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# On multigraphic and potentially multigraphic sequences

*Dedicated to the memory of Antal Iványi*

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**Abstract.** An $r$-graph(or a multigraph) is a loopless graph in which no two vertices are joined by more than $r$ edges. An $r$-complete graph on $n$ vertices, denoted by $K_n^{(r)}$, is an $r$-graph on $n$ vertices in which each pair of vertices is joined by exactly $r$ edges. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be $r$-graphic if it is realizable by an $r$-graph on $n$ vertices. An $r$-graphic sequence $\pi$ is said to be potentially $S_{L,M}^{(r)}$-graphic if it has a realization containing $S_{L,M}^{(r)}$ as a subgraph. We obtain conditions for an $r$-graphic sequence to be potentially $S_{L,M}^{(r)}$-graphic. These are generalizations from split graphs to $p$-tuple $r$-split graph.

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# 1 Introduction

For a positive integer $r$, an $r$-graph(or multigraph) is a loopless graph in which no two vertices are joined by more than $r$ edges. An $r$-complete graph on $n$ vertices, denoted by $K_n^{(r)}$, is an $r$-graph on $n$ vertices in which each pair of vertices is joined by exactly $r$ edges. Clearly, $K_n^{(1)} = K_n$. A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of non-negative integers is said to be $r$-graphic if it is the degree sequence of an $r$-graph $G$ on $n$ vertices, and such an $r$-graph $G$ is referred to as a realization of $\pi$. We take $\sigma(\pi) = \sum_{i=1}^{n} d_i$. For graph theoretical notations and definitions we refer to [9].

Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of non-negative integers with $d_1 \leq \sum\limits_{i=2}^{n} \min\{r, d_i\}$. Define $\pi_k^{'} = (d_1', d_2', \dots, d_{n-1}')$ to be the non-increasing rearrangement of the sequence obtained from

$$(d_1, d_2, \dots, d_{k-1}, d_{k+1}, \dots, d_n)$$

by reducing by 1 the remaining largest terms that have not been reduced $r$ times, and repeating the procedure $d_k$ times. $\pi_k^{'}$ is called the residual sequence obtained from $\pi$ by laying off $d_k$.

The following three results due to Chungphaisian [2] are generalizations from 1-graphs to $r$-graphs of three well-known results, one by Erdős and Gallai [3], one by Kleitman and Wang [6] and one by Fulkerson, Hoffman and Mcandrew [5].

**Theorem 1** [2] *Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of non-negative integers, where $\sigma(\pi)$ is even. Then $\pi$ is $r$-graphic if and only if for each positive integer $t \leq n$,*

$$\sum_{i=1}^{t} d_i \leq rt(t-1) + \sum_{i=t+1}^{n} \min\{rt, d_i\}.$$

**Theorem 2** [2] *$\pi$ is $r$-graphic if and only if $\pi_k'$ is $r$-graphic.*

Let the subgraph $H$ on the vertices $v_i, v_j, v_k, v_l$ of a multigraph $G$ contain the edges $v_iv_j$ and $v_kv_l$. The operation of deleting these edges and introducing a pair of new edges $v_iv_l$ and $v_jv_k$, or $v_iv_k$ and $v_jv_l$ is called an elementary degree preserving transformation. If this operation is performed $r$ times on the same edge set, it is called $r$-exchange.

**Theorem 3** [2] *Let $\pi$ be an $r$-graphic sequence, and let $G$ and $G'$ be realizations of $\pi$. Then there is a sequence of $r$-exchanges, $E_1, \ldots, E_k$ such that the application of these $r$-exchanges to $G$ in order will result in $G'$.*

An $r$-graphic sequence $\pi$ is said to be potentially $K_{m+1}^{(r)}$ if there exists a realization of $\pi$ containing $K_{m+1}^{(r)}$ as a subgraph. If $\pi$ has a realization $G$ containing $K_{m+1}^{(r)}$ on the $m+1$ vertices of highest degree in $G$, then $\pi$ is said to be potentially $A_{m+1}^{(r)}$-graphic. As a special case of Lemma 2.1 in [13], Yin showed that an $r$-graphic sequence is potentially $K_{m+1}^{(r)}$-graphic if and only if it is potentially $A_{m+1}^{(r)}$-graphic.

