Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) 152–166

DOI: 10.2478/ausm-2018-0013

Totally geodesic property of the unit tangent sphere bundle with g-natural metrics

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Abstract. In this paper, we consider the tangent bundle of a Riemannian manifold (M, g) with g-natural metrics and among all of these metrics, we specify those with respect to which the unit tangent sphere bundle with induced g-natural metric is totally geodesic. Also, we equip the unit tangent sphere bundle T_1M with g-natural contact (paracontact) metric structures, and we show that such structures are totally geodesic K-contact (K-paracontact) submanifolds of TM, if and only if the base manifold (M, g) has positive (negative) constant sectional curvature. Moreover, we establish a condition for g-natural almost contact B-metric structures on T_1M such that these structures be totally geodesic submanifolds of TM.

1 Introduction

One of the classical research fields rising in both mathematics and physics is the notion of totally geodesic submanifold. This geometric motif has still remained a topic of debate in some various branches of physics such as string theory and cosmology as well as in differential geometry. In recent years, many

2010 Mathematics Subject Classification: 53A45, 53C40, 53D15, 58A10
Key words and phrases: natural metrics, sphere bundle, totally geodesic submanifold

mathematicians and physicists have placed this notion in the center of attention and have obtained some important results (see for example [9, 4]).

On the other hand the motif of lifted metric on the tangent bundle of Riemannian manifolds is widely considered as an interesting field by many mathematicians. This notion was first introduced by Sasaki and in recent years his works have generated strong motivation for other mathematicians to study and develop this concept on the tangent bundles of Riemannian manifolds. In [3], the authors introduced the notion of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g). In the framework of g-natural metrics on the tangent bundle and tangent sphere bundle of a Riemannian manifold (M, g), Abbassi, et al. have made significant contributions (see for example [2, 3]).

The other fundamental motif in differential geometry of manifolds, given by Sasaki in [8], is the notion of the almost contact structure. As a counterpart of the almost contact metric structure, the notion of the almost contact B-metric structure has been an interesting research field for many geometrists in differential geometry of manifolds and geometric properties of such structures have been studied frequently (see for example [6]).

The aim of this paper is to specify all of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g), such that with respect to them the unit tangent sphere bundle with induced g-natural metric is totally geodesic. The work is organized in the following way. In Section 2, we begin with a study on the concept of g-natural metrics on the tangent bundle and unit tangent sphere bundle of a Riemannian manifold (M, g) and we provide some necessary information about the mentioned spaces. We proceed in Section 3, to describe and study the totally geodesic property of the unit tangent sphere bundle and then we present the main theorem of the paper. In other words, we determine some conditions for the g-natural metric G on the tangent bundle TM, such that the unit tangent sphere bundle T_1M with the induced g-natural metric $\widetilde{G}$ from G is totally geodesic. In the next two sections, we equip the unit tangent sphere bundle T_1M with g-natural contact metric and paracontact metric structures, and we show that there is a direct correlation between sectional curvature of M and K-contact and K-paracontact totally geodesic property of T_1M. Also, we obtain a condition for a g-natural almost contact B-metric structure on T_1M such that this structure be totally geodesic submanifold of TM.

2 g-natural metric on sphere bundle

We provide some necessary information on g-natural metrics on the tangent bundle and unit tangent sphere bundle in this section.

2.1 g-natural metrics on the tangent bundle

We consider the $(n+1)$-dimensional Riemannian manifold (M, g) and denoting by ∇ its Levi-Civita connection, the tangent space $TM_{(x,u)}$ of the tangent bundle TM at a point (x, u) splits as

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces with respect to ∇. The *horizontal lift* of $X \in M_x$ to $(x, u) \in TM$ is a unique vector $X^h \in \mathcal{H}_{(x,u)}$ such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Moreover, for $X \in M_x$, the *vertical lift* of vector X is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Needless to say, 1-forms df on M are considered as functions on TM (i.e., $(df)(x, u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. As a result of this explanation, one can write each tangent vector $Z \in (TM)_{(x,u)}$ in the form $Z = X^h + Y^v$, where $X, Y \in M_x$, are uniquely determined vectors. Also, the geodesic flow vector field on TM is uniquely determined by $u^h_{(x,u)} = u^i(\frac{\partial}{\partial x^i})^h_{(x,u)}$, for any point $x \in M$ and $u \in TM_x$, with respect to the local coordinates $\{\frac{\partial}{\partial x^i}\}$ on M. In [2], the authors bring up a discussion on g-natural metrics on tangent bundle TM of a Riemannian manifold (M, g), including the following characterization.

