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Fundamental theorem of calculus under weaker forms of primitive

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Abstract. In this paper we will present abstract versions of fundamental theorem of calculus (FTC) in the setting of Kurzweil - Henstock integral for functions taking values in an infinite dimensional locally convex space. The result will also be dealt with weaker forms of primitives in a widespread setting of integration theories generalising Riemann integral.

1 Introduction and preliminaries

The (FTC) theorem is one of the celeberated results of classical analysis. The result establishes a relation between the notions of integral and derivative of a function. In its origional form FTC asserts that: if for the function $F : [a, b] \longrightarrow \mathbb{R}$, $F'(t)$ exists and $F'(t) = f(t)$ and if $f(t)$ if integrable then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Let us recall that a (tagged) partition of the interval $[a, b]$ is a finite set of non-overlapping subintervals $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$, where $a = x_0 < x_1 < \cdots <$

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$x_n = b$ and t_i's are the tags attached to each subinterval $[x_{i-1}, x_i]$. The norm or the mesh of the partition is define to be

$$|\mathcal{P}| = \max_{1 \le i \le n} (t_i - t_{i-1}).$$

Definition 1 A (bounded) function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be Riemann integrable if: $\exists\ x \in \mathbb{R}$ such that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that for each (tagged) partition $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ of $[a, b]$ with $|P| < \delta$

$$|S(f, \mathcal{P}) - x| \le \epsilon,$$

where $S(f, \mathcal{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$ is the Riemann sum of f corrsponding to the partition $\mathcal{P}$: The (unique) vector x, to be denoted by $\int_a^b f(t)dt$ shall be called the Riemann integral of f over $[a, b]$.

Theorem 1 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is differentiable on* $[a, b]$ *and* $f'(t)$ *is (Riemann) integrable then*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

In the preceeding therem the assumption of the integrability of the derivative $f'(t)$ is unavoidable. Below we give an explicate of FTC in the setting of Kurzweil – Henstock integral, where the integrability of the derivative comes for free.

Recalling that a gauge is a positive function $\delta : [a, b] \longrightarrow (0, \infty)$ and a partition $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is said to be δ-fine if $[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))\ \forall\ 1 \le i \le n$.

Definition 2 [2], [5] A function $f : [0, 1] \longrightarrow \mathbb{R}$ is said to be Kurzweil – Henstock integrable if there exists $x \in \mathbb{R}$ such that the following is true: for any $\epsilon > 0$, there exists a gauge $\delta(t) > 0$ on $[a, b]$ such that if $\{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is any δ-fine (tagged) partition of $[a, b]$ then

$$|S(f, \mathcal{P}) - x| \le \epsilon,$$

where x is the integral of f and $S(f, \mathcal{P}) = \sum\limits_{i=1}^n f(t_i)(x_i - x_{i-1})$ is the Riemann sum, symbolically we write $f \in KH([0, 1])$.

The (KH–integral) integral is defined in almost the same way as Riemann integral through Riemann sums. The only difference is in defining the δ here it is assumed to be a positive function instead of a constant. The only technicality to be taken care of and the definition to make sense is that we must have a δ-fine partition for every gauge. Pierre Cousin [3] gives the existence of such a partition for every gauge $\delta(t)$ in the form of so called Cousin's lemma. Before proceeding further, we would like to show that the KH – integral sub- sumes Riemann integral properly through the famous Dirchlet's function $f : [0, 1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

We know that f is not Riemann integrable. Here we will show that f is KH – integrable. Let $\epsilon > 0$ be given and set

$$\delta(x) = \begin{cases} 1, & x \text{ is irrational} \\ \frac{\epsilon}{2^{i+1}}, & x = q_i, i \geq 1 \end{cases}$$

where q_i is the enumeration of rationals in $[0, 1]$. Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a δ-fine partition of $[0, 1]$. If t_i is not rational, the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum of f with respect to $\mathcal{P}$ is 0. If t_i is rational and $t_i = q_j$ for some j, the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum is less than $2\delta(q_j) = \frac{\epsilon}{2^{j+1}}$. Thus we have

$$\left|\sum_{i=0}^{n-1} f(t_i)(x_i - x_{i-1})\right| < 2\sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \epsilon$$

which shows that $f \in \mathsf{KH}([0, 1])$. This is the most common example of a bounded function which is not Riemann integrable. But it turns out to be KH – integrable and furnishes a comparison between the two theories of integration.

