Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) 134–151

DOI: 10.2478/ausm-2018-0012

An efficient numerical method for solving nonlinear Thomas-Fermi equation

Kourosh Parand
Department of Computer Sciences,
Department of Cognitive Modelling,
Institute for Cognitive and Brain Sciences,
Shahid Beheshti University, Iran
email: k_parand@sbu.ac.ir

Kobra Rabiei
Department of Computer Sciences,
Shahid Beheshti University, Iran
email: K.rabiee@alzahra.ac.ir

Mehdi Delkhosh
Department of Computer Sciences,
Shahid Beheshti University, Iran
email: mehdidelkhosh@yahoo.com

Abstract. In this paper, the nonlinear Thomas-Fermi equation for neutral atoms by using the fractional order of rational Chebyshev functions of the second kind (FRC2), $FU_n^{\alpha}(t, L)$, on an unbounded domain is solved, where L is an arbitrary parameter. Boyd (*Chebyshev and Fourier Spectral Methods, 2ed, 2000*) has presented a method for calculating the optimal approximate amount of L and we have used the same method for calculating the amount of L. With the aid of quasilinearization and FRC2 collocation methods, the equation is converted to a sequence of linear algebraic equations. An excellent approximation solution of $y(t)$, $y'(t)$, and $y'(0)$ is obtained.

1 Introduction

In this section, the introduction of numerical methods used for solving equations in unbounded domains is expressed. Furthermore, the mathematical model of Thomas-Fermi equation is introduced.

2010 Mathematics Subject Classification: 34B16, 34B40, 65N35
Key words and phrases: Thomas-Fermi equation, fractional order of rational Chebyshev functions, quasilinearization method, collocation method, unbounded domain

1.1 The problems on unbounded domains

There are several numerical methods for solving differential equations on unbounded domains, such as:

1. *Finite difference method (FDM)*: One of the oldest and the simplest methods for solving differential equations is using the FDM approximations for derivatives. The FDMs are in a class of the discretization methods [2].

2. *Finite element method (FEM)*: One of the important methods used for solving the boundary value problems for partial differential equations is the finite element method [2].

3. *Meshfree methods*: Meshfree methods are those that do not require a connection between nodes of the simulation domain, i.e. a mesh, but are rather based on the interaction of each node with all its neighbors [3]. The use of Radial Basis Functions (RBFs) in meshless methods is very common in solving differential equations [4, 5]. This approach has recently received a great deal of attention from researchers [6, 7].

4. *Spectral methods*: Several approaches in Spectral methods have been proposed for solving the problems on unbounded domains:

   (a) Using functions such as Hermite, Sinc, Laguerre, and Bessel functions that are defined on the unbounded domains. This approach investigated by Parand et al. [8, 9], Funaro & Kavian [10], and Guo & Shen [11].

   (b) Mapping an unbounded equation to a bounded equation. Authors of [12, 13] have applied this approach in their works.

   (c) Replacing unbounded domains with $[-B, B]$ or $[0, B]$ by choosing B sufficiently large. This method is named domain truncation [14, 15].

   (d) Mapping the bounded basic functions to the unbounded basic functions. In this approach, the basic functions on a bounded domain convert to the functions on an unbounded domain. For example, Boyd [16] introduced a new spectral basis, called rational Chebyshev functions, on the unbounded domain by mapping on the Chebyshev polynomials, and also in Refs. [17, 18, 19]. There are three important mappings for this approach:

(A) Algebraic mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = \frac{bx+aL}{x+L}$ convert to functions on an unbounded domain $x \in [0, \infty)$, where L is an arbitrary parameter [21].

(B) Exponential mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = b + (a - b)e^{-\frac{x}{L}}$ convert to functions on an unbounded domain $x \in [0, \infty)$ [20].

(C) Logarithmic mapping: basic functions on a bounded domain $t \in [a, b]$ by using the transformation of $t = a+(b-a)\tanh(2\frac{x}{L})$ convert to functions on an unbounded domain $x \in [0, \infty)$.

In this paper, a Spectral method is introduced to solve unbounded problems by using the fractional order of rational Chebyshev orthogonal functions of the second kind.

1.2 The Thomas-Fermi equation

The Thomas-Fermi equation is an important nonlinear singular differential equation which is defined on semi-infinite domain [22, 23]:

$$\begin{aligned} \frac{d^2y(t)}{dt^2} - \frac{1}{\sqrt{t}}y^{\frac{3}{2}}(t) = 0, \qquad & t \in [0, \infty), \\ y(0) = 1, \qquad & y(\infty) = 0. \end{aligned} \tag{1}$$

The nonlinear Thomas-Fermi equation appears in the problem of determining the effective nuclear charge in heavy atoms, therefore, many great scholars were considered it, such as Fermi [24], Feynman (physics) [25], and Slater (chemistry) [26].

The initial slope $y'(0)$ is difficult for computing by any means and plays an important role in determining many properties of the physical of Thomas-Fermi atom [27]. It determines the energy of a neutral atom in Thomas-Fermi approximation:

$$E = \frac{6}{7}\left(\frac{4\pi}{3}\right)^{\frac{2}{3}} Z^{\frac{7}{3}}y'(0), \tag{2}$$

where Z is the nuclear charge.

