of (M, N) if for all $f \in E \setminus X$

$$r(X \cup f) + s(X \cup f) \geq r(X) + s(X) + 1.$$

Further, a subset $X \subseteq E$ to be a *cyclic flat* of (M, N) if X is a flat of (M, N) and there is no element $e \in X$ such that e is a coloop in both $M|X$ and $N|X$. In other words, we say that X is a cyclic flat of the matroid pair (M, N) defined on E with rank functions r and s respectively if X is a flat of (M, N) and for all $e \in X$,

$$r(X \setminus e) + s(X \setminus e) > r(X) + s(X) - 2.$$

A classic result in Matroid Theory is as follows. X is a cyclic flat of a matroid M if and only if $E \setminus X$ is a cyclic flat of M^*. See [16] for an introduction to Matroid Theory. The next result extends this property to matroid pairs. If $P = (M, N)$ is a matroid pair, we denote by P^* the matroid pair (M^*, N^*).

Theorem 1 *Let P be a matroid pair. Then X is a cyclic flat of P if and only if $E \setminus X$ is a cyclic flat of P^*.*

Proof. First recall that if M is a matroid defined on E with rank function r, and r^* denotes the rank function of the dual matroid M^*, then for all $X \subseteq E$, we have

$$r^*(X) = |X| + r(E \setminus X) - r(E). \tag{1}$$

Suppose that X is a cyclic flat of P but $E \setminus X$ is not a cyclic flat of P^*. Then, either $E \setminus X$ is not a flat of (M^*, N^*) or $E \setminus X$ contains an element f which is a coloop in both $M^*|(E \setminus X)$ and $N^*|(E \setminus X)$.

Assume that $Y = E \setminus X$ is not a flat of (M^*, N^*). Then, for some element $g \in X$, we have

$$r^*(Y \cup g) + s^*(Y \cup g) < r^*(Y) + s^*(Y) + 1.$$

Therefore, by equation (1), we get

$$r(X \setminus g) + s(X \setminus g) < r(X) + s(X) - 1.$$

Equivalently,

$$r(X \setminus g) + s(X \setminus g) = r(X) + s(X) - 2.$$

Thus the element g is a coloop in both $M|X$ and $N|X$. Therefore X is not a cyclic flat of P, a contradiction.

If $Y = E \setminus X$ contains an element f which is coloop of Y in both M^* and N^* then $r^*(Y \setminus f) = r^*(Y) - 1$.

By equation (1), we get

$$|Y \setminus f| + r(E \setminus (Y \setminus f)) - r(E) = |Y| + r(E \setminus Y) - r(E) - 1.$$

Thus $r(X \cup f) = r(X)$. Similarly $s(X \cup f) = s(X)$.

Hence

$$r(X \cup f) + s(X \cup f) = r(X) + s(X) < r(X) + s(X) + 1.$$

Therefore, X is not a flat of P, a contradiction. To prove the converse one only needs to swap the roles of X and $E \setminus X$. $\square$

In the sequel, we write $(ijkl) \leq (i'j'k'l')$ if $i \leq i'$ and $j \leq j'$ and $k \leq k'$ and $l \leq l'$, and we write $(ijkl) = (i'j'k'l')$ if $i = i'$ and $j = j'$ and $k = k'$ and $l = l'$. We say that $(ijkl)$ and $(i'j'k'l')$ are *incomparable* if some indices in $(ijkl)$ are strictly superior to the corresponding indices in $(i'j'k'l')$ and some other indices in $(ijkl)$ are strictly inferior to the corresponding indices in $(i'j'k'l')$. For all $X \subseteq E$ let $\mathrm{cor}_M(X)$ denote the integer $r(E) - r(X)$, $\mathrm{nul}_M(X)$ denote $|X| - r(X)$, $\mathrm{cor}_N(X)$ denote the integer $s(E) - s(X)$ and $\mathrm{nul}_N(X)$ denote $|X| - s(X)$. Let $\mathcal{E}_{ijkl}$ denote the family of subsets $X \subseteq E$ such that $\mathrm{cor}_M(X) = i$, $\mathrm{nul}_M(X) = j$, $\mathrm{cor}_N(X) = k$, $\mathrm{nul}_N(X) = l$.