Proposition 1 [2] *Let* (M, g) *be a Riemannian manifold and* G *be the g-natural metric on* TM*. Then there are six smooth functions* $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$*, such that for every* $u, X, Y \in M_x$*, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g(X, Y) + (\beta_1 + \beta_3)(r^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2)g(X, Y) + \beta_2(r^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g(X, Y) + \beta_1(r^2)g(X, u)g(Y, u), \end{cases} \tag{1}$$

where $r^2 = g(u, u)$.

As a prime example of Riemannian g-natural metrics on the tangent bundle, we express the Sasaki metric obtained from Proposition 1 with

$$\alpha_1(t) = 1, \qquad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0.$$

2.2 g-natural metric on the unit tangent sphere bundle

Let (M, g) be a Riemannian manifold. The hyperspace

$$T_1M = \{(x, u) \in TM \mid g_x(u, u) = 1\},$$

in TM, is called the unit tangent sphere bundle over the Riemannian manifold (M, g). Denoting by $(T_1M)_{(x,u)}$, the tangent space of T_1M at a point $(x, u) \in T_1M$, we have

$$(T_1M)_{(x,u)} = \{X^h + Y^v | X \in M_x, Y \in \{u\}^{\perp} \subset M_x\}.$$

A g-natural metric on T_1M, is any metric $\widetilde{G}$, induced on T_1M by a g-natural metric G on TM. Using [5], we know that $\widetilde{G}$ is completely determined by the values of four real constants, namely

$$a = \alpha_1(1), \qquad b = \alpha_2(1), \qquad c = \alpha_3(1), \qquad d = \beta(1) = (\beta_1 + \beta_3)(1).$$

Let (M, g) be a $(2n + 1)$-dimensional Riemannian manifold. Considering an orthogonal basis $\{X_0 = u, X_1, \dots, X_n\}$ on $x \in M$, we define $X_0^h = u^h$. The metric $\widetilde{G}$ on T_1M is completely determined by

$$\begin{cases} \widetilde{G}_{(x,u)}(X_i^h, X_j^h) = (a + c)g_x(X_i, X_j) + dg_x(X_i, u)g_x(X_j, u), \\ \widetilde{G}_{(x,u)}(X_i^h, Y_j^v) = bg_x(X_i, Y_j), \\ \widetilde{G}_{(x,u)}(Y_i^v, Y_j^v) = ag_x(Y_i, Y_j), \end{cases} \tag{2}$$

at any point $(x, u) \in T_1M$, for all $X_i, Y_j \in M_x$, with Y_j orthogonal to u [5].

Taking into account $\phi = a(a+c+d) - b^2$, using the Schmidt's orthogonalization process and some standard calculations, it can be shown that whenever $\phi \neq 0$, the following vector field on TM is normal to T_1M and is unitary at any point of T_1M for all $(x, u) \in TM$

$$N^G_{(x,u)} = \frac{1}{\sqrt{|(a + c + d)\phi|}}[-bu^h + (a + c + d)u^v].$$

Moreover, for a vector $X \in M_x$ at $(x, u) \in T_1M$, the *tangential lift* X^{t_G} with respect to G is defined as the tangential projection of the vertical lift of X to (x, u) with respect to N^G, in other words

$$X^{t_G} = X^v - \frac{\phi}{|\phi|}G_{(x,u)}(X^v, N^G_{(x,u)})N^G_{(x,u)} = X^v - \sqrt{\frac{|\phi|}{|a + c + d|}}g_x(X, u)N^G_{(x,u)}. \tag{3}$$

