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Induced label graphoidal graphs

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Abstract. Let G be a non-trivial, simple, finite, connected and undirected graph of order n and size m. An induced acyclic graphoidal decomposition ($IAGD$) of G is a collection ψ of non-trivial and internally disjoint induced paths in G such that each edge of G lies in exactly one path of ψ. For a labeling $f : V \to \{1, 2, 3, \ldots, n\}$, let $\uparrow G_f$ be the directed graph obtained by orienting the edges uv of G from u to v, provided $f(u) < f(v)$. If the set ψ_f of all maximal directed induced paths in $\uparrow G_f$ with directions ignored is an induced path decomposition of G, then f is called an induced graphoidal labeling of G and G is called an induced label graphoidal graph. The number $\eta_{il} = \min\{|\psi_f| : f \text{ is an induced graphoidal labeling of } G\}$ is called the induced label graphoidal decomposition number of G. In this paper we introduce and study the concept of induced graphoidal labeling as an extension of graphoidal labeling.

1 Introduction

By a graph $G = (V, E)$, we mean a non-trivial, simple, finite, connected and undirected graph. The order and size of G are denoted by n and m respectively. For terms not defined here we refer to [7].

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A *decomposition* of a graph G is a collection ψ of its subgraphs such that every edge of G lies in exactly one member of ψ. The significance of graph decomposition problems arise from the rigor of their theoretical treatise as well as their application to a variety of fields such as coding theory, bio-informatics and various types of networks. Among the decomposition problems explored by researchers we have path decomposition introduced by Harary [8], where each element of ψ is a path and various types of path decompositions like unrestricted path cover [9], graphoidal cover [1], simple graphoidal cover [4] and so on. A detailed review of the results pertaining to graphoidal decomposition along with an array of open problem is given in [3].

Definition 1 [1] *A* graphoidal decomposition (GD) *of a graph* G *is a collection* ψ *of non-trivial paths and cycles of* G *such that*

(i) Every vertex of G *is an internal vertex of at most one member of* ψ.

(ii) Every edge of G *is in exactly one member of* ψ.

Note that by an internal vertex of a path P we mean the vertices of P other than its end vertices. A *GD* wherein no member is a cycle is called an acyclic graphoidal decomposition (*AGD*) which was introduced by Arumugam and Suresh Susheela [6]. By demanding the members of an *AGD* to be induced paths, Arumugam [2] coined another variation namely *induced acyclic graphoidal decomposition (IAGD)*. The minimum cardinality of the respective graphoidal decompositions are denoted by η, η_a and η_{ia}. The study of the parameter η_{ia} initiated by Ratan Singh and Das [10] was further extended by I. Sahul Hamid and M. Joseph [13].

Linking graphoidal decompositions and vertex labeling, Acharya and Sampathkumar [1] conceptualized the idea of graphoidal labeling as follows.

Definition 2 [1] *Let* $G = (V, E)$ *be a graph with* n *vertices and let* $f : V \to \{1, 2, \ldots, n\}$ *be a labeling of the vertices of* G. *Orient the edges* uv *from* u *to* v *provided* $f(u) < f(v)$. *Such an orientation is called a* low-to-high *orientation of* G *with respect to the given labeling* f. *By* $\uparrow G_f$ *we mean, G together with the labeling* f *with respect to which the edges of* G *are oriented from low -to- high. Let* $\pi^*(\uparrow G_f)$ *be the set of all maximal directed paths in* $\uparrow G_f$ *and* $\pi(\uparrow G_f)$ *be the set of all members of* $\pi^*(\uparrow G_f)$ *with directions ignored. We say that* f *is a* graphoidal labeling *of* G *if* $\pi(\uparrow G_f)$ *is a graphoidal decomposition of* G *and if* G *admits such a labeling* f *of its vertices, then* G *is called a* label graphoidal graph (LGG).

If G is a label graphoidal graph with a graphoidal labeling f, then the graphoidal decomposition $\pi(\uparrow G_f)$ is called the *label graphoidal decomposition of* G *with respect to the labeling* f and is denoted by ψ_f.

The following result due to Acharya and Sampathkumar [1] gives a complete characterization of label graphoidal graphs.

Recall that a vertex of a directed graph with in-degree zero is called a *source* and a vertex of out-degree zero is called a *sink*.

Theorem 3 [1] *Suppose* G *has a graphoidal labeling* f. *Then any vertex* v *of* G *with degree* > 2 *is either a sink or a source in* $\uparrow G_f$.