For these reasons, the problem has been studied by many researchers and has been solved by different techniques where a number of them are listed in Table 1, in this table, the calculated value of $y'(0)$ by researchers is shown.

The rest of the paper is constructed as follows: the FRC2s and their properties are expressed in section 2. The methodology is explained in section 3. In section 4, results and discussions of the method are shown. Finally, a conclusion is provided.

2 Fractional order of rational Chebyshev functions of the second kind

In this section, the definition of the fractional order of rational Chebyshev functions of the second kind (FRC2s) and some theorems for them is provided.

2.1 The FRC2s definition

Using some transformations, some researchers have generalized the Chebyshev polynomials to semi-infinite or infinite domains, for example the rational Chebyshev functions on the semi-infinite domain [28], the rational Chebyshev functions on an infinite domain [1], and the generalized fractional order of the Chebyshev functions (GFCF) on finite interval $[0, \eta]$ [29, 30, 31] are introduced by using transformations $x = \frac{t-L}{t+L}$, $x = \frac{t}{\sqrt{t^2+L}}$, and $x = 1 - 2(\frac{t}{\eta})^\alpha$, respectively.

In the proposed work, by new transformation $x = \frac{t^\alpha - L}{t^\alpha + L}$, $L > 0$ on the Chebyshev polynomials of the second kind, the fractional order of rational Chebyshev functions of the second kind on domain $[0, \infty)$ is introduced, which is denoted by $FU_n^\alpha(t, L) = U_n(\frac{t^\alpha - L}{t^\alpha + L})$ where L is a numerical parameter.

The $FU_n^\alpha(t, L)$ can be calculated by using the following relation:

$$FU_0^\alpha(t, L) = 1, \qquad FU_1^\alpha(t, L) = 2\frac{t^\alpha - L}{t^\alpha + L},$$
$$FU_{n+1}^\alpha(t, L) = 2\frac{t^\alpha - L}{t^\alpha + L} FU_n^\alpha(t, L) - FU_{n-1}^\alpha(t, L), \quad n = 1, 2, \cdots, \tag{3}$$

and we can also calculate:

$$FU_n^\alpha(t, L) = \sum_{k=0}^{n} \beta_{n,k}(t^\alpha + L)^{-k}, \tag{4}$$

where

$$\beta_{n,k} = (-4L)^k \frac{(n + k + 1)!}{(n - k)!(2k + 1)!} \qquad \text{and} \qquad \beta_{0,k} = 1.$$

2.2 Approximation of functions

Any function of continuous and differentiable $y(t)$, $t \in [0, \infty)$, can be expanded as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n \ \mathrm{FU}_n^{\alpha}(t, L),$$

where the coefficients a_n can be obtained by:

$$a_n = \frac{8\alpha L^{\frac{3}{2}}}{\pi} \int_0^{\infty} \mathrm{FU}_n^{\alpha}(t, L) \ y(t) \ w(t) dt, \qquad n = 0, 1, 2, \cdots .$$

In the numerical methods, we have to use first $(m+1)$-terms FRC2s and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m} a_n \ \mathrm{FU}_n^{\alpha}(t, L). \tag{5}$$

Theorem 1 *The FRC2,* $\mathrm{FU}_n^{\alpha}(t, L)$*, has precisely* n *real simple zeros on the interval* $(0, \infty)$ *in the form*

$$t_k = \left(L \frac{1 + \cos\left(\frac{k\pi}{n+1}\right)}{1 - \cos\left(\frac{k\pi}{n+1}\right)} \right)^{\frac{1}{\alpha}}, \qquad k = 1, 2, ..., n.$$

Proof. Chebyshev polynomial of the second kind $U_n(x)$ has n real simple zeros [1]:

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \qquad k = 1, 2, ..., n.$$

Therefore $U_n(x)$ can be written as

$$U_n(x) = (x - x_1)(x - x_2)...(x - x_n).$$

Using transformation $x = \frac{t^{\alpha} - L}{t^{\alpha} + L}$ yields to

$$\mathrm{FU}_n^{\alpha}(t, L) = \left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L}\right) - x_1\right)\left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L}\right) - x_2\right)...\left(\left(\frac{t^{\alpha} - L}{t^{\alpha} + L}\right) - x_n\right),$$

so, the real zeros of $\mathrm{FU}_n^{\alpha}(t, L)$ are $t_k = \left(L \frac{1+x_k}{1-x_k}\right)^{\frac{1}{\alpha}}$. □

Theorem 2 *The FRC2s are orthogonal on domain* $[0,\infty)$ *for all* $L>0$ *with positive weight function* $w(t)=\frac{t^{\frac{3}{2}\alpha-1}}{(t^\alpha+L)^3}$ *as follows:*

$$\int_0^\infty FU_n^\alpha(t,L)\ FU_m^\alpha(t,L)\ w(t)\ dt=\frac{\pi}{8\alpha L^{\frac{3}{2}}}\delta_{mn}, \tag{6}$$

where δ_{mn} *is the Kronecker delta.*

Proof. The Chebyshev polynomials of the second kind $U_n(x)$ are orthogonal as [1]:

$$\int_{-1}^1 U_n(x)\ U_m(x)\ \sqrt{1-x^2}\ dx=\frac{\pi}{2}\delta_{mn}.$$

Now, by using transformation $x=\frac{t^\alpha-L}{t^\alpha+L}, L>0$ on the integral, the theorem can be proved. □

3 The methodology

The quasi-linearization method (QLM) based on the Newton-Raphson method has introduced by Bellman and Kalaba [32, 33]. Some researchers have used this method in their works [34, 35, 36, 37].