Also, if $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$. Assuming that $b = 0$, the tangential lift X^{t_G} and the classical tangential lift X^t defined for the case of the Sasaki metric coincide. In the most general case, we have

$$X^{t_G} = X^t + \frac{b}{a+c+d} g(X,u)u^h.$$

Remark 1 [5] *The tangential lift* u^{t_G} *to* $(x,u) \in T_1M$ *of the vector* u *is given by* $u^{t_G} = \frac{b}{a+c+d}u^h$*, that is,* u^{t_G} *is a horizontal vector. Therefore, the tangent space* $(T_1M)_{(x,u)}$ *of* T_1M *at* (x,u) *is spanned by vectors of the form* X^h *and* Y^{t_G} *as follows,*

$$(T_1M)_{(x,u)} = \{X^h + Y^{t_G} | X \in M_x, Y \in \{u\}^\perp \subset M_x\}, \tag{4}$$

hence, the operation of tangential lift from M_x *to a point* $(x,u) \in T_1M$ *will always be applied* ***only*** *to those vectors of* M_x *which are orthogonal to* u*.*

Taking into account Remark 1, the Riemannian metric $\widetilde{G}$ on T_1M, induced from G, is completely determined by the following identities.

$$\begin{cases} \widetilde{G}(X_1^h, X_2^h) = (a+c)g_x(X_1,X_2) + dg_x(X_1,u)g_x(X_2,u), \\ \widetilde{G}(X_1^h, Y_1^{t_G}) = bg_x(X_1,Y_1), \\ \widetilde{G}(Y_1^{t_G}, Y_2^{t_G}) = ag_x(Y_1,Y_2), \end{cases}$$

where $X_i, Y_i \in M_x$, for $i = 1,2$ with Y_i orthogonal to u. It should be noted that by the above equations, horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$, if and only if $b = 0$. Further details about g-natural metrics on the tangent bundle can be found in [5]. Here, we present the following propositions.

Proposition 2 [1] *Let* (M,g) *be a Riemannian manifold,* ∇ *its Levi-Civita connection and* R *its curvature tensor. Let* G *be the* g*-natural metric on* TM *given by* (1) *with* $a > 0$*,* $\alpha = a(a+c) - b^2 > 0$*, and* $\phi(t) = a(a+c+t\beta(t)) - b^2 > 0$*, for all* $t \in [0,\infty)$*. Then the Levi-Civita connection* $\overline{\nabla}$ *of* (TM,G) *is characterized by*

1.

$$\begin{aligned}(\overline{\nabla}_{X^h}Y^h)_{(x,u)} = &\Big\{(\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] \\ &+ \frac{b\beta(1)}{2\alpha}[g(X_x,u)Y_x + g(Y_x,u)X_x] + \frac{b}{\alpha\phi}[a^2\beta(1)g(R(X_x,u)Y_x,u) \\ &+ (\alpha\beta'(1) - a\beta^2(1))g(X_x,u)g(Y_x,u)]u\Big\}^h + \Big\{\frac{b^2}{\alpha}R(X_x,u)Y_x\end{aligned}$$

$$- \frac{a(a+c)}{2\alpha} R(X_x, Y_x)u - \frac{(a+c)\beta(1)}{2\alpha}[g(Y_x, u)X_x + g(X_x, u)Y_x]$$
$$+ \frac{1}{\alpha\phi}[-ab^2\beta(1)g(R(X_x, u)Y_x, u) + (-\alpha(a+c+\beta(1))\beta'(1)$$
$$+ b^2\beta^2(1))g(Y_x, u)g(X_x, u)]u\Big\}^v,$$

2.

$$(\overline{\nabla}_{X^h} Y^v)_{(x,u)} = \Big\{ - \frac{a^2}{2\alpha} R(Y_x, u)X_x + \frac{a\beta(1)}{2\alpha} g(X_x, u)Y_x$$
$$+ \frac{a}{2\alpha\phi}[a^2\beta(1)g(R(X_x, u)Y_x, u) + \alpha\beta(1)g(X_x, Y_x) + (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^h + \Big\{(\nabla_X Y)_x + \frac{ab}{2\alpha} R(Y_x, u)X_x$$
$$- \frac{b\beta(1)}{2\alpha} g(X_x, u)Y_x + \frac{b}{2\alpha\phi}[-\alpha\beta(1)g(X_x, Y_x) - a^2\beta(1)g(R(X_x, u)Y_x, u)$$
$$- (2\alpha\beta'(1) - a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^v,$$

3.