The $r$-join (complete product) of two $r$-graphs $G_1$ and $G_2$ is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and the edge set consisting of all edges of $G_1$ and $G_2$ together with the edges joining each vertex of $G_1$ with every vertex of $G_2$ by exactly $r$ edges. Let $K_l^{(r)}$ and $K_m^{(r)}$ be complete $r$-graphs with $l$ and $m$ vertices respectively, that is the complete graphs having exactly $r$ edges between every two vertices. The $r$-split graph of $K_l^{(r)}$ and $\overline{K}_m^{(r)}$ denoted by $\overline{S}_{l,m}^{(r)}$ is the graph $K_l^{(r)} \vee \overline{K_m^{(r)}}$ having $l+m$ vertices, where $\overline{K_m^{(r)}}$ (having no edges) is the complement of $K_m^{(r)}$. [14]. If $\pi$ has a realization $G$ containing $\overline{S}_{l,m}$ on the $l+m$ vertices of highest degree in $G$, then $\pi$ is said to be potentially $\overline{A}_{l,m}$-graphic.

The following two results due to Yin [13] are generalizations from 1-graphs to $r$-graphs of two well-known results given by A. R. Rao [12].

**Theorem 4** [13] *Let $n \geq l+1$ and $\pi = (d_1, d_2, \ldots, d_n)$ be an $r$-graphic sequence with $d_{l+1} \geq rl$. Then $\pi$ is potentially $A_{l+1}^{(r)}$-graphic if and only if $\pi_{l+1}$ is $r$-graphic.*

**Theorem 5** [13] *Let $n \geq l+1$ and $\pi = (d_1, d_2, \ldots, d_n)$ be an $r$-graphic sequence with $d_{l+1} \geq 2rl-1$, then $\pi$ is potentially $K_{l+1}^{(r)}$.*

An extremal problem for 1-graphic sequences to be potentially $K_l^{(1)}$-graphic was considered by Erdős, Jacobson and Lehel [4] and solved by Li et al. [7, 8]. Yin [13] generalized this extremal problem and the Erdős-Jacobson-Lehel conjecture from 1-graphs to $r$-graphs.

In 2014, the authors [10] proved the following assertion.

**Theorem 6** [10] *If $G_1$ is a realization of $\pi_1 = (d_1^1, \ldots, d_m^1)$ containing $K_p$ as a subgraph and $G_2$ is a realization of $\pi_2 = (d_1^2, \ldots, d_n^2)$ containing $K_q$ as a subgraph, then the degree sequence $\pi = (d_1, \ldots, d_{m+n})$ of the join of $G_1$ and $G_2$ is potentially $K_{p+q}$-graphic.*

The following two results for simple graphs are due to Yin [14].

**Theorem 7** [14] *$\pi$ is potentially $\overline{A}_{l,m}$-graphic if and only if $\pi_l$ is graphic.*

**Theorem 8** [14] *Let $n \geq l + m$ and let $\pi = (d_1, d_2, \cdots, d_n)$ be a non-increasing graphic sequence. If $d_{l+m} \geq 2l + m - 2$, then $\pi$ is potentially $\overline{A}_{l,m}$-graphic.*

A condition for a graphic sequence $\pi$ to be potentially $K_4 - e$ graphic can be found in [11], where $K_4 - e$ is the graph obtained from the complete graph $K_4$ by deleting one edge $e$.

## 2 Bounds on the sum of squares of degrees of a multigraph

From the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n} |a_i b_i| \leq \Big( \sum_{i=1}^{n} |a_i|^2 \Big)^{\frac{1}{2}} \Big( \sum_{i=1}^{n} |b_i|^2 \Big)^{\frac{1}{2}},$$

Taking $a_i = d_i$ and $b_i = 1$, we have $\big( \sum\limits_{i=1}^{n} d_i \big)^2 \leq n \sum\limits_{i=1}^{n} d_i^2$ which implies $\frac{1}{n}\big( \sum\limits_{i=1}^{n} d_i \big)^2 \leq \big( \sum\limits_{i=1}^{n} d_i^2 \big)$. From this and the hand shaking Lemma $\sum\limits_{i=1}^{n} d_i = 2|E|$, we have $\frac{4|E|^2}{n} = \frac{1}{n}\big( \sum\limits_{i=1}^{n} d_i \big)^2 \leq \sum\limits_{i=1}^{n} d_i^2$.

Now we have the following observation, the proof is by using the same argument as in Theorem 1 of [1].

**Lemma 9** *For an $r$-graph $G$, $\sum\limits_{i=1}^{n} d_i^2 \leq |E|\big(r(n-2) + \frac{2|E|}{n-1}\big)$.*

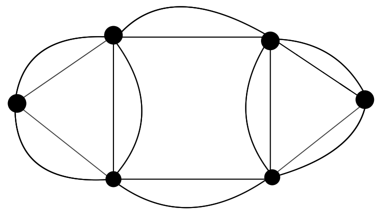

Figure 1: A 2-graph

**Remark 10** *From Lemma 9, we observe that*

$$\frac{4|E|^2}{n} \le \sum_{i=1}^{n} d_i^2 \le |E|\big(r(n-2) + \frac{2|E|}{n-1}\big).$$

The following example shows that the equality does not hold in the above inequality.