Occasionally the linear ordinary differential equation that gets from the QLM at each iteration does not solve analytically. Hence we can use the Spectral methods to approximate the solution.

The QLM for Thomas-Fermi equation (1) is as follows:

$$\frac{d^2y_{n+1}}{dt^2}-\frac{3}{2\sqrt{t}}(y_n(t))^{1/2}y_{n+1}(t)=-\frac{1}{2\sqrt{t}}(y_n(t))^{3/2}, \tag{7}$$

$$y_{n+1}(0)=1,\qquad y_{n+1}(\infty)=0, \tag{8}$$

where $n=0,1,2,3,\cdots$.

The QLM iteration requires an initialization or "initial guess" $y_0(t)$. We assume that $y_0(t)\equiv 1$, i.e. the initial guess satisfies in the boundary condition at zero. Mandelzweig and Tabakin in Ref. [38] have shown that if the initial function is true in one of the conditions of (8) then the QLM is convergent.

Baker has shown that the solution of Eq. (1) is generated by the powers of $t^{\frac{1}{2}}$ as follows [39]:

$$\begin{aligned} y(t)=1+Bt+\frac{4}{3}t^{\frac{3}{2}}+\frac{2}{5}Bt^{\frac{5}{2}}+\frac{1}{3}t^3+\frac{3}{70}B^2t^{\frac{7}{2}}+\frac{2}{15}Bt^4 \\ +\frac{4}{63}\Big(\frac{2}{3}-\frac{1}{16}B^3\Big)t^{\frac{9}{2}}+\cdots, \end{aligned} \tag{9}$$

for this reason, in Eq. (3), we assume that $\alpha = \frac{1}{2}$.

We apply the FRC2s collocation method to solve the linear ordinary differential equations at each iteration Eq. (7) with boundary conditions (8).

Approximation of functions $y_{n+1}(t)$ by using Eq. (5) is shown by $y_{m,n+1}(t)$. Now, for applying the collocation method, we construct the residual function for the Thomas-Fermi equation by substituting $y_{m,n+1}(t)$ for $y(t)$ in Eq. (1):

$$\mathrm{RES}_n^m(t) = \frac{d^2}{dt^2}\left(y_{m,n+1}(t)\right) - \frac{1}{\sqrt{t}}\left(y_{m,n+1}(t)\right)^{\frac{3}{2}}. \tag{10}$$

In this study, the roots of the FRC2s in the semi-infinite domain $[0, \infty)$ (Theorem 1) have been used as collocation points. Also, consider that all of the computations have been done by Maple 2015.

Boyd in Ref. [1] has provided the method of the experimental trial-and-error for calculating the approximation of the optimal value of L:

*"**The experimental trial-and-error method** (Optimizing infinite Interval Map Parameter) (Page 377 in Ref. [1]):*
Plot the coefficients a_i versus degree on a log-linear plot. If the graph abruptly flattens at some m, then this implies that L is TOO SMALL for the given m, and one should increase L until the flattening is postponed to $i = m$."

It must be noted that the optimal value of L is dependent on m.

Fig. 1 presents the graph of the coefficients of $\log(|a_i|)$ for different values of L, $m = 200$ and $n = 50$, according to the above experimental trial-and-error method, the approximation optimal amount of L is about 21.

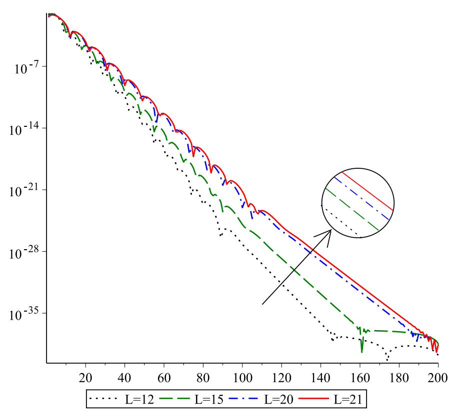


Figure 1: Graph of logarithm of coefficients $|a_i|$ with $m = 200$, $n = 50$, and different values of L, for calculating an approximation optimal value of L

Bellman & Kalaba [32] and Mandelzweig & Tabakin [38] proved the convergence of the QLM. Let $\delta y_{n+1}(t) \equiv y_{n+1}(t) - y_n(t)$, then it can show that $\| \delta y_{n+1} \| \leq k \| \delta y_n \|^2$ where k is a positive real constant [38]. Therefore, the convergence rate is of the order of 2, i.e. $O(h^2)$. We can also obtain for $(n+1)$-th iteration:

$$\| \delta y_{n+1} \| \leq (k \| \delta y_1 \|)^{2^n} / k. \tag{11}$$

Furthermore, it can be hoped that even if the initial guess is not appropriate, then after a while the solution converges [32].