The next result, proved in [11], is instrumental in the proof of Theorem 2.4.

Lemma 1 *Let (M, N) be a matroid pair defined on E. If $\mathcal{E}_{ijkl}$ is not empty then q_{ijkl} is positive and*

$$q_{ijkl} = |\mathcal{E}_{ijkl}| + \sum_{Y \notin \mathcal{E}_{ijkl}} \binom{\mathrm{cor}_M(Y)}{i} \binom{\mathrm{nul}_M(Y)}{j} \binom{\mathrm{cor}_N(Y)}{k} \binom{\mathrm{nul}_N(Y)}{l}$$

where the sum is only over the subsets Y such that $(i'j'k'l') > (ijkl)$.

A *corner* of $T(M;x,y)$ is a coefficient T_{ij} such that $T_{ij} > 0$ and there is no other positive coefficient $T_{i'j'}$ with $(i'j') > (ij)$. Corners of a Tutte polynomial $T(M;x,y)$ convey much information about the matroid M. In [5], Brylawski proved that if T_{ij} is a corner of $T(M;x,y)$, then T_{ij} counts the sets of corank i and nullity j and each such set is a cyclic flat of M. This result is strengthened in [13] as follows.

Theorem 2 [13, Theorem 4.11] *Suppose that* $T_{ij} > 0$ *for a matroid* M*. Then the following are equivalent.*

(i) T_{ij} *is a corner of* $T(M;x,y)$*.*

(ii) *Every set of corank* i *and nullity* j *is a cyclic flat.*

(iii) T_{ij} *counts the sets of corank* i *and nullity* j*.*

We extend Theorem 2 to matroid pairs as follows. A coefficient q_{ijkl} is called a *corner* in $Q(M,N;x,y,u,v)$ if

(i) $q_{ijkl} \neq 0$

(ii) $q_{i'j'k'l'} = 0$ for all $(i'j'k'l')$ such that $(i'j'k'l') > (ijkl)$.

Theorem 3 *Let* (M,N) *be a matroid pair defined on* E*. If* q_{ijkl} *is a corner of* $Q(M,N;x,y,u,v)$ *then every* $X \in \mathcal{E}_{ijkl}$ *is a cyclic flat of* (M,N)*.*

Proof. Suppose that q_{ijkl} is a corner of $Q(M,N)$ and $X \in \mathcal{E}_{ijkl}$. Suppose that X is not a cyclic flat of (M,N). Then, either X is not a flat of (M,N) or X contains an element e which is a coloop of X in both M and N.

Suppose that X is not a flat of (M,N). Then there is an element $e \in E \setminus X$ such that

$$r(X \cup e) + r(X \cup e) < r(X) + s(X) + 1.$$

Equivalently

$$r(X \cup e) + r(X \cup e) = r(X) + s(X).$$

Thus $r(X \cup e) = r(X)$ and $s(X \cup e) = s(X)$. Now, consider $X \cup e$.

$$\mathrm{cor}_M(X \cup e) = \mathrm{cor}_M(X), \quad \mathrm{cor}_N(X \cup e) = \mathrm{cor}_N(X)$$

$$\mathrm{nul}_M(X \cup e) = \mathrm{nul}_M(X) + 1, \quad \mathrm{nul}_N(X \cup e) = \mathrm{nul}_N(X) + 1.$$

Thus, if $X \in \mathcal{E}_{ijkl}$, then $\mathcal{E}_{ij'kl'}$ where $j' = j+1, l' = l+1$ is not empty. Hence by Lemma 1, $q_{i,j',k,l'} \neq 0$. Thus q_{ijkl} is not a corner as $(ij'kl') > (ijkl)$. Contradiction.