Theorem 4 [1] *A graph* G *is label graphoidal if and only if every odd cycle in* G *has a vertex of degree 2.*

Motivated by the observation that label graphoidal decompositions of a given graph may vary according to the nature of the graphoidal labeling, Arumugam and Sahul Hamid [5] defined the concept of label graphoidal decomposition number and obtained some fundamental results.

Definition 5 [5] *Let* G *be a label graphoidal graph. The* label graphoidal decomposition number $\eta_l(G)$ *is defined to be* $\eta_l(G) = \min\{|\psi_f| : f \text{ is a graphoidal labeling of } G\}$.

The following result obtained by Arumugam and Sahul Hamid [5] gives the value of η_l for a tree T in terms of its size and the number of vertices of degree 2.

Theorem 6 [5] *Let* T *be a tree with* b *vertices of degree 2. Then* $\eta_l(T) = m - b$.

2 Induced label graphoidal graphs

In this section we introduce the concept of induced label graphoidal graph and investigate its properties.

Definition 7 *Let* $G = (V, E)$ *be a graph of order* n *and size* m *and* $f : V \to \{1, 2, \ldots, n\}$ *be a labeling of the vertices of* G. *Let* $\uparrow G_f$ *be the* oriented graph *obtained by orienting the edges* uv *from* u *to* v*, provided* $f(u) < f(v)$ *(this orientation given to the edges of* G *is called a* low-high *orientation). If the set* ψ_f *of all maximal directed induced paths in* $\uparrow G_f$ *with directions ignored is an induced path decomposition of* G*, then* f *is said to be an induced graphoidal labeling of* G *and* G *is called an induced label graphoidal graph (ILGG).*

Example 8 *Consider the cycle* $C_n = (v_1, v_2, \ldots, v_n, v_1)$ *where* $n \geq 4$*. Define* $f : V \to \{1, 2, \ldots, n\}$ *by*

$$f(v_1) = 2,$$
$$f(v_2) = 1,$$
$$f(v_i) = i \text{ when } 3 \leq i \leq n.$$

Then $\psi_f = \{(v_2, v_1, v_n), (v_2, v_3, \ldots, v_n)\}$ *is an induced decomposition of* G*, proving that* f *is an induced graphoidal labeling and thus* G *is an induced label graphoidal graph. Certainly, triangles are not induced label graphoidal graphs and thus the cycle* C_n *is an induced label graphoidal graph only when* $n \geq 4$*.*

Lemma 9 *If* f *is an induced graphoidal labeling of a graph* G*, then any vertex of* G *with degree greater than two is either a sink or a source in* $\uparrow G_f$*.*

Proof. Suppose f is an induced graphoidal labeling of G. Assume if possible, that there exists a vertex u in G with $\deg u > 2$ such that both in-degree and out-degree of u in $\uparrow G_f$ are positive. Consider three neighbors , say x, y and z of the vertex u. Then the orientation of the edges xu, yu and zu will be one of the following.

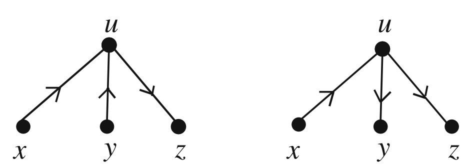

Figure 1

Concerning the first case, ψ_f will have a path P in which (x, u, z) is a section and similarly, there exists a path Q in ψ_f such that (y, u, z) is a section. Hence ψ_f is no longer a decomposition as the edge yu lies both in P and in Q. We will arrive at a similar contradiction in the later case as well. Hence every vertex of G with degree greater than two is either a sink or a source in $\uparrow G_f$. □

Theorem 10 *If* f *is an induced graphoidal labeling, then* ψ_f *is an induced acyclic graphoidal decomposition.*

Proof. Suppose f is an induced graphoidal labeling of G. By definition ψ_f is an induced path decomposition of G and moreover no member of ψ_f is a cycle, which implies that ψ_f is an induced acyclic path decomposition of G. Further it follows from Lemma 9 that every vertex v of G having degree at least three is either a source or a sink in $\uparrow G_f$ so that the vertex v is exterior to ψ_f. Hence every vertex v in G is an internal vertex of at most one path in ψ_f. Thus ψ_f is an induced acyclic path decomposition of G such that every vertex of G is an internal vertex of at most one path in ψ_f and so ψ_f is an *IAGD* of G. $\square$

Remark 11 *Since ψ_f is an induced acyclic graphoidal decomposition of a graph G when f is an induced graphoidal labeling, ψ_f in particular is a graphoidal decomposition of G. That is,* induced label graphoidal graphs are label graphoidal graphs*. But a label graphoidal graph need not be induced (for example consider the triangle). At the same time there are label graphoidal graphs which are induced label graphoidal as well while a labeling which served as a graphoidal labeling need not necessarily be an induced graphoidal labeling of the graph.*

For example, the labeling of cycle C_4 as in Figure 2 is a graphoidal labeling, but not an induced graphoidal labeling.