Theorem 2 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is differentiable on* $[a, b]$ *then* $f'(t)$ *is* (*Kurzweil – Henstock*) *integrable and*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

A slight modification in the definition of δ makes an immense impact and if we take it to be a constant we get the Riemann integral. It is quite remarkable that the simple idea of replacing δ by a positive function $\delta(t)$ leads to a powerfull generalization of Riemann integral. The convergence theorems of the Lebesgue integral hold true in the setting of KH – integral and more importantly FTC holds in its full generality without the assumption of integrability of the derivative [1].

Definition 3 [2] Let $F, f : [a, b] \longrightarrow \mathbb{R}$, we say that:
(i) F is primitive of f on $[a, b]$ if $F'(x)$ exists and $F'(x) = f(x)$ for all $x \in [a, b]$.
(ii) F is a-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a null set $E \subset [a, b]$.
(iii) F is c-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a countable set $E \subset [a, b]$.
(iv) F is f-primitive of f on $[a, b]$ if F is continuous, $F'(x)$ exists and $F'(x) = f(x)$ outside a finite set $E \subset [a, b]$.

In the following example we show that the proof of Theorem 2 can be re-designed to permit one point of non-differentiability.

Example 1 Define $f : [0, 1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

f is not bounded on $[0, 1]$. If we take $F(x) = 2x$ for $x \in [0, 1]$ then F is continuous on $[0, 1]$ and $F'(x) = f(x)$ for all $x \in (0, 1]$ but $F'(0)$ does not exist. Hence F is an f-primitive of f on $[0, 1]$ with the exceptional set $E = \{0\}$. Now, if $t \in (0, 1]$ and $\epsilon > 0$ we can choose $\delta(t)$ in such a way that the conclusion of FTC holds true for F. To tackle with the point of exception 0 we choose $\delta(0) = \frac{\epsilon^2}{4}$ so that if $0 \le v \le \delta(0)$, then $F(v) - F(0) = 2\sqrt{v} \le \epsilon$.
Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition of $[0, 1]$ that is δ-fine. If all of the tags belong to $(0, 1]$ the proof of Theorem 2 applies without any change. However, if the first tag $t_1 = 0$ then the first term in the Riemann sum $S(f, \mathcal{P})$ is equal to $f(0)(x_1 - x_0) = 0$. Also we have

$$|F(x_1) - F(x_0) - f(0)(x_1 - x_0)| = |F(x_1)| = 2\sqrt{x_1} \le \epsilon.$$

We now apply the argument given in Theorem 2 to the remaining terms to obtain

$$\left| \sum_{i=2}^{n} F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right| < \epsilon.$$

Therefore on adding these terms we have

$$|F(1) - F(0) - S(f, \mathcal{P})| \le \epsilon + \epsilon = 2\epsilon$$

Since ϵ is arbitrary we conclude that $f \in KH([0, 1])$ and that

$$\int_0^1 f(t)dt = F(1) - F(0) = 2.$$

The argument of the above theorem can easily be carried over to any exceptional set of finitely many points and the conclusion of the theorem is sought for an f – primitive.
As a significant extension below we present a version of FTC where the conclusion holds true for a countably infinite (exceptional) set.