4 Results and discussion

Calculating the amount of $y'(0)$ of Thomas-Fermi potential is very important for determining many physical properties of Thomas-Fermi atom.

Comparison of methods: Zaitsev et al. [40] showed that the Adams-Bashforth and Runge-Kutta methods to solve this equation on unbounded domains are ill-conditioned, hence, researchers have used the methods of numerical and semi-analytical for solving the equation, and some researchers can calculate very good solutions. For example, authors of [55, 57, 58, 59, 60, 61, 64, 68, 70] used the analytical methods for solving the equation and Amore et al. [68] were able to calculate the best solution using Pade-Hankel method, correct to 26 decimal places. Authors of [54, 56, 62, 63, 65, 66, 67] used the numerical methods for solving the equation and Parand & Delkhosh [73] were able to calculate the best solution using the combination of the quasilinearization method and the fractional order of rational Chebyshev collocation method, correct to 37 decimal places. In numerical methods, there is usually a numerical arbitrary parameter which selected by authors. Such as, in [54] the parameter is chosen 0.258497 to accuracy 10^{-6}, in [56] is chosen 0.93799968 to accuracy 10^{-8}, in [63] is chosen 0.62969503 to accuracy 10^{-6}, in [65] is chosen 0.0958885 to accuracy 10^{-7}, and in [67] is chosen 1.588071 to accuracy 10^{-7}. Here we choose $L = 21$ to accuracy 10^{-37}.

Table 1 presents some of the calculated values of $y'(0)$ of Thomas-Fermi potential by some researchers. It is clear that some researchers were able to calculate good solution and accuracy. The last three rows present the best solution obtained by the present method for different values of m.

Table 1: Comparison of the obtained values of $y'(0)$ by researchers, inaccurate digits are **bold**.

Author/Authors	Obtained value of $y'(0)$
Fermi (1928) [24]	-1.58
Baker (1930) [39]	-1.588**558**
Bush and Caldwell (1931) [41]	-1.58**9**
Miranda (1934) [42]	-1.5880**464**
Slater and Krutter (1935) [26]	-1.5880**8**
Feynman et al. (1949) [25]	-1.588**75**
Kobayashi et al. (1955) [43]	-1.58807**0972**
Mason (1964) [44]	-1.5880710
Laurenzi (1990) [45]	-1.588**588**
MacLeod (1992) [46]	-1.5880710226
Wazwaz (1999) [47]	-1.58807**6779**
Epele et al. (1999) [48]	-1.588**102**
Esposito (2002) [49]	-1.588
Liao (2003) [50]	-1.58**712**
Khan and Xu (2007) [51]	-1.58**6494973**
El-Nahhas (2008) [52]	-1.5**5167**
Yao (2008) [53]	-1.5880**04950**
Parand and Shahini (2009) [54]	-1.58807**02966**
Marinca and Herianu (2011) [55]	-1.5880**659888**
Oulne (2011) [56]	-1.5880710**34**
Abbasbandy and Bervillier (2011) [57]	-1.58807102261137531271 8**9**
Fernandez (2011) [58]	-1.588071022611375313
Zhu et al. (2012) [59]	-1.58807**411**
Turkylmazoglu (2012) [60]	-1.5880**1**
Zhao et al. (2012) [61]	-1.5880710226
Boyd (2013) [62]	-1.588071022611375312718684 5
Parand et al. (2013) [63]	-1.58807**0339**
Marinca and Ene (2014) [64]	-1.588071**9992**
Kilicman et al. (2014) [65]	-1.588071**347**
Jovanovic et al. (2014) [66]	-1.588071022**811**
Bayatbabolghani & Parand(2014)[67]	-1.588071
Amore et al. (2014) [68]	-1.58807102261137531271868450**8**
Fatoorehchi & Abolghasemi(2014)[69]	-1.58807**6818**
Liu and Zhu (2015) [70]	-1.58807**2**
Parand et al. (2016) [71]	-1.58807102261137531271868450 9
Parand et al. (2016) [72]	-1.58807102261137531271868450 9423
Parand and Delkhosh (2017) [73]	-1.5880710226113753127186845094239501095
Parand and Delkhosh (2017) [74]	-1.588071022611375312718684509
This paper [m=200]	-1.5880710226113753127186845094239501093
" " [m=100]	-1.5880710226113753127186845094239
" " [m=50]	-1.588071022611375312728

Table 2 presents the absolute errors in the calculation of $y'(0)$ for different values of m and the obtained results are compared with the best solution calculated in Ref. [73].