$$(\overline{\nabla}_{X^v} Y^h)_{(x,u)} = \Big\{ - \frac{a^2}{2\alpha} R(X_x, u)Y_x + \frac{a\beta(1)}{2\alpha} g(Y_x, u)X_x$$
$$+ \frac{a}{2\alpha\phi}[a^2\beta(1)g(R(X_x, u)Y_x, u) + \alpha\beta(1)g(X_x, Y_x) + (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^h + \Big\{ \frac{ab}{2\alpha} R(X_x, u)Y_x - \frac{b\beta(1)}{2\alpha} g(Y_x, u)X_x$$
$$+ \frac{b}{2\alpha\phi}[-\alpha\beta(1)g(X_x, Y_x) - a^2\beta(1)g(R(X_x, u)Y_x, u) - (2\alpha\beta'(1)$$
$$- a\beta^2(1))g(X_x, u)g(Y_x, u)]u\Big\}^v,$$

4.

$$(\overline{\nabla}_{X^v} Y^v)_{(x,u)} = 0,$$

for all vector fields X, Y *on* M *and* $(x, u) \in TM$, *where* $g_x(u, u) = 1$.

Proposition 3 [1] *At* $(x,u) \in T_1M$, *the Levi-Civita connection* $\widetilde{\nabla}$ *on* T_1M *is given by*

1.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^h}Y^h)_{(x,u)} &= \Big\{(\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] + \frac{bd}{2\alpha}[g(X_x,u)Y_x \\
&+ g(Y_x,u)X_x] + \frac{b}{\alpha(a+c+d)}[(ad+b^2)g(R(X_x,u)Y_x,u) \\
&- d(a+c+d)g(X_x,u)g(Y_x,u)]u\Big\}^h + \Big\{\frac{b^2}{\alpha}R(X_x,u)Y_x \\
&- \frac{a(a+c)}{2\alpha}R(X_x,Y_x)u - \frac{(a+c)d}{2\alpha}[g(Y_x,u)X_x + g(X_x,u)Y_x] \\
&+ \frac{1}{\alpha}[-b^2 g(R(X_x,u)Y_x,u) + d(a+c)g(Y_x,u)g(X_x,u)]u\Big\}^{t_G},
\end{aligned}
$$

2.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^h}Y^{t_G})_{(x,u)} &= \Big\{-\frac{a^2}{2\alpha}R(Y_x,u)X_x + \frac{ad}{2\alpha}g(X_x,u)Y_x \\
&+ \frac{1}{2\alpha(a+c+d)}[a(ad+b^2)g(R(X_x,u)Y_x,u) + \alpha d g(X_x,Y_x)]u\Big\}^h \\
&+ \Big\{(\nabla_X Y)_x + \frac{ab}{2\alpha}R(Y_x,u)X_x - \frac{bd}{2\alpha}g(X_x,u)Y_x - \frac{ab}{2\alpha}g(R(X_x,u)Y_x,u)u\Big\}^{t_G},
\end{aligned}
$$

3.

$$
\begin{aligned}
(\widetilde{\nabla}_{X^{t_G}}Y^h)_{(x,u)} &= \Big\{-\frac{a^2}{2\alpha}R(X_x,u)Y_x + \frac{ad}{2\alpha}g(Y_x,u)X_x \\
&+ \frac{1}{2\alpha(a+c+d)}[a(ad+b^2)g(R(X_x,u)Y_x,u) + \alpha d g(X_x,Y_x)]u\Big\}^h \\
&+ \Big\{\frac{ab}{2\alpha}R(X_x,u)Y_x - \frac{bd}{2\alpha}g(Y_x,u)X_x - \frac{ab}{2\alpha}g(R(X_x,u)Y_x,u)u\Big\}^{t_G},
\end{aligned}
$$

4.

$$
(\widetilde{\nabla}_{X^{t_G}}Y^{t_G})_{(x,u)} = 0,
$$

for all $(x,u) \in T_1M$ *and* X, Y *on* M *satisfying* (4).