Theorem 3 *If* $f : [a, b] \longrightarrow \mathbb{R}$ *has a* c *– primitive* F *on* $[a, b]$ *then* $f \in KH[a, b]$ *and*

$$\int_a^b f(t)dt = F(b) - F(a)$$

In the preceeding theorem the conclusion holds true for a c – primitive that is if the exceptional set is taken to be a countably infinite set. We know that every countable set is a null set. So, it is natural to ask whether the gap between countable and the null set can be bridged. More precisely, can we replace above theorem by the assertion: if F is continuous function on $[a, b]$ and there exists a null set E such that $F'(x) = f(x)$ for all $x \in [a, b] - E$ then $f \in KH([0, 1])$ and

$$\int_a^b f(t)dt = F(b) - F(a).$$

On this account it becomes inevitable to discuss the the so-called Cantor-Lebesgue function on $[0, 1]$ the construction of the function is given as:
Define

$$\Lambda : [0, 1] \longrightarrow \mathbb{R}$$

by

$$\Lambda(x) = \lim_{n \longrightarrow \infty} \Lambda_n(x)$$

where $\Lambda_n(x)$ is taken to be $\frac{1}{2^n}$ on the left out intervals of $[0, 1]$ while constructing the Cantor set, $\Lambda_n(0) = 0$ and $\Lambda_n(1) = 1$.
It is easy to see that Λ is a continuous non-decreasing function and its derivative $\Lambda'(x) = 0$ for all points of $[0, 1]$ outside the Cantor set.
Now comming back to the question raised above we see that $\Lambda'(x)$ exists and $\Lambda'(x) = \Lambda(x)$ outside a (Cantor) null set. But

$$\int_0^1 \Lambda' = 0 \neq 1 = \Lambda(1) - \Lambda(0)$$

2 FTC – for functions taking values in a Frechet space

We begin this section by giving a formal definition of the Kurzweil – Henstock integral also known as gauge integral or generalized Riemann integral for functions taking values in a complete metrizable locally convex space known as Frechet space [7]. In this section X will denote a Frechet space, $p(X)$ a family of seminorms on X.

Definition 4 A function $f : [0, 1] \longrightarrow X$ is said to be Kurzweil – Henstock integrable if there exists $x \in X$ for which the following is true: for any $\epsilon > 0$, and a seminorm $p \in p(X)$ there exists a gauge $\delta_{\epsilon,p} > 0$ on $[a, b]$ such that if $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is any $\delta_{\epsilon,p}$-fine (tagged) partition of $[a, b]$ then

$$p\left(S(f, \mathcal{P}) - x\right) \leq \epsilon,$$

where x is the integral of f and $S(f, \mathcal{P}) = \sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ is the Riemann sum, symbolically we write $f \in KH([0, 1], X)$.

Lemma 1 *Let* $F : [a, b] \to X$ *be differentiable at a point* $t \in [a, b]$*, then given* $\epsilon > 0$ *there exists* $\delta_{\epsilon,p}(t) > 0$ *such that if* $u, v \in [a, b]$ *satisfy*

$$t - \delta_{\epsilon,p}(t) \leq u \leq t \leq v \leq t + \delta_{\epsilon,p}(t)$$

then

$$p\left(F(v) - F(u) - F'(t)(v - u)\right) \leq \epsilon(v - u)$$

Proof. By definition of the derivative at $t \in [0, 1]$, we have, given $\epsilon > 0$ there exists $\delta_{\epsilon,p}(t) > 0$, such that

$$p\left(\frac{F(z) - F(t)}{z - t} - F'(t)\right) \leq \epsilon, \text{ for } |z - t| \leq \delta_{\epsilon,p}(t), z \in [a, b]$$

$$p\left(F(z) - F(t) - F'(t)(z - t)\right) \leq \epsilon|z - t| \text{ for all } z \in [a, b]$$

with

$$|z - t| \leq \delta_{\epsilon,p}(t).$$

In particular, if we pick $u \le t$ and $v \ge t$ in this interval around t and note that $v - t \ge 0$ and $t - u \ge 0$, then we have

$$\begin{aligned} p\left(F(v) - F(u) - F'(t)(v-u)\right) &= p\left(\left(F(v) - F(t) - F'(t)(v-t)\right)\right. \\ &\quad \left.-\left(F(u) - F(t) - F'(t)(t-u)\right)\right) \\ &\le p\left(F(v) - F(t) - F'(t)(v-u)\right) \\ &\quad + p\left(F(u) - F(t) - F'(t)(t-u)\right) \\ &\le \epsilon(v-t) + \epsilon(t-u) \\ &= \epsilon(v-u) \end{aligned}$$

which implies,

$$p\left(F(v) - F(u) - F'(t)(v-u)\right) \le \epsilon(v-u).$$

□

Now we will present the Frechet space analogue of FTC.