**Example 11** *Consider the* 2*-graph as shown in Figure 1.*

Here, $\frac{4|E|^2}{n} = \frac{4\times 16^2}{6} = \frac{512}{3} < 4^2 + 6^2 + 6^2 + 6^2 + 6^2 + 4^2 = 176 < 16\big(2(6-2) + \frac{2\times 16}{6-1}\big) = \frac{1152}{5}$.

Now, we have the following result.

**Lemma 12** *A multigraph* $G$ *is regular if and only if* $\frac{4|E|^2}{n} = \sum\limits_{i=1}^{n} d_i^2$.

**Proof.** Suppose an $r$-graph $G$ is regular of degree $b$. Then $2|E| = nb$ and $d_i = b$ for all $i = 1, 2, \ldots, n$. We know that $\sum\limits_{i=1}^{n} d_i^2 = nb^2$ and $\frac{4|E|^2}{n} = \frac{1}{n} 4 \frac{1}{4} n^2 b^2 = nb^2$. These together give $\sum\limits_{i=1}^{n} d_i^2 = \frac{4|E|^2}{n}$.

Conversely, suppose that $\sum\limits_{i=1}^{n} d_i^2 = \frac{4|E|^2}{n}$. Then $\frac{4}{n}|E|^2 = \sum\limits_{i=1}^{n} d_i^2$. This implies that $\frac{1}{n}(d_1^2 + d_2^2 + \ldots + d_n^2 + 2(d_1 d_2 + d_1 d_3 + \ldots + d_1 d_n) + \ldots + 2(d_{n-2} d_{n-1} + d_{n-2} d_n) + 2(d_{n-1} d_n)) - (d_1^2 + d_2^2 + \ldots + d_n^2) = 0$, which on simplification gives

$\frac{1}{n}\bigg((d_1-d_2)^2+(d_1-d_3)^2+\ldots+(d_1-d_n)^2+(d_2-d_3)^2+(d_2-d_4)^2+\ldots+$
$(d_2-d_n)^2+\ldots+(d_{n-1}-d_n)^2\bigg)=0$. From this, we see that each term on left side is non-negative for every $i$, $j$ and right side is equal to zero. Therefore the above equation is possible when $d_i=d_j$ for every $i$, $j=1,2,\ldots,n$ and hence G is a regular graph. □

Now, we have the following observation.

**Lemma 13** *Let* $G$ *be an* $r$*-graph with* $n>2$ *vertices. Then* $G$ *is a complete graph* $K_n^r$ *if and only if* $\frac{4|E|^2}{n}=\sum\limits_{i=1}^{n}d_i^2=|E|\big(r(n-2)+\frac{2|E|}{n-1}\big)$.

**Proof.** First we note that an $r$-graph G is a complete $r$-graph if and only if $|E|=\frac{1}{2}rn(n-1)$. Moreover, we know that $|E|=\frac{1}{2}nr(n-1)$, which implies that $2|E|(n-2)+2|E|n=nr(n-1)(n-2)+2|E|n$ and on simplication gives $\frac{4|E|^2}{n}=|E|\big(r(n-2)+\frac{2|E|}{n-1}\big)$. Thus the result follows. □

The following result partially answers the question raised in Remark 10.

**Theorem 14** *A bipartite multigraph* $G=K_{l,m}^{(r)}$*, where* $m>1$*, is an* $r$*-star graph* $K_{1,n-1}^{(r)}$ *if and only if* $\sum\limits_{i=1}^{n}d_i^2=|E|\big(r(n-2)+\frac{2|E|}{n-1}\big)$.

**Proof.** Let $K_{l,m}^{(r)}$ be an r-complete bipartite graph, where $m>1$, $n=l+m$ and $|E|=rlm$. There are $l$-vertices each of whose degree is $r\times m$ and $m$ vertices each of whose degree is $r\times l$, so $\sum\limits_{i=1}^{n}d_i^2=l(rm)^2+m(rl)^2=lr^2m^2+mr^2l^2=r^2(lm^2+ml^2)=r^2lm(l+m)$. Therefore, we have $|E|\big(r(n-2)+\frac{2|E|}{n-1}\big)=rlm\big(r(l+m-2)+\frac{2rlm}{n-1}\big)=r^2lm\big(\frac{l^2+m^2+4lm-3l-3m+2}{l+m-1}\big)$. Therefore, $r^2lm(l+m)=r^2lm\big(\frac{l^2+m^2+4lm-3l-3m+2}{l+m-1}\big)$, which gives $l=1$. Hence the result follows. □