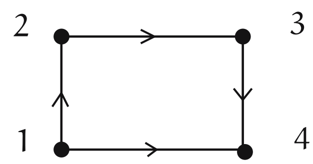

Figure 2

Theorem 12 *Induced label graphoidal graphs are triangle-free.*

Proof. If G is a graph of order n having a triangle $C = (u, v, w, u)$, then under any labeling $f : V(G) \to \{1, 2, \ldots, n\}$ of G, the orientation of the edges of C will be one of the forms given in Figure 3.

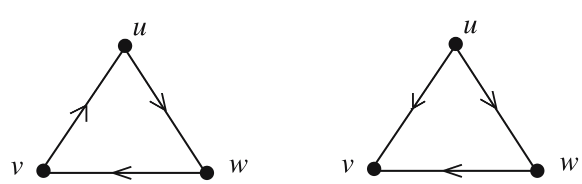

Figure 3

In the first case the path (u, w, v) is a section of a path in ψ_f and consequently ψ_f cannot be an induced decomposition. Similarly in the second case also ψ_f contains a path having (u, w, v) as its section. Hence ψ_f will no longer be an induced graphoidal decomposition of G so that G is not induced label graphoidal. □

Corollary 13 *Complete graphs and wheels are not induced label graphoidal.*

Theorem 14 *Bipartite graphs are induced label graphoidal.*

Proof. Let G be a bipartite graph with the bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$. Define $f : V(G) \to \{1, 2, \ldots, (r+s)\}$ by $f(x_i) = i;\ \mathtt{for\ all}\ i = 1, 2, \ldots, r$ and $f(y_j) = (j+r);\ \mathtt{for\ all}\ j = 1, 2, \ldots, s$. Then every vertex of X is a source and that of Y is a sink in $\uparrow G_f$ so that $\psi_f = E(G)$; which of course is an induced graphoidal decomposition and thus f is an induced graphoidal labeling of G. □

Corollary 15 *Every label graphoidal graph with $\delta \geq 3$ is an induced label graphoidal graph.*

Proof. Let G be a label graphoidal graph with $\delta \geq 3$. Then Theorem 4 implies that G does not contain any odd cycle. Hence G is bipartite and so the result follows from Theorem 14. □

Theorem 16 *Induced subgraph of an induced label graphoidal graph is induced label graphoidal.*

Proof. Let G be an induced label graphoidal graph and let H be an induced subgraph of G. Let f be an induced graphoidal labeling of G. Let g be the labeling of H obtained from f by assigning the label 1 to the vertex of H which receives the minimum among the labels of the vertices of H under f and using the label 2 to the vertex receiving the next minimum on the label under f and so on. We claim that g is an induced graphoidal labeling of H. Since H is an induced subgraph of G, any member of ψ_g is either a member of ψ_f or a section of a member of ψ_f and so each member in ψ_g is an induced path in H. Hence ψ_g is an induced graphoidal decomposition of H and consequently g is an induced graphoidal labeling of H. □

Remark 17 *It follows from Theorem 16 that a graph which is not induced label graphoidal cannot be an induced subgraph of an induced label graphoidal graph. Since triangles are not induced label graphoidal graphs, any induced label graphoidal graph is triangle free which in fact is Theorem 12.*

The following theorem completely characterizes the induced label graphoidal graphs.

Theorem 18 *A graph* G *is induced label graphoidal if and only if* G *is triangle-free such that every odd cycle of* G *contains a vertex of degree 2.*

Proof. Suppose G is an induced label graphoidal graph. Remark 11 says then that G is a label graphoidal graph and so the required conditions follow from Theorem 12 and Theorem 4. Therefore, we need to verify only the converse.

Let G be a triangle-free graph such that every odd cycle in G contains a vertex of degree two. Suppose G does not contain any odd cycle. Then G is a bipartite graph and hence by Theorem 14, G is induced label graphoidal.