Theorem 4 *Let* X *be a Frechet space. If* $f : [a,b] \to X$ *has a primitive* F *i,e.,* $F : [a,b] \to X$ *is differentiable at every point of* $[a,b]$ *and* $F' = f$ *on* $[a,b]$ *then* $f \in KH([a,b],X)$ *and*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Proof. Since $F'(t)$ exists for every $t \in [a,b]$ and $F'(t) = f(t)$, given $\epsilon > 0$ there exists $\delta_{\epsilon,p}(t) > 0$ such that

$$p\left(\frac{F(z) - F(t)}{z - t} - f(t)\right) \le \epsilon,$$

for

$$|z - t| \le \delta_{\epsilon,p}(t), z \in [a,b]$$

which implies,

$$p\left(F(z) - F(t) - F'(t)(z-t)\right) \le \epsilon|z-t| \text{ for all } z \in [a,b].$$

Therefore by Lemma 1, if $a \le u \le t \le v \le b$ and $0 < v - u \le \delta_\epsilon(t)$, then

$$p(F(v) - F(u) - f(t)(v-u)) \le \epsilon|v-u|.$$

If $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a $\delta_{\epsilon,p}$-fine partition of $[a, b]$ then the telescoping sum $F(b) - F(a) = \sum_{i=1}^{n}\{F(x_i) - F(x_{i-1})\}$ satisfies the approximation

$$\begin{aligned} p\left(S(f,\mathcal{P}) - (F(b) - F(a))\right) &= p\left(\sum_{i=1}^{n}(x_i - x_{i-1})f(t_i) - \left(F(x_i) - F(x_{i-1})\right)\right) \\ &\leq \sum_{i=1}^{n} p\left((x_i - x_{i-1})f(t_i) - \left(F(x_i) - F(x_{i-1})\right)\right) \\ &\leq \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) \\ &= \epsilon(b - a) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary letting $\epsilon \to 0$, we get $f \in HK([a, b], X)$ and

$$\int_a^b f(t)dt = F(b) - F(a).$$

□

Theorem 5 *If* $f : [a, b] \longrightarrow X$ *has a* c*-primitive* F *on* $[a, b]$ *the* $f \in KH([a, b], X)$ *and*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Proof. Let $E = \{c_k\}_{k=1}^{\infty}$ be the exceptional set for the c-primitive. Since E is countable, it is a null set and without loss of generality we may suppose that $f(c_k) = 0$. We shall define a gauge $\delta_{\epsilon,p}$ on $[a, b]$. Given $\epsilon > 0$ if $t \in [a, b] - E$ we take $\delta_{\epsilon,p}$ as in Lemma 1. For $t \in E, t = c_k$ for some $k \in N$. Since F is continuous on $[a, b]$ we can choose $\delta_{\epsilon,p}(c_k) > 0$ such that

$$p\left(F(z) - F(c_k)\right) \leq \frac{\epsilon}{2^{k+2}} \ \forall\, z \in [a, b]$$

that satisfy

$$|z - c_k| \leq \delta_{\epsilon,p}(c_k).$$

Thus a gauge is defined on $[a, b]$.
Now let $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a $\delta_{\epsilon,p}$-fine partition of $[a, b]$. If none of the tags belong to E, then the proof given in the Theorem 4 applies without any change. However if $c_k \in E$ is the tag of some subinterval then,

$$p\left(F(x_i) - F(x_{i-1}) - f(c_k)(x_i - x_{i-1})\right)$$

$$\begin{aligned}
&\leq p\big(F(x_i)-F(c_k)\big)+p\big(F(c_k)-F(x_{i-1})\big)+p\big(f(c_k)(x_i-x_{i-1})\big)\\
&\leq \frac{\epsilon}{2^{k+2}}+\frac{\epsilon}{2^{k+2}}\\
&= \frac{\epsilon}{2^{k+1}}
\end{aligned}$$