Suppose that X has an element e which is a coloop in both M and N. Consider the subset $X \setminus e$. Then,

$$\mathrm{cor}_M(X \setminus e) = \mathrm{cor}_M(X) + 1, \quad \mathrm{cor}_N(X \setminus e) = \mathrm{cor}_N(X) + 1$$
$$\mathrm{nul}_M(X \setminus e) = \mathrm{nul}_M(X), \quad \mathrm{nul}_N(X \setminus e) = \mathrm{nul}_N(X).$$

Thus, if $X \in \mathcal{E}_{ijkl}$, then $\mathcal{E}_{i'jk'l}$ where $i' = i+1, k' = k+1$ is not empty. Hence by Lemma 1, $q_{i',j,k',l} \neq 0$. Hence q_{ijkl} not a corner. □

Theorem 2, which is a strengthening of the result of Brylawski can not be generalised to matroid pairs. Indeed, Theorem 2 says, among other things, that if X is a cyclic flat of M of corank i and nullity j, then T_{ij} is not a corner if and only if there exists a subset Y such that $\mathrm{cor}(Y) = i$ and $\mathrm{nul}(Y) = j$ but Y is not a cyclic flat. But we have an example of a matroid pair (M, N) where all $X \in \mathcal{E}_{ijkl}$ are cyclic flats of (M, N) but q_{ijkl} is not a corner of $Q(M, N)$. Indeed, consider the matroid pair given in Figure 1.

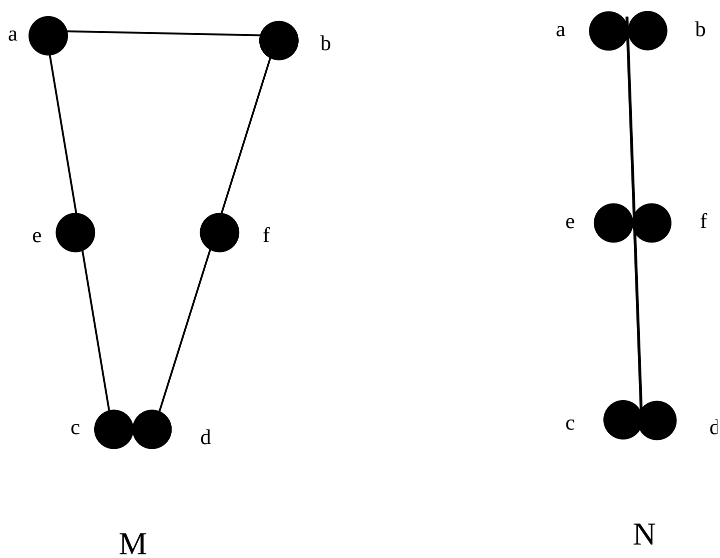

Figure 1: Example where $X \in \mathcal{E}_{ijkl}$ are cyclic flats of (M, N) but q_{ijkl} is not a corner of $Q(M, N)$.

The subset $\{a, b\} \in \mathcal{E}_{1011}$ is a cyclic flat of (M, N). The subset $\{c, d\} \in \mathcal{E}_{2111}$, thus $\mathcal{E}_{2111}$ is not empty. The subset $\{c, d\}$ is also a cyclic flat of (M, N). Since $(2111) > (1011)$, then q_{1011} is not a corner of $Q(M, N; x, y, u, v)$.

Suppose there is a subset $X \in \mathcal{E}_{1011}$ such that X is not a cyclic flat of (M, N). Since $\mathrm{cor}_M(X) = \mathrm{cor}_N(X) = \mathrm{nul}_N(X) = 1$ and $\mathrm{nul}_M(X) = 0$, such an X can be only $\{a, b\}$ or $\{e, f\}$. But they are both cyclic flats of N. Hence they are cyclic flats of (M, N). Contradiction. Therefore (M, N) does not contain such a subset X.

References

[1] D. Benard, A. Bouchet and A. Duchamp, *On the Martin and Tutte polynomials*, Technical Report, Département d'Informatique, Université du Maine, Le Mans, France, May 1997.