Table 2: Absolute errors of $y'(0)$ for different values of m and iterations

m	L_{opt}	10th Iter.	20th Iter.	30th Iter.	40th Iter.	50th Iter.
25	0.5	3.970e-08	3.939e-08	3.939e-08	3.939e-08	3.939e-08
75	5	6.667e-13	3.926e-18	5.878e-24	4.065e-30	4.646e-30
100	7	6.524e-13	5.656e-20	4.026e-24	8.314e-31	1.976e-33
175	19	6.524e-13	1.065e-25	1.349e-27	4.240e-31	1.908e-34
200	21	6.524e-13	3.477e-25	6.072e-29	9.237e-32	1.974e-37

Table 3: Obtained values of $y(t)$ by the present method for different values t

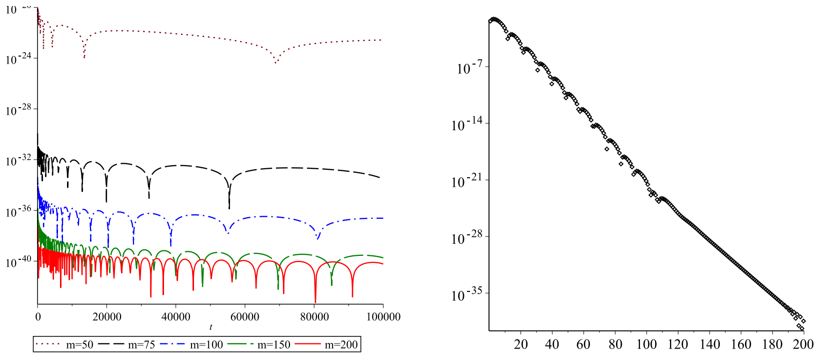
t	$y(t)$	t	$y(t)$	t	$y(t)$
0.25	0.7552014653133312	5	7.880777925136990e-2	125	5.423519678389911e-5
0.50	0.6069863833559799	6	5.942294925042258e-2	150	3.263396444625690e-5
0.75	0.5023468464123686	7	4.609781860449858e-2	175	2.115958647941346e-5
1.00	0.4240080520807056	8	3.658725526467680e-2	200	1.450180349694576e-5
1.25	0.3632014144595141	9	2.959093527054687e-2	300	4.548571953616680e-5
1.50	0.3147774637004581	10	2.431429298868086e-2	400	1.979732628112504e-5
1.75	0.2754513279960917	15	1.080535875582389e-2	500	1.034077168199939e-5
2.00	0.2430085071611195	20	5.784941191566940e-3	1000	1.351274773541057e-7
2.25	0.2158946265761301	25	3.473754416765632e-3	2000	1.733984751613821e-8
2.50	0.1929841234580007	50	6.322547829849047e-4	3000	5.189408334513832e-9
3.00	0.1566326732164958	75	2.182104320497469e-4	5000	1.130926706343084e-9
4.00	0.1084042569189077	100	1.002425681394073e-4	10000	1.42450045099559e-10

Tables 3 and 4 present the obtained results of $y(t)$ and $y'(t)$ by the present method for different values of t.

Table 4: Obtained values of $y'(t)$ by the present method for different values t

t	$y'(t)$	t	$y'(t)$	t	$y'(t)$
0.25	-0.7223069849102349	5	-2.356007495470051e-2	125	-1.202665391336449e-6
0.50	-0.4894116125745380	6	-1.586754953340707e-2	150	-6.091399478608917e-7
0.75	-0.3583068801675136	7	-1.114253181486708e-2	175	-3.410947673774533e-7
1.00	-0.2739890515933062	8	-8.088602969645474e-3	200	-2.057532316475268e-7
1.25	-0.2157941303007336	9	-6.033074714457392e-3	300	-4.365949618530290e-8
1.50	-0.1737387990139451	10	-4.602881871269254e-3	400	-1.436682305996181e-8
1.75	-0.1423209371968936	15	-1.515323082023606e-3	500	-6.034363442475256e-9
2.00	-0.1182431916254876	20	-6.472543327776920e-4	1000	-3.98801070822799e-10
2.25	-0.0994093212014470	25	-3.240429977697511e-4	2000	-2.57608536992070e-11
2.50	-0.0844261867988090	50	-3.249890204825881e-5	3000	-5.15300117644723e-12
3.00	-0.0624571308541209	75	-7.777974714283007e-6	5000	-6.75339712163883e-13
4.00	-0.0369437578241234	100	-2.739351068678330e-6	10000	-4.26161647550093e-14

Fig. 2 presents the graphs of the residual errors of RES_n^m of Eq. (10) with $m = 50, 75, 100, 150, 200$ and $n = 50$, and the logarithm of coefficients $|a_i|$ with $m = 200$ and $n = 50$, for showing the convergence of the method. It can see that the residual errors are very small value, about 10^{-39}.



(a) The residual errors (b) The logarithm of coefficients $|a_i|$

Figure 2: Graphs of the residual errors for different values of m and the logarithm of coefficients $|a_i|$, for showing the convergence of the method.