Now each point of E can be the tag of at most two subintervals in $\mathcal{P}$ therefore for each $t_i \in E$ we have the following inequality satisfied

$$\sum_{t_i\in E} p\big(F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\big) \leq \sum_{k=1}^{\infty}\frac{\epsilon}{2^k}=\epsilon.$$

Also for $t_i \notin E$, we have from Lemma 1

$$\sum_{t_i\notin E} p\big(F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\big) \leq \epsilon\sum_{t_i\notin E}(x_i-x_{i-1}) \leq \epsilon(b-a).$$

Now $\mathcal{P}$ is $\delta_{\epsilon,p}$-fine, therefore we have

$$|F(b)-F(a)-S(f,\mathcal{P})| \leq \epsilon(b-a)$$

Letting $\epsilon \longrightarrow 0$, we conclude that $f \in KH([0,1],X)$ with integral $F(b)-F(a)$ which proves the theorem. $\square$

3 FTC – some interesting situations in vector integration

As pointed out in the Section 1 conclusion of the above theorem does not hold true even for a real valued function if the exceptional set E is taken to be a null set. But the problem has been dealt with and the conclusion sought, in the setting of Bochner integral by C. Volintiru [6] with the assumption that the Hausdorff measure of the image of E under F is 0.
Let (M,d) be a metric space and $A \subset M$. Let C_i be a covering of A with $\mathrm{diam}(C_i) \leq \delta\ \forall\ i$. Let $\mathcal{C}(A,\delta)$ be the collection of all such coverings of A. Now for $\alpha > 0$, define

$$h_\alpha^\delta(A)=\inf\left(\sum_i(\mathrm{diam}C_i)^\alpha : (C_i)\in\mathcal{C}(A,\delta)\right).$$

Then

$$h_\alpha(A)=\lim_{\delta\longrightarrow 0}h_\alpha^\delta(A)=\sup_{\delta>0}h_\alpha^\delta(A)$$

gives an outer measure on the power set of M which is countably additive on the σ-field of Borel subsets of M. This measure is known as Hausdorff measure.

Definition 5 A measurable function $f : \Omega \longrightarrow X$ (X a Banach space) is said to be Bochner integrable if there exists a sequence of simple function (f_n) such that

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0$$

In this case, $\int_E f d\mu$ is defined for each $E \in \sum$ by

$$\int_E f d\mu = \lim_n \int_E f_n d\mu$$

where $\int_E f_n$ is defined in the ususal way.

Theorem 6 *A measurable function* $f : \Omega \longrightarrow X$ *is Bochner integrable if and only if*

$$\int_\Omega \|f\| d\mu < \infty$$

Theorem 7 *If* F *is the* a*-primitive of the function* $f : [a, b] \longrightarrow X$ *such that* $h_1\big(F(E)\big) = 0$ (E *being the exceptional set*)*. Then* f *is measurable and if we assume the integrability of* f*, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Here we would like to pose following questions which appear to be open!

Problem 1 *Does the conclusion of FTC hold true in the setting of KH – integral for the exceptional set E to be a null set with the assumption that Hausdorff measure of* $F(E)$ *is taken to be zero.*

Problem 2 *Can we have an analogue of Theorem 7 in the setting of a more general class of integral (which subsumes KH–integral and Bochner integral) known as Pettis integral [4].*

There are two aspects of FTC to ponder upon. One about integrating the derivatives (what we have discussed) other differentiating the indefinite integrals. A lot of work has been done with regard to this aspect of FTC in the setting of KH–integral [2]. Below we state a result of paramount importance on diferentiating integrals and then conclude with a problem which appears to be open.

Theorem 8 *If* $f \in KH([a, b])$ *then any indefinite integral* F *is continuous on* $[a, b]$ *and an* a*-primitive of* f *that is* $F'(t) = f(t)\ \forall\ t \in [a, b] - E$*, where* E *is a null set.*

Problem 3 *If* $f : [a, b] \longrightarrow X$ *is Pettis integrable. Does the conclusion of above theorem hold true?*

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