## 3 Potentially r-graphic sequences

**Definition 15** *Let* $\overline{S}_{r_1,s_1}^{(r)},\overline{S}_{r_2,s_2}^{(r)},\ldots,\overline{S}_{r_p,s_p}^{(r)}$ *be* $r$*-split graphs, respectively with* $r_1+s_1$*,* $r_2+s_2$*,…,* $r_p+s_p$ *vertices. Let* $L=\sum_{i=1}^{p}r_i$ *and* $M=\sum_{i=1}^{p}s_i$*. Then* *the* $p$*-tuple* $r$*-split graph, denoted by* $S^{(r)}_{L,M}$*, is the graph*

$$S^{(r)}_{L,M} = S^{(r)}_{\sum_{i=1}^{p} r_i,\ \sum_{i=1}^{p} s_i} = \overline{S}^{(r)}_{r_1,s_1} \vee \overline{S}^{(r)}_{r_2,s_2} \vee \ldots \vee \overline{S}^{(r)}_{r_p,s_p}.$$

Clearly $S^{r}_{L,M}$ has vertex set $\bigcup_{i=1}^{p} V(\overline{S}^{(r)}_{r_i,s_i})$ and the edge set consists of all edges of $\overline{S}^{(r)}_{r_1,s_1}, \overline{S}^{(r)}_{r_2,s_2}, \ldots, \overline{S}^{(r)}_{r_p,s_p}$ together with the edges joining each vertex of $\overline{S}^{(r)}_{r_i,s_i}$ with every vertex of $\overline{S}^{(r)}_{r_j,s_j}$ by exactly $r$-edges for every $i$, $j$ with $i \neq j$.

An $r$-graphic sequence $\pi$ is said to be potentially $S^{(r)}_{L+M}$-graphic if there exists a realization of $\pi$ containing $S^{(r)}_{L+M}$ as a subgraph. If $\pi$ has a realization $G$ containing $S^{(r)}_{L+M}$ on the $L+M$ vertices of highest degree in $G$, then $\pi$ is said to be potentially $A^{(r)}_{L+M}$-graphic.

Let $n \geq L+M$ and let $\pi = (d_1, \ldots, d_n)$ be a non-increasing sequence of non-negative integers with $d_L \geq r(L+M)-1$ and $d_{L+M} \geq rL$. We define sequences $\pi_1, \ldots, \pi_L$ as follows. Construct the sequence

$$\pi_1 = (d_2 - r, \ldots, d_L - r, d_{L+1} - r, \ldots, d_{L+M} - r, d^1_{L+M+1}, \ldots, d^1_n)$$

from $\pi$ by reducing 1 from the largest term that have not been already reduced $r$ times, and then reordering the last $n-L-M$ terms to be non-increasing. For $2 \leq i \leq r$, construct

$$\pi_i = (d_{i+1} - ir,\ \ldots,\ d_L - ir, d_{L+1} - ir,\ \ldots, d_{L+M} - ir,\ d^i_{L+M+1},\ \ldots, d^i_n)$$

from

$$\begin{aligned}\pi_{i-1} =&(d_i - (i-1)r, \ldots, d_L - (i-1)r, d_{L+1} - (i-1)r, \ldots,\\ &d_{L+M} - (i-1)r, d^{i-1}_{L+M+1}, \ldots, d^{i-1}_n)\end{aligned}$$

by deleting $d_i - (i-1)r$, reducing the first $d_i - (i-1)r$ remaining terms of $d_{i-1}$ by one that have not been already reduced $r$ times, and then reordering the last $n-L-M$ terms to be non-increasing.

We start with the following lemma.

**Lemma 16** *If* $\pi = (d_1, d_2, \ldots, d_n)$ *is the graphic sequence of* $S^{(r)}_{L,M}$, *then*

$$\pi = \left(\left(\sum_{i=1}^{m} r(r_i+s_i-1)\right)^{r_j}, \left(\sum_{i=1}^{m} rr_i + \sum_{i=1,\ i\neq j}^{m} rs_i\right)^{s_j}\right), \quad \text{for } \ j = 1, 2, \ldots, m.$$