5 Conclusion

In this paper, the combination of the methods of the quasilinearization and the FRC2s collocation is used for constructing an approximation of the solution of the nonlinear singular Thomas-Fermi equation on unbounded domain. The present method has several advantages. For example, for the first time, the fractional order of rational Chebyshev functions of the second kind (FRC2s) has been introduced as a new basic for Spectral methods. The fractional basis were used to solve an ordinary differential equation and this provides an insight into an important issue. The roots of the FRC2s are used on unbounded domain $[0, \infty)$ as collocation points for solving Thomas-Fermi equation and the problem does not convert to a bounded domain. Some researchers solved the equation by changing the variables in this equation [58, 62] or domain truncation [38] but we solved the problem without any changing on variables or domain in this equation. An approximate optimal value of L is calculated. The convergence of the obtained results is shown. The accurate solutions for $y(t)$, $y'(t)$ and $y'(0)$ by 200 collocation points are obtained. This article provided a good history of solving Thomas-Fermi equation by other researchers and the numerical methods to solve equations in unbounded domains.

References

[1] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Second Edition, DOVER Publications, Mineola, New York, (2000).

[2] W. Bu,Y. Ting, Y. Wu, J. Yang, Finite difference/finite element method for two-dimensional space and time fractional blochtorrey equations, *J. Comput. Phys.*, **293** (2015), 264–279.

[3] K. Parand, S. Abbasbandy, S. Kazem, A. R. Rezaei, An improved numerical method for a class of astrophysics problems based on radial basis functions, *Phys. Scripta*, **83** (1) (2011), 015011, 11 pages.

[4] K. Parand, M. Hemami, Numerical Study of Astrophysics Equations by Meshless Collocation Method Based on Compactly Supported Radial Basis Function, *Int. J. Appl. Comput. Math.*, **3** (2) (2017), 1053–1075.

[5] R. Franke, Scattered data interpolation: Tests of some methods, *Math. Comput.*, **38** (1982), 181–200.

[6] J. A. Rad, K. Parand, Pricing American options under jump-diffusion models using local weak form meshless techniques, *Int. J. Comp. Math.*, (2016) 10.1080/00207160.2016.1227434.

[7] J. A. Rad, K. Parand, Numerical pricing of American options under two stochastic factor models with jumps using a meshless local Petrov-Galerkin method, *Appl. Numer. Math.*, **115** (2017), 252–274.

[8] K. Parand, P. Mazaheri, M. Delkhosh, A. Ghaderi, New numerical solutions for solving Kidder equation by using the rational Jacobi functions, *SeMA J.*, (2017) doi:10.1007/s40324-016-0103-z.

[9] K. Parand, M. Nikarya, J. A. Rad, Solving non-linear Lane-Emden type equations using Bessel orthogonal functions collocation method, *Celest. Mech. Dyn. Astr.*, **116** (2013), 97–107.

[10] D. Funaro and O. Kavian, approximation of some diffusion evolution equations in unbounded domains by Hermite functions, *Math. Comput.*, **57** (1991), 597–619.

[11] B. Y. Guo, J. Shen, Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval, *Numer. Math.* **86**(4) (2000), 635–654.

[12] B. Y. Guo, Jacobi Approximations in Certain Hilbert Spaces and Their Applications to Singular Differential Equations, *J. Math. Anal. Appl.*, **243** (2000), 373–408.

[13] J. A. Rad, K. Parand, L. V. Ballestra, Pricing European and American options by radial basis point interpolation, *Appl. Math. Comput.*, **251** (2015), 363–377.

[14] J. A. Rad, K. Parand, S. Abbasbandy, Pricing European and American Options Using a Very Fast and Accurate Scheme: The Meshless Local Petrov-Galerkin Method, *P. Natl Acad. Sci. India Section A: Phys. Sci.*, **85** (3) (2015), 337–351.

[15] M. Delkhosh, M. Delkhosh, M. Jamali, Introduction to Green's Function and its Numerical Solution, *Middle-East J. Sci. Res.*, **11** (7) (2012), 974–981.

[16] J. P. Boyd,Orthogonal rational functions on a semi-infinite interval, *J. Comput. Phys.*, **70** (1987), 63–88.

[17] K. Parand, M. Dehghan, F. Baharifard, Solving a laminar boundary layer equation with the rational Gegenbauer functions, *Appl. Math. Model.*, **37** (2013), 851–863.

[18] J. A. Rad, S. Kazem, M. Shaban, K. Parand, A. Yildirim, Numerical solution of fractional differential equations with a Tau method based on Legendre and Bernstein polynomials, *Math. Method. Appl. Sci.*, **37** (3) (2014), 329–342.

[19] K. Parand, L. Hossein, Numerical approach of flow and mass transfer on nonlinear stretching sheet with chemically reactive species using rational Jacobi collocation method, *Int. J. Numer. Method. H. F. F.*, **23** (5) (2013), 772–789.

[20] F. Baharifard, S. Kazem, K. Parand, Rational and Exponential Legendre Tau Method on Steady Flow of a Third Grade Fluid in a Porous Half Space, *Int. J. Appl. Comput. Math.*, **2** (4) (2016), 679–698.

[21] K. Parand, S. Khaleqi, The rational Chebyshev of Second Kind Collocation Method for Solving a Class of Astrophysics Problems, *Eur. Phys. J. Plus*, **131** (24), (2016).