3 Totally geodesic property of the sphere bundle

We consider a submanifold M of a (pseudo) Riemannian manifold $(\overline{M}, \overline{g})$. The (pseudo) Riemannian metric $\overline{g}$ induces a (pseudo) Riemannian metric g on the submanifold M. Then (M, g) is also called a (pseudo) Riemannian submanifold of $(\overline{M}, \overline{g})$. A submanifold M of a (pseudo) Riemannian manifold $(\overline{M}, \overline{g})$ is called totally geodesic if any geodesic on the submanifold M with its induced (pseudo) Riemannian metric g is also a geodesic on $(\overline{M}, \overline{g})$. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and (M, g) respectively. The shape tensor or second fundamental form tensor II is a symmetric tensor field which can be defined as follows

$$\mathrm{II}(X, Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

for all vector fields X, Y on M. The (pseudo) Riemannian submanifold M is totally geodesic provided its shape tensor vanishes, i.e. $\mathrm{II} = 0$ [7]. Here, we provide the main theorem of this paper.

Theorem 1 *The unit tangent sphere bundle* $(T_1M, \widetilde{G})$ *is a totally geodesic submanifold of* (TM, G) *if and only if* G *is a* g*-natural metric on* TM *with* $b = 0$ *and* $\beta'(1) = 0$.

Proof. First, notice that (4) yields that the tangent space of T_1M at (x, u) can be written as

$$(T_1M)_{(x,u)} = \mathrm{span}(u^h) \oplus \{X^h | X \perp u\} \oplus \{Y^v | Y \perp u\}. \tag{5}$$

Now, we compute the coefficients of the fundamental tensor II as follows. Taking into account Proposition 2 and Proposition 3 and (3) we get

$$\begin{aligned}
\mathrm{II}_{(x,u)}(X^h, Y^h) &= (\overline{\nabla}_{X^h} Y^h)_{(x,u)} - (\widetilde{\nabla}_{X^h} Y^h)_{(x,u)} \\
&= \left[\frac{ba^2\beta(1)}{\alpha\phi} - \frac{b(ad + b^2)}{\alpha(a + c + d)}\right] g(R(X_x, u)Y_x, u)u^h \\
&\quad + \left[-\frac{ab^2\beta(1)}{\alpha\phi} + \frac{b^2}{\alpha}\right] g(R(X_x, u)Y_x, u)u^v,
\end{aligned}$$

$$\begin{aligned}
\mathrm{II}_{(x,u)}(X^v, Y^h) = &\left[\frac{a^3\beta(1)}{2\alpha\phi} - \frac{a(ad + b^2)}{2\alpha(a + c + d)}\right] g(R(X_x, u)Y_x, u)u^h \\
&- \frac{b\alpha\beta(1)}{2\alpha\phi} g(X_x, Y_x)u^v + \left[-\frac{ba^2\beta(1)}{2\alpha\phi} + \frac{ab}{2\alpha}\right] g(R(X_x, u)Y_x, u)u^v,
\end{aligned}$$

$$\begin{aligned}\mathrm{II}_{(x,u)}(u^h,u^h) &= \left[\frac{b(\alpha\beta'(1)-a\beta^2(1))}{\alpha\phi}+\frac{bd}{\alpha}\right]u^h\\ &+\left[\frac{(-\alpha(a+c+\beta(1))\beta'(1)+b^2\beta^2(1))}{\alpha\phi}\right]u^v,\end{aligned}$$

$$\mathrm{II}_{(x,u)}(X^v,Y^v)=\mathrm{II}_{(x,u)}(u^h,Y^h)=\mathrm{II}_{(x,u)}(X^v,u^h)=0,$$

for all X, Y satisfying (5) where $\alpha = a(a+c)-b^2$, $\phi = a(a+c+d)-b^2$ and $\beta(1) = d$. Therefore, the second fundamental form II vanishes if and only if the following system of equations

$$\begin{cases} \dfrac{ba^2\beta(1)}{\alpha\phi}=\dfrac{b(ad+b^2)}{\alpha(a+c+d)}, \quad \dfrac{ab^2\beta(1)}{\alpha\phi}=\dfrac{b^2}{\alpha}, \quad \dfrac{a^3\beta(1)}{2\alpha\phi}=\dfrac{a(ad+b^2)}{2\alpha(a+c+d)},\\[2ex] \dfrac{b\alpha\beta(1)}{2\alpha\phi}=0, \qquad \dfrac{ba^2\beta(1)}{2\alpha\phi}=\dfrac{ab}{2\alpha}, \qquad \dfrac{b(\alpha\beta'(1)-a\beta^2(1))}{\alpha\phi}=-\dfrac{bd}{\alpha},\\[2ex] \dfrac{-\alpha(a+c+\beta(1))\beta'(1)+b^2\beta^2(1)}{\alpha\phi}=0, \end{cases} \tag{6}$$

Satisfies. Standard calculations show that this system of equations satisfies if and only if $b = 0$ and $\beta'(1) = 0$. Hence, $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if G is a g-natural metric on TM with $b = 0$ and $\beta'(1) = 0$. □

As immediate consequences of this theorem, we have the following corollaries.