**Proof.** To prove the result we use induction on $m$.
For $m = 1$, the result is obviously true. For $m = 2$, we have $S^{(r)}_{\sum_{i=1}^{2} r_i, \sum_{i=1}^{2} s_i}$.
Therefore for every $i = 1, 2, \ldots, r_1$ and $i = 1, 2, 3, \ldots = r_2$ and $j = 1, 2, 3, \ldots, s_1$ and $j = 1, 2, 3, \ldots, s_2$

$$\overline{d_i} = d_i + r(r_2 + s_2) \tag{1}$$

and

$$\overline{d_j} = r(r_1 + r_2 + s_2), \tag{2}$$

where $\overline{d_i}$ and $\overline{d_j}$ are respectively the degree of $\overline{v_i^{th}}$ and $\overline{v_j^{th}}$ vertex in $S_{r_1+r_2, s_1+s_2}$ and $d_i$ is the degree of $i^{th}$ vertex in $K_{r_1}$. Equations (1) and (2) hold for every i, j. Thus the graphic sequence $\pi^2$ of $S_{r_1+r_2,\ s_1+s_2}$ is

$$\begin{aligned}\pi^2 = &\Bigg(\Big(r(r_1+s_1-1) + r(r_2+s_2\Big)^{r_1}, \Big(r(r_1+s_1-1)+r(r_2+s_2\Big)^{r_2},\\ &\Big(r(r_1+r_2+s_2)\Big)^{s_1}, \Big(r(r_1+r_2+s_2)\Big)^{s_2}\Bigg)\\ = &\left(\left(\sum_{i=1}^{2} r(r_i+s_i-1)\right)^{r_j}, \left(\sum_{i=1}^{m} rr_i + \sum_{i=1,\ i\neq j}^{m} rs_i\right)^{r_j}\right), \quad \text{for } \ j = 1, 2.\end{aligned}$$

This shows that the result is true for $m = 2$. Assume that the result holds for $m = k-1$, therefore for all $j = 1,\ 2,\ \cdots,\ k-1$,

$$\pi^{k-1} = \left(\left(\sum_{i=1}^{k-1} r(r_i+s_i-1)\right)^{r_j}, \left(\sum_{i=1}^{k-1} rr_i + \sum_{i=1,\ i\neq j}^{k-1} rs_i\right)^{r_j}\right), \quad \text{for } \ j = 1, 2.$$

Now for $m = k$,

$$\begin{aligned} G &= S^{(r)}_{r_1,s_1} \vee S^{(r)}_{r_2,s_2} \vee \ldots \vee S^{(r)}_{r_{k-1},s_{k-1}} \vee S^{(r)}_{r_k,s_k}\\ &= A \vee S^{(r)}_{r_k,s_k}, \qquad \text{where} \quad A = S^{(r)}_{r_1,s_1} \vee S^{(r)}_{r_2,s_2} \vee \ldots \vee S^{(r)}_{r_{k-1},s_{k-1}}.\end{aligned}$$

Since the result is proved for all $m = k-1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that result holds for $m = k$ also. That is,

$$\pi = \left(\left(\sum_{i=1}^{k} r(r_i + s_i - 1)\right)^{r_j}, \left(\sum_{i=1}^{k} rr_i + \sum_{i=1,\ i\neq j}^{k} rs_i\right)^{s_j}\right), \quad \text{for } \ j = 1, 2, \dots, k$$

This proves the lemma. □

**Lemma 17** *A non-increasing integer sequence $\pi = (d_1, \dots, d_n)$ is potentially $A^{(r)}_{L,M}$-graphic if and only if it is potentially $S^{(r)}_{L,M}$-graphic.*

**Proof.** We only need to prove that if $\pi = (d_1, \dots, d_n)$ is potentially $S^{(r)}_{L,M}$-graphic, then it is potentially $A^{(r)}_{L,M}$-graphic. We choose a realization $G$ of $\pi$ with vertex set $V(G) = \{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$, the induced $r$-subgraph $G[\{v_1, \dots, v_{L+M}\}]$ of $\{v_1, \dots, v_{L+M}\}$ in $G$ contains $S^{(r)}_{L,M}$ as its $r$-subgraph and $|V(K^{(r)}_L) \cap \{v_1, \dots, v_L\}|$ is maximum. Denote $H = G[\{v_1, \dots, v_{L+M}\}]$. If $|V(K^{(r)}_L) \cap \{v_1, \dots, d_L\}| = L$, that is, $V(K^{(r)}_L) = \{v_1, \dots, v_L\}$, then $\pi$ is potentially $A^{(r)}_{L,M}$-graphic. Assume that $|V(K^{(r)}_L) \cap \{v_1, \dots, v_L\}| < L$. Then there exists $v_i \in \{v_1, \dots, v_L\} \setminus V(K^{(r)}_L)$ and a $v_j \in V(K^{(r)}_L) \setminus \{v_1, \dots, v_L\}$. Let $A = N_H(v_j) \setminus (\{v_i\} \cup N_H(v_i))$ and $B = N_G(v_i) \setminus (\{v_j\} \cup N_G(v_j))$. Since $d_G(v_i) \geq d_G(v_j)$, we have $|B| \geq |A|$. Let $C$ be any subset of $B$ such that $|C| = |A|$. Now form a new realization $G'$ of $\pi$ by a sequence of $r$-exchanges to the $r$-edges of the star centralized at $v_j$ with end vertices in $A$ with the non $r$-edges of the star centralized at $v_j$ with end vertices in $C$, and by a sequence of $r$-exchange the $r$-edges of the star centralized at $v_i$ with end vertices in $C$ with the non $r$-edges of the star centralized at $v_i$ with end vertices in $A$. It is easy to see that $G'$ contains $S^{(r)}_{L,M}$ on $\{v_1, \dots v_{L+M}\}$ so that $|V(K^{(r)}_L) \cap \{v_1, \dots, v_L\}|$ is larger than that of $G$, which contradicts to the choice of $G$. □