[22] L. H. Thomas, The calculation of atomic fields, *Math. Proc. Cambridge*, **23** (1927), 542–548.

[23] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, 1967.

[24] E. Fermi, Eine statistische Methode zur Bestimmung einiger Eigenschaften des Atoms und ihre Anwendung auf die Theorie des periodischen Systems der Elemente, *Z. Phys.*, **48** (1928), 73–79.

[25] R. P. Feynman, N. Metropolis, E. Teller, Equations of State of Elements Based on the Generalized Fermi-Thomas Theory, *Phys. Rev.*, **75** (10) (1949), 1561–1573.

[26] J. C. Slater, H. M. Krutter, The Thomas-Fermi method for metals, *Phys. Rev.*, **47** (1935), 559–568.

[27] B. J. Laurenzi, An analytic solution to the Thomas-Fermi equation, *J. Math. Phys.*, **10** (1990), 2535–2537.

[28] A. Saadatmandi, M. Dehghan, Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method, *Numer. Method. Part. D. E.*, **26** (1) (2010), 239–252.

[29] A. H. Bhrawy, A. S. Alofi, The operational matrix of fractional integration for shifted Chebyshev polynomials, *Appl. Math. Lett.*, **26** (2013), 25–31.

[30] K. Parand, M. Delkhosh, M. Nikarya, Novel orthogonal functions for solving differential equations of arbitrary order, *Tbilisi Math. J.*, **10** (1) (2017), 31–55

[31] K. Parand, M. Delkhosh, Operational Matrices to Solve Nonlinear Volterra-Fredholm Integro-Differential Equations of Multi-Arbitrary Order, *Gazi Uni. J. Sci.*, **29** (4) (2016), 895–907.

[32] R. E. Bellman, R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-Value Problems*, Elsevier Publishing Company, New York, 1965.

[33] R. Kalaba, *On nonlinear differential equations, the maximum operation and monotone convergence*, RAND Corporation, P-1163, 1957.

[34] K. Parand, M. Delkhosh, An Efficient Numerical Solution of Nonlinear Hunter-Saxton Equation, *Commun. Theor. Phy.*, **67** (5) (2017), 483–492

[35] R. Krivec, V. B. Mandelzweig, Quasilinearization approach to computations with singular potentials, *Comput. Phys. Comm.*, **179** (12) (2008), 865–867.

[36] E. Z. Liverts, V. B. Mandelzweig, Analytical computation of amplification of coupling in relativistic equations with Yukawa potential, *Ann. Phys-New York*, **324** (2) (2009), 388–407.

[37] K. Parand, M. M. Moayeri, S. Latifi, M. Delkhosh, A numerical investigation of the boundary layer flow of an Eyring-Powell fluid over a stretching sheet via rational Chebyshev functions, *Euro. Phy. J. Plus*, **132** (7) (2017), 325.

[38] V. B. Mandelzweig, F. Tabakinb, Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs, *Comput. Phys. Commun.*, **141** (2001), 268–281.

[39] E. B. Baker, The application of the Fermi-Thomas statistical model to the calculation of potential distribution in positive ions, *Quart. Appl. Math.*, **36** (1930), 630–647.

[40] N. A. Zaitsev, I. V. Matyushkin, D. V. Shamonov, Numerical Solution of the Thomas-Fermi Equation for the Centrally Symmetric Atom, *Russ. Microelectronics*, **33** (2004), 303–309.

[41] V. Bush, S. H. Caldwell, Thomas-Fermi equation solution by the differential analyzer, *Phys. Rev.*, **38** (1931), 1898–1902.

[42] C. Miranda, Teorie e metodi per l'integrazione numerica dell'equazione differenziale di Fermi, Memorie della Reale Accademia d'Italia, Classe di scienze fisiche, *Math. Nat.*, **5** (1934), 285–322.

[43] S. Kobayashi, T. Matsukuma, S. Nagi, K. Umeda, Accurate value of the initial slope of the ordinary T-F function, *J. Phys. Soc. Japan*, **10** (1955), 759–762.

[44] J. C. Mason, Rational approximations to the ordinary Thomas-Fermi function and its derivative, *Proc. Phys. Soc.*, **84** (1964), 357–359.

[45] B. J. Laurenzi, An analytic solution to the Thomas-Fermi equation, *J. Math. Phys.*, **31** (1990) 2535-2537.

[46] A. J. MacLeod, Chebyshev series solution of the Thomas-Fermi equation, *Comput. Phys. Commun.*, **67** (1992), 389–391.

[47] A-M. Wazwaz, The modified decomposition method and Pade approximates for solving the Thomas-Fermi equation, *Appl. Math. Comput.*, **105** (1999), 11–19.

[48] L. N. Epele, H. Fanchiotti, C. A. G. Canal, J. A. Ponciano, Pade approximate approach to the Thomas-Fermi problem, *Phys. Rev. A*, **60** (1999), 280–283.

[49] S. Esposito, Majorana solution of the Thomas-Fermi equation, *Am. J. Phys.*, **70** (2002), 852–856.