Corollary 1 *The Sasaki metric obtained from (1) for*

$$\alpha_1(t)=1, \quad \alpha_2(t)=\alpha_3(t)=\beta_1(t)=\beta_2(t)=\beta_3(t)=0,$$

satisfies the conditions $b = 0$ *and* $\beta'(1) = (\beta_1+\beta_3)'(1) = 0$. *Therefore, the unit tangent sphere bundle* T_1M *is a totally geodesic submanifold of* (TM, g_S).

Corollary 2 *The Cheeger-Gromoll metric* g_{CG}, *as a classical example of* g-*natural metrics on the tangent bundle, is obtained for*

$$\alpha_1(t)=\beta_1(t)=-\beta_3(t)=\frac{1}{1+t}, \qquad \alpha_2(t)=\beta_2(t)=0, \qquad \alpha_3(t)=\frac{t}{1+t}.$$

So we have $b = \alpha_2(1) = 0$ *and* $\beta'(1) = (\beta_1+\beta_3)'(1) = 0$. *Hence,* T_1M *with induced* g-*natural metric is a totally geodesic submanifold of* (TM, g_{CG}).

Corollary 3 *Metrics of Cheeger-Gromoll type* $h_{m,r}$ *are obtained from* (1) *when*

$$\alpha_1(t) = \frac{1}{(1+t)^m}, \qquad \alpha_3(t) = 1 - \alpha_1(t),$$
$$\alpha_2(t) = \beta_2(t) = 0, \qquad \beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m},$$

where $m \in \mathbb{R}$ *and* $r \geqslant 0$*. Obviously, these metrics satisfy* $b = 0$ *and* $\beta'(1) = 0$*. Hence, the unit tangent sphere bundle* T_1M *with induced* g*-natural metric is a totally geodesic submanifold of* $(TM, h_{m,r})$*.*

Corollary 4 *Kaluza-Klein metrics, are obtained from (1) for*

$$\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0.$$

Thus, Kaluza-Klein metrics satisfy $b = 0$ *and* $\beta'(1) = (\beta_1 + \beta_3)'(1) = 0$ *and therefore,* T_1M *with induced* g*-natural metric is a totally geodesic submanifold of* TM *with Kaluza-Klein metrics.*

4 g-natural contact and paracontact metric structures on tangent sphere bundle

In this section, we equip the unit tangent sphere bundle T_1M with g-natural contact metric and paracontact metric structures, and we show that there is a direct correlation between sectional curvature of M and K-contact and K-paracontact totally geodesic property of T_1M.

A $(2n+1)$-dimensional manifold M is called a contact manifold if it admits a global 1 form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M and a unique vector field ξ such that $\eta(\xi) = 1$ and $d\eta(\xi, .) = 0$. In addition, a Riemannian metric g is said to be an associated metric if there exists a tensor φ, of type $(1,1)$, such that

$$\eta = g(\xi, .), \qquad d\eta = g(\xi, .), \qquad \varphi^2 = -I + \eta \otimes \xi.$$

Moreover, a Riemannian metric g is said to be compatible with the contact structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M.

Now, (η, g, ξ, φ) is called a contact metric structure and $(M, \eta, g, \xi, \varphi)$ a contact metric manifold.

The following proposition determines the g-natural contact metric structure on unit tangent sphere bundle.