We use the Havel-Hakimi procedure to test whether or not an $r$-graphic sequence $\pi$ is potentially $A^{(r)}_{L,M}$-graphic.

**Theorem 18** *For $r \geq 1$ and $n \geq 1$, an $r$-graphic sequence $\pi = (d_1, \dots, d_n)$ is potentially $A^{(r)}_{L,M}$-graphic if and only if $\pi_L$ is $r$-graphic.*

**Proof.** Assume that $\pi$ is potentially $A^{(r)}_{L,M}$-graphic. Then $\pi$ has a realization $G$ with the vertex set $V(G) = \{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$ for $(1 \leq i \leq n)$

and $G$ contains $S^{(r)}_{L,M}$ on the vertices $v_1, \dots, v_{L+M}$, where $L + M \le n$, so that $V^{(r)}(K_L) = \{v_1, \dots, v_L\}$ and $V(\overline{K}^{(r)}_M) = \{v_{L+1}, \dots, v_{L+M}\}$. By applying a sequence of $r$-exchanges to $G$ in order we will show that there is one such realization $G'$ such that $G' \setminus v_1$ has degree sequence $\pi_1$. If not, we may choose such a realization $H$ of $r$-graphic sequence $\pi$ such that the number of vertices adjacent to $v_1$ in $\{v_{L+M+1}, \dots, v_{d_1+1}\}$ is maximum. Let $v_i \in \{v_{L+M+1}, \dots, v_{d_1+1}\}$ and assume that there is no edge between $v_1$ and $v_i$ and let $v_j \in \{v_{d_1+2}, \dots, v_n\}$ and there are $r$ edges between $v_1$ and $v_j$. We may assume that $d_i > d_j$. Hence there is a vertex $v_t, t \ne i, j$ such that there are $r$ edges between $v_i$ and $v_t$ and no edge between $v_j$ and $v_t$. Clearly $G = \left(H \setminus \{v_1^{(r)} v_j, v_i^{(r)} v_t\}\right) \bigcup \{v_1^{(r)} v_i, v_j^{(r)} v_t\}$ (where $v_i^{(r)} v_j$ means that there are $r$ edges between $v_i$ and $v_j$) is a realization of $\pi$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$, $G$ contains $S^{(r)}_{L,M}$ on $v_1, \dots, v_{L+M}$ with $V^{(r)}(K_L) = \{v_1, \dots, v_L\}$ and $V(\overline{K}^{(r)}_M) = \{v_{L+1}, \dots, v_{L+M}\}$ and $G$ has the number of vertices adjacent to $v_1$ in $\{v_{L+M+1}, \dots, v_{d_1+1}\}$ larger than that of $H$. This contradicts the choice of $H$. Repeating this procedure, we can see that $\pi_i$ is potentially $A^{(r)}_{L-i}$-graphic successively for $i = 2, \dots, L$. In particular, $\pi_L$ is $r$-graphic.

Conversely, suppose that $\pi_L$ is $r$-graphic and is realized by a graph $G_L$ with a vertex set $V(G_L) = \{v_{L+1}, \dots, v_n\}$ such that $d_{G_L}(v_i) = d_i$ for $L + 1 \le i \le n$. For $i = L, L-1 \dots, 1$ form $G_{i-1}$ from $G_i$ by adding a new vertex $v_i$ that is adjacent to each of $v_{i+1}, \dots, v_{L+M}$ with $r$-edges and also to the vertices of $G_i$ with degrees $s^{i-1}_{L+M+1} - r, \dots, d^{i-1}_{d_i+1} - r$. Then for each $i$, $G_i$ has degrees given by $\pi_i$ and $G_i$ contains $S^{(r)}_{L-i,M}$ on $L+M-i$ vertices $v_{i+1}, \dots, v_{L+M}$ whose degrees are $d_{i+1} - ir, \dots, d_{L+M} - ir$ so that $V(K^{(r)}_{L-i}) = \{v_{i+1}, \dots, v_L\}$ and $V(\overline{K}^{(r)}_M) = \{v_{L+1}, \dots, v_{L+M}\}$. In particular, $G_0$ has degrees given by $\pi$ and contains $S^{(r)}_{L,M}$ on $L + M$ vertices $v_1, \dots, v_{L+M}$ whose degrees are $d_1, \dots, d_{L+M}$ so that $V(K^{(r)}_L) = \{v_1, \dots, v_L\}$ and $V(\overline{K}^{(r)}_M) = \{v_{L+1}, \dots, v_{L+M}\}$. Hence the result follows. □