[50] S. Liao, An explicit analytic solution to the Thomas-Fermi equation, *Appl. Math. Comput.*, **144** (2003), 495–506.

[51] H. Khan, H. Xu, Series solution to the Thomas-Fermi equation, *Phys. Let. A*, **365** (2007), 111–115.

[52] A. El-Nahhas, Analytic Approximations for Thomas-Fermi Equation, *Acta Phys. Pol. A*, **114** (4) (2008), 913–918.

[53] B. Yao, A series solution to the Thomas-Fermi equation, *Appl. Math. Comput.*, **203** (2008), 396–401.

[54] K. Parand, M. Shahini, Rational Chebyshev pseudospectral approach for solving Thomas-Fermi equation, *Phys. Let. A*, **373** (2009), 210–213.

[55] V. Marinca, N. Herisanu, An optimal iteration method with application to the Thomas-Fermi equation, *Cent. Eur. J. Phys.*, **9** (2011), 891–895.

[56] M. Oulne, Variation and series approach to the Thomas-Fermi equation, *Appl. Math. Comput.*, **218** (2011), 303–307.

[57] S. Abbasbandy, C. Bervillier, Analytic continuation of Taylor series and the boundary value problems of some nonlinear ordinary differential equations, *Appl. Math. Comput.*, **218** (2011), 2178–2199.

[58] F. M. Fernandez, Rational approximation to the Thomas-Fermi equations, *Appl. Math. Comput.*, **217** (2011), 6433–6436.

[59] S. Zhu, H. Zhu, Q. Wu, Y. Khan, An adaptive algorithm for the Thomas-Fermi equation, *Numer. Algor.*, **59** (2012), 359–372.

[60] M. Turkyilmazoglu, Solution of the Thomas-Fermi equation with a convergent approach, *Commun. Nonlinear. Sci. Numer. Simulat.*, **17** (2012), 4097–4103.

[61] Y. Zhao, Z. Lin, Z. Liu, S. Liao, The improved homotopy analysis method for the Thomas-Fermi equation, *Appl. Math. Comput.*, **218** (2012), 8363–8369.

[62] J. P. Boyd, Rational Chebyshev series for the Thomas-Fermi function: Endpoint singularities and spectral methods, *J. Comput. Appl. Math.*, **244** (2013), 90–101.

[63] K. Parand, M. Dehghanb, A. Pirkhedri, The Sinc-collocation method for solving the Thomas-Fermi equation, *J. Comput. Appl. Math.*, **237** (2013), 244–252.

[64] V. Marinca, R. D. Ene, Analytical approximate solutions to the Thomas-Fermi equation, *Cent. Eur. J. Phys.*, **12** (7) (2014), 503–510.

[65] A. Kilicman, I. Hashimb, M. Tavassoli Kajani, M. Maleki, On the rational second kind Chebyshev pseudospectral method for the solution of the Thomas-Fermi equation over an infinite interval, *J. Comput. Appl. Math.*, **257** (2014), 79–85.

[66] R. Jovanovic, S. Kais, F. H. Alharbi, Spectral Method for Solving the Nonlinear Thomas-Fermi Equation Based on Exponential Functions, *J. App. Math.*, **2014** (2014), Article ID 168568, 8 pages.

[67] F. Bayatbabolghani, K. Parand, Using Hermite Function for Solving Thomas-Fermi Equation, *Int. J. Math. Comput. Phys. Elect. Comp. Eng.*, **8**(1) (2014), 123–126.

[68] P. Amore, J. P. Boyd, F. M. Fernandez, Accurate calculation of the solutions to the Thomas-Fermi equations, *Appl. Math. Comput.*, **232** (2014), 929–943.

[69] H. Fatoorehchi, H. Abolghasemi, An Explicit Analytic Solution to the Thomas-Fermi Equation by the Improved Differential Transform Method, *Acta Phys. Pol. A*, **125** (5) (2014), 1083–1087.

[70] C. Liu, S. Zhu, Laguerre pseudospectral approximation to the Thomas-Fermi equation, *J. Comput. Appl. Math.*, **282** (2015), 251–261.

[71] K. Parand, H. Yousefi, M. Delkhosh, A. Ghaderi, A Novel Numerical Technique to Obtain an Accurate Solution of the Thomas-Fermi Equation, *Eur. Phys. J. Plus*, **131** (2016), 228.

[72] K. Parand, A. Ghaderi, M. Delkhosh, H. Yousefi, A new approach for solving nonlinear Thomas-Fermi equation based on fractional order of rational Bessel functions, *Electron. J. Differential Equations*, **2016** (331) (2016), 1–18.

[73] K. Parand, M. Delkhosh, Accurate solution of the Thomas-Fermi equation using the fractional order of rational Chebyshev functions, *J. Comput. Appl. Math.*, **317** (2017), 624–642.

[74] K. Parand, M. Delkhosh, New Numerical Solution For Solving Nonlinear Singular Thomas-Fermi Differential Equation, *Bull. Belg. Math. Soc. Simon Stevin*, **24** (3) (2017), 457–476.

Received: August 24, 2017