Proposition 4 [1] *Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle. Let $\widetilde{G}$ be a g-natural metric on T_1M given by (2). The set $(\widetilde{G}, \eta, \varphi, \xi)$ described by (7)-(10) is a family of contact metric structures over T_1M.*

$$\xi = u^h, \tag{7}$$

$$\eta(X^h) = g(X, u), \qquad \eta(X^{t_G}) = bg(X, u), \tag{8}$$

$$\begin{cases} \varphi(X^h) = \dfrac{1}{2\alpha}[-bX^h + (a+c)X^{t_G} + \dfrac{bd}{a+c+d}g(X, u)u^h], \\ \varphi(X^{t_G}) = \dfrac{1}{2\alpha}[-aX^h + bX^{t_G} + \dfrac{\phi}{a+c+d}g(X, u)u^h], \end{cases} \tag{9}$$

$$4\alpha = a + c + d = 1. \tag{10}$$

A K-contact manifold is a contact metric manifold $(M, g, \eta, \varphi, \xi)$ such that the characteristic vector field ξ is a Killing vector field with respect to g. We refer to [1] for more information on K-contact manifolds. Now, we provide the following statement.

Theorem 2 *Let $\widetilde{G}$ be a Riemannian g-natural metric on T_1M and $(T_1M, \widetilde{G})$ be a totally geodesic submanifold of (TM, G). The contact metric manifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-contact if and only if the base manifold (M, g) has positive constant sectional curvature $\frac{a+c}{a}$.*

Proof. $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-contact manifold if and only if the characteristic vector field ξ is a Killing vector field with respect to $\widetilde{G}$ and according to Theorem 2 in [1], ξ is Killing vector field if and only if $b = 0$ and (M, g) has constant sectional curvature $\frac{a+c}{a} > 0$. Using Theorem 1, the unit tangent sphere bundle $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if $b = 0$ and $\beta'(1) = 0$. Consequently, totally geodesic submanifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$

of (TM, G) is K-contact if and only if the base manifold (M, g) has positive constant sectional curvature $\frac{a+c}{a} > 0$. $\square$

Analogous to the contact cases, a $(2n+1)$-dimensional manifold (M, g) is called a paracontact manifold if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\eta = g(\xi, .), \qquad d\eta = g(\xi, .), \qquad \varphi^2 = I - \eta \otimes \xi.$$

Also, a pseudo-Riemannian metric g is said to be compatible with the paracontact structure if

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all X, Y vector fields on M.

Now we report the following statement from [5].

Proposition 5 [5] *Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle. Let $\widetilde{G}$ be a g-natural metric on T_1M given by (2). The set $(\widetilde{G}, \eta, \varphi, \xi)$ described by (11)-(14) is a family of paracontact metric structures over T_1M.*

$$\xi = u^h, \tag{11}$$

$$\eta(X^h) = g(X, u), \qquad \eta(X^{t_G}) = bg(X, u), \tag{12}$$

$$\begin{cases} \varphi(X^h) = \dfrac{1}{2\alpha}[-bX^h + (a+c)X^{t_G} + \dfrac{bd}{a+c+d} g(X, u)u^h], \\[2ex] \varphi(X^{t_G}) = \dfrac{1}{2\alpha}[-aX^h + bX^{t_G} + \dfrac{\phi}{a+c+d} g(X, u)u^h], \end{cases} \tag{13}$$

$$-4\alpha = a + c + d = 1. \tag{14}$$

A paracontact metric structure (φ, g, η, ξ) is said to be K-paracontact if ξ is a Killing vector filed.

Remark 2 *According to [5], in order to construct a paracontact metric structure with an associated g-natural metric on the unit tangent sphere bundle T_1M, it requires to $a + c + d > 0$ and $\alpha < 0$. It deduces from $\alpha < 0$ that the induced g-natural metric $\widetilde{G}$ on T_1M is a non-degenerate pseudo-Riemannian metric. It can be shown that for $\alpha < 0$ and $\phi > 0$, Proposition 2 and Proposition 3 and consequently Theorem 1 remain true.*

Here, we have the following.

Theorem 3 *Let $\widetilde{G}$ be a pseudo-Riemannian g-natural metric on T_1M and $(T_1M, \widetilde{G})$ be a totally geodesic submanifold of (TM, G). The paracontact metric manifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-paracontact manifold if and only if the base manifold (M, g) has negative constant sectional curvature $\frac{a+c}{a} < 0$.*

Proof. The paracontact metric manifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ is K-paracontact if and only if ξ is a Killing vector field with respect to pseudo-Riemannian metric $\widetilde{G}$. It concludes from Theorem 3 of [5] that ξ is Killing vector field if and only if $b = 0$ and M has negative sectional curvature $\frac{a+c}{a}$. Moreover, by Theorem 1, $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if $b = 0$ and $\beta'(1) = 0$. Consequently, totally geodesic submanifold $(T_1M, \widetilde{G}, \eta, \varphi, \xi)$ of (TM, G) is K-paracontact if and only if the base manifold (M, g) has negative constant sectional curvature $\frac{a+c}{a} < 0$. □

5 g-natural almost contact b-metric structures on unit tangent sphere bundle

In this section, we establish a condition for a g-natural almost contact B-metric structure on T_1M such that this structure be a totally geodesic submanifold of TM.