The following is a sufficient condition for an $r$-graphic sequence to be potentially $A^{(r)}_{L,M}$-graphic.

**Theorem 19** *Let* $n \ge L + M$ *and let* $\pi = (d_1, \dots, d_n)$ *be an* $r$*-graphic sequence. If* $d_{L+M} \ge 2rL + rM - 2$*, then* $\pi$ *is potentially* $A^{(r)}_{L,M}$*-graphic.*

**Proof.** Let $n \ge L + M$ and let $\pi = (d_1, \dots, d_n)$ be a non-increasing $r$-graphic sequence with $d_{L+M} \ge 2rL + rM - 2$. By using the argument similar to Theorem 8, $\pi$ is potentially $K_L^{(r)}$-graphic and hence by Lemma 17, $A_L^{(r)}$-graphic. Therefore, we assume that $G$ is a realization of $\pi$ with a vertex set $V(G) = (v_1, \ldots, v_n)$ such that $d_G(v_i) = d_i$, $(1 \le i \le n)$ and $G$ contains $K_L^{(r)}$ on $(v_1, \ldots, v_L)$, that is, $V(K_L^{(r)}) = \{v_1, \ldots, v_L\}$ and

$$t = e_G(\{v_1, \ldots, v_{r_1}, \ldots, v_L\}, \{v_{L+1}, \ldots, v_{L+s_1}, \ldots, v_{L+M}\})$$

(that is, the number of edges between $\{v_1, \ldots, v_L\}$ and $\{v_{L+1}, \ldots, v_{L+M}\}$) is maximum. If $t = rLM + rs_1s_2 + s_j \sum_{i=1}^{j-1} rs_i$, for $j = 3, 4, \ldots, p$, then the cardinality of the edge set of $S_{L,M}^{(r)}$ is same as $t$ and therefore $G$ contains $S_{L,M}^{(r)}$ on the vertices $v_1, v_2, \ldots, v_{L+M}$ with $V^{(r)}(K_M) = \{v_1, v_2, \ldots, v_L\}$ and

$$V(\overline{K^{(r)}}_M) = \{v_{L+1}, v_{L+2}, \ldots, v_{L+s_1}, \ldots, v_{L+M}\}.$$

In other-words, $\pi$ is potentially $\overline{A}_{L,M}^{(r)}$-graphic. Assume that $t < \{rLM + rs_1s_2 + s_j \sum_{i=1}^{j-1} rs_i\}$, for $j = 3, 4, \ldots, p$. Then there exists a $v_k \in \{v_1, v_2, \ldots, v_{s_i}\}$ and $v_m \in \{v_{s_i+1}, v_{s_i+2}, \ldots, v_{s_i+s_j}\}, (i \ne j)$ such that $v_k^r v_m \notin E(G)$. Let

$$A = N_{G \setminus \{v_{s_i+1}, v_{s_i+2}, \ldots, v_{s_i+s_j}\}}(v_k) \setminus N_{G \setminus \{v_1, v_2, \ldots, v_{s_i}\}}(v_m)$$

and

$$B = N_{G \setminus \{v_{s_i+1}, v_{s_i+2}, \ldots, v_{s_i+s_j}\}}(v_k) \cap N_{G \setminus \{v_1, v_2, \ldots, v_{s_i}\}}(v_m).$$

Then $e_G(x, y) = r$ for $x \in N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)$ and $y \in N_{G\setminus\{v_1,\ldots,v_{L+M}\}}(v_k)$. Otherwise, if $e_G(x, y) < r$, then $G' = (G \setminus \{v^{(r)}y, v_m^{(r)}x\}) \cup \{v_k^{(r)}v_m, x^{(r)}y\}$ is a realization of $\pi$ and contains $\overline{S}_{L,M}^{(r)}$ on $v_1, \ldots, v_{L+M}$ with $V(K_L^{(r)}) = \{v_1, \ldots, v_L\}$ and $(\overline{K}_M^{(r)}) = \{v_{L+1}, \ldots, v_{L+M}\}$ such that

$$e_{G'}(\{v_1, \ldots, v_L\}, \{v_{L+1}, \ldots, v_{L+M}\}) > t,$$

which contradicts the choice of $G$. Thus $B$ is $r$-complete. We consider the following cases.