[2] E. D. Bolker and H. Crapo, How to brace a one-story building, *Environ. Plann. B Plann. Des.*, **4** (2) (1977), 125–152.

[3] T. Brylawski, A combinatorial perspective on the Radon convexity theorem, *Geom. Ded.*, **5** (1976), 459–466.

[4] T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 1–44.

[5] T. H. Brylawski, *The Tutte polynomial, part 1: General theory.* In A. Barlotti (ed). Matroid Theory and Its Applications, (C.I.M.E. 1980), Liguori, Naples, (1982), 125–275.

[6] T. H. Brylawsky and J. Oxley, *The Tutte polynomial and its applications*, Matroid Application, N. L. White (Ed), Cambridge University Press, Cambridge, U.K, (1992), 123–225.

[7] H. H. Crapo, The Tutte polynomial, *Aeq. Math.*, **3** (1969), 211–229.

[8] J. A. Ellis-Monaghan and I. Moffatt, The Las Vergnas polynomial for embedded graphs, *Europ. J. Comb.*, **50** (2015), 97–114.

[9] G. Etienne and M. Las Vergnas, External and internal elements of a matroid basis, *Disc. Math.*, **179** (1998), 111–119.

[10] K. K. Kayibi, S. Pirzada, On the p-bases of Matroid perspectives, *Disc. Math.*, **339** (6) (2016), 1629–1639.

[11] K. K. Kayibi, On the coefficients of the linking polynomial, *Creative Mathematics and Informatics*, **25** (2) (2016), 177–182.

[12] W. Kook, V. Reiner and D. Stanton, A convolution formula for the Tutte polynomial, *J. Comb. Theo. Ser. B*, **76** (2) (1999), 297–300.

[13] J. W. Leo, On coefficients of the Tutte polynomial, *Disc. Math.*, **184** (1998), 121–125.

[14] J. G. Oxley and G. Whittle, A characterization of Tutte invariants of 2-polymatroids, *J. Combin. Theory Ser. B* **59** (1993), 210–244.

[15] J. G. Oxley and G. Whittle, T*utte invariants for 2-polymatroids*, Contemporary Mathematics **147**, Graph Structure Theory, (eds. N. Robertson and P. Seymour), (1995), pp. 9–20.

[16] J. G. Oxley, *Matroid Theory*, Oxford University Press, New-York, (1992).

[17] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Can. J. Math.*, **6** (1947), 80–91.

[18] A. Recski, Maps of matroids with applications, *Disc. Math.*, **303** (2005), 175–185.

[19] M. Las Vergnas, The Tutte polynomial of a morphism of matroids I. set-pointed matroids and matroid perspectives, *Ann. Inst. Fourier, Grenoble*, **49** (3) (1999), 973–1015.

[20] M. Las Vergnas, On The Tutte polynomial of a morphism of matroids, *Ann. Disc. Math*, **8** (1980), 7–20.

[21] M. Las Vergnas, Extensions normale d'un matroide, polynôme de Tutte d'un morphisme, *C. R. Acad. Sci. Paris Sér A*, **280** (1975), 1479–1482.

[22] M. Las Vergnas, Acyclic and totally cyclic orientations of combinatorial geometries, *Disc. Math.*, **20** (1977), 51–61.

[23] M. Las Vergnas, *Eulerian circuits of 4-valent graphs imbedded in surfaces*, in L. Lovasz and V. Sos (eds), Algebraic Methods in Graph Theory, North-Holland, (1981) 451–478.

[24] D. J. A. Welsh and K. K. Kayibi, A linking polynomial of two matroids, *Adv. Appl. Math.*, **32** (2004), 391–419.

[25] T. Zaslavsky, *Facing up to arrangemens: face-count formulas for partitions of spaces by hyperplanes*, Published by Mem. Amer. Math. Soc., Volume 1, Issue 1 **154** (1975).

Received: December 10, 2017