A $(2n+1)$-dimensional manifold M has an almost contact B-metric structure if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0,$$
$$\eta \circ \varphi = 0, \quad g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).$$

Now we consider the unit tangent sphere bundle of a Riemannian manifold (M, g) with g-natural metric, and we equip it with an almost contact B-metric structure denoted briefly by $(T_1M, \varphi, \xi, \eta, \tilde{G})$, and also a basis $\{X^h, X^{t_G}, \xi\}$ such that $X^h, X^{t_G} \perp \xi$, with respect to $\tilde{G}$, where $\xi = u^h$. An almost contact structure on T_1M is defined by

$$\eta(X^h) = \eta(X^{t_G}) = 0, \;\; \eta(\xi) = 1, \;\; \varphi(X^h) = X^{t_G}, \quad \varphi(X^{t_G}) = -X^h, \quad \varphi(\xi) = 0.$$

In order to construct an almost contact B-metric structure with an associated g-natural metric on the unit tangent sphere bundle T_1M, it requires to $a +$

$c + d > 0$ and $\alpha < 0$. It deduces from $\alpha < 0$ that the induced g-natural metric $\widetilde{G}$ on T_1M is a non-degenerate pseudo-Riemannian metric. It can be shown that for $\alpha < 0$ and $\phi > 0$, Proposition 2 and Proposition 3 and consequently Theorem 1 still remain true. Also, pseudo-Riemannian metric $\widetilde{G}$ must be of signature $(n, n+1)$ or $(n+1, 1)$, therefore, it requires to $b = 0$. Hence, the adapted g-natural metric on the unit tangent sphere bundle T_1M with almost contact B-metric structure is of following form

$$\begin{cases} \widetilde{G}_{(x,u)}(X_i^h, X_j^h) = (a+c)g_x(X_i, X_j) + dg_x(X_i, u)g_x(X_j, u), \\ \widetilde{G}_{(x,u)}(X_i^h, Y_j^{t_G}) = 0, \\ \widetilde{G}_{(x,u)}(Y_i^{t_G}, Y_j^{t_G}) = ag_x(Y_i, Y_j), \end{cases} \tag{15}$$

for all vector fields X, Y on M with $Y \perp u$. Also, we have following relations

$$\tilde{G}(\varphi X_i^h, \varphi X_j^h) = -\tilde{G}(X_i^h, X_j^h), \qquad \tilde{G}(\varphi X_i^{t_G}, \varphi X_j^{t_G}) = -\tilde{G}(X_i^{t_G}, X_j^{t_G}),$$

which give that $\tilde{G}$ is a B-metric. As a result of these relations we have $a + c = -a$. Notice that using $b = 0$ and $a+c = -a$, we conclude that $\tilde{G}$ is of signature $(n, n+1)$ or $(n+1, 1)$. Now we provide the following statement.

Theorem 4 *The unit tangent sphere bundle* $(T_1M, \widetilde{G}, \varphi, \eta, \xi)$ *equipped with a* g*-natural almost contact B-metric structure is a totally geodesic submanifold of* (TM, G) *if and only if* G *is a* g*-natural metric on* TM *with* $\beta'(1) = 0$.

Proof. Taking into the account Theorem 1, the unit tangent sphere bundle $(T_1M, \widetilde{G})$ is a totally geodesic submanifold of (TM, G) if and only if G is a g-natural metric on TM with $b = 0$ and $\beta'(1) = 0$. Also, using (15) for a g-natural almost contact B-metric $\widetilde{G}$ we have $b = 0$. Hence, $(T_1M, \widetilde{G}, \varphi, \eta, \xi)$ is a totally geodesic submanifold of (TM, G) if and only if G is a g-natural metric on TM with $\beta'(1) = 0$. □

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Received: April 7, 2017