Let $A = \emptyset$. Then $2rL + rM - 2 \le d_k = d_G(v_k) < rL + rM - 1 + r|B|$, and so $|B| \ge rL$. Since each vertex in $N_{G\setminus v_1,\ldots,v_L}(v_m)$ is adjacent to each vertex in $B$ by $r$ edges and $|N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)| \ge 2rL + rM - 2 = rL + rM - 1$. It can be easily seen that the $r$ induced subgraph of $N_{G\setminus\{v_1,\ldots,v_L\}}(v_m) \cup \{v_m\}$ in $G$ contains $\overline{S}_{L,M}^{(r)}$

as a subgraph. Thus $\pi$ is potentially $\overline{A}_{L,M}^{(r)}$- graphic.

Let $A \neq \emptyset$. Let $a \in A$. If there are $x, y \in N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)$ such that $e_G(x,y) < r$ then $G' = (G \setminus \{v_m^{(r)}x, v_m^{(r)}y, v_k^{(r)}a\}) \cup \{v_k^{(r)}v_m, a^{(r)}v_m, x^{(r)}y\}$ is a realization of $\pi$ and contains $\overline{S}_{L,M}^{(r)}$ on $v_1, \ldots, v_{L+M}$ with $V(K_L^{(r)}) = \{v_1, \ldots, v_L\}$ and $V(\overline{K}_M^{(r)}) = \{v_{L+1}, \ldots, v_{L+M}\}$ such that $e_{G'}(\{v_1, \ldots, v_L\}, \{v_{L+1}, \ldots, v_{L+M}\}) > t$ which contradicts the choice of G. Thus $N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)$ is r-complete. Since

$$|N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)| \geq rL + rM - 1 \quad \text{and} \quad e_G(v_m, z) = r,$$

for any $z \in N_{G\setminus\{v_1,\ldots,v_L\}}(v_m)$, it is easy to see that the induced r-subgraph of $N_{G\setminus\{v_1,\ldots,v_L\}}(v_m) \cup \{v_m\}$ in G is r-complete, and so contains $\overline{S}_{L,M}^{(r)}$ as a r-subgraph. Thus $\pi$ is potentially $\overline{A}_{L,M}^{(r)}$-graphic. □

**Theorem 20** *If* $\pi = (d_1, d_2, \ldots, d_n)$ *is an* r*-graphic sequence such that* $\sigma(\pi)$ *is at least* $(n^2 - 3n + 8)r$*, then* $\pi$ *is potentially* $K_4^{(r)}$*-graphic.*

**Proof.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be an r-graphic sequence such that $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ and $\sigma(\pi) = (n^2 - 3n + 8)r$. Suppose G is a graphical realization of $\pi$ and $e(G)$ is the size of G. Then $2e(G) = \sigma(\pi)$ and $2e(G^c) = nb(n-1) - \sigma(\pi) = nr(n-1) - (n^2 - 3n + 6)r = r(2n - 6)$, so that $e(G^c) = r(n-3)$, where $G^c$ is the complement of the r-graph G. An extremal problem is r-graph G is obtained by deleting $r(n-3)$ independent edges from the complete r-graph $K_n^{(r)}$ of order n. Hence the largest vertex number of independent sets in $G^c$ is 3. This implies that the largest possible complete r-subgraph of G is of order 3. As $1 \leq n - 3 \leq 3$. Hence there is no complete r-subgraph of order 4 in G. Therefore, we have

$$\sigma(K_4^{(r)}, n) \geq (n^2 - 3n + 6)r + 2r = (n^2 - 3n + 8)r$$

Now Suppose that $\pi = (d_1, d_2, \ldots, d_n)$ is r-graphic sequence with $d_1 \geq d_2 \geq \ldots \geq d_n \geq r$ and $\sigma(\pi) \geq (n^2 - 3n + 8)r$. Then every graphical realization G of $\pi$ is obtained by removing at most $r(n-4)$ edges from the r-complete graph $K_n^{(r)}$. Hence the maximal complete subgraph of G has order at least $n - (n-4) = 4$. Thus G is potentially $K_4^{(r)}$. In other words,

$$\sigma(K_4^{(r)}, n) \leq (n^2 - 3n + 8)r \tag{3}$$

Combining (3) and (4), the result follows. □

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