Now assume that G contains an odd cycle. Consider the collection $B = \{b_1, b_2, \ldots, b_k\}$ of vertices of degree two in G with minimum cardinality whose removal results in a graph with no odd cycles. Let $H = < V(G) \setminus B >$. Then H is a connected graph without any odd cycles as the vertices of degree two lying on a cycle of G cannot be cut-vertices of G. Hence H is a bipartite graph. Let (X, Y) be the bipartition of H with $X = \{x_1, x_2, \ldots, x_{n_1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n_2}\}$. Now, define a labeling $f : V(G) \to \{1, 2, \ldots, n\}$ where $n = n_1 + n_2$ as follows.

$$f(x_i) = i, \text{ for all } i = 1, 2, \ldots, n_1$$
$$f(b_i) = n_1 + i \text{ for all } i = 1, 2, \ldots, k \text{ and}$$
$$f(y_i) = n_1 + k + i \text{ for all } i = 1, 2, \ldots, n_2.$$

Then one can observe that in $\uparrow G_f$ the vertices in X are sources and the vertices in Y are sinks. Also, every vertex in B is adjacent from exactly one vertex of X and adjacent to exactly one vertex of Y. Hence ψ_f consists of the edges joining a vertex of X and that of Y along with the paths of length two connecting a vertex in X and a vertex of Y having a vertex of B as an internal vertex. Now, as G is triangle-free, the initial and terminal vertices of a path of length two in ψ_f are not adjacent which implies that such a path in ψ_f is an induced path in G. Thus ψ_f is an *IAGD* of G proving that f is an induced graphoidal labeling of G. □

Corollary 19 *A label graphoidal graph* G *is induced label graphoidal if and only if* G *is triangle-free.*

Proof. Follows from Theorem 4 and Theorem 18. □

3 Induced label graphoidal decomposition number

For a graph G admitting a graphoidal labeling f, the labeling function f need not necessarily be unique; which in turn implies that G possesses more than one graphoidal decomposition. Motivated by this observation, Arumugam and Sahul Hamid [5] introduced the concept of label graphoidal decomposition number. We extend this notion further with respect to induced graphoidal decompositions.

Definition 20 *Let G be an induced label graphoidal graph. Then the* induced label graphoidal decomposition number $\eta_{il}(G)$ *is defined to be*

$$\eta_{il}(G) = \min\{|\psi_f| : f \text{ is an induced graphoidal labeling of } G\}.$$

Example 21 *(i) A graph G is given in Figure 4 with two different labelings. Observe that the two different labelings of the graph G yield two different induced acyclic graphoidal decompositions. For the first labeling f_1 we have $\psi_{f_1} = E(G)$ whereas in the second case the decomposition corresponding to the given labeling f_2 is given by $\psi_{f_2} = \{(1,2),(1,4,6),(3,6),(1,5,6)\}$. However, it can easily be verified that $\eta_{il}(G) = 4$.*

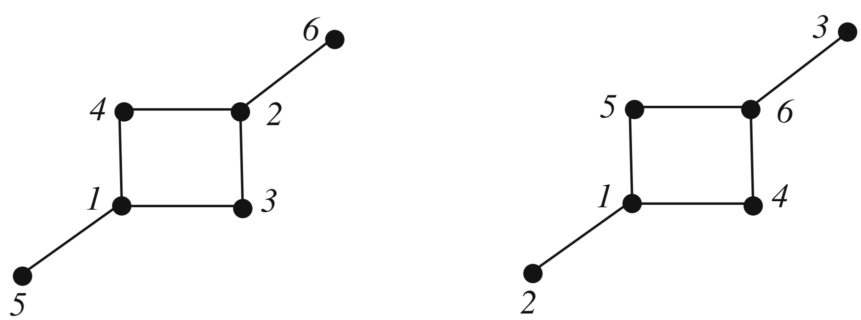

Figure 4

(ii) It follows immediately from the definition that $\eta_{il}(C_n) = 2$ for all $n \geq 4$ and $\eta_{il}(P_n) = 1$ for all $n \geq 2$.

(iii) If G is a graph with $\delta(G) \geq 3$, by Lemma 2.3 every vertex of G is either a sink or a source in $\uparrow G_f$ under any induced graphoidal labeling f of G and consequently $\eta_{il}(G) = m$.

(iv) *Consider the complete bipartite graph* $K_{r,s}$ *where* $2 \leq r \leq s$*, and let* (X, Y) *be the bipartition where* $X = \{x_1, x_2, \ldots, x_r\}$ *and* $Y = \{y_1, y_2, \ldots, y_s\}$*. If* $r \geq 3$*, then* $\delta(K_{r,s}) = r \geq 3$ *and so* $\eta_{il}(K_{r,s}) = rs$ *as observed above. When* $r = 2$*, define a labeling* $f : V(K_{r,s}) \to \{1, 2, \ldots, (r+s)\}$ *by*

$$f(x_1) = 1$$
$$f(x_2) = s + 2 \text{ and}$$
$$2 \leq f(y_j) \leq (s+1), \text{for each } j \text{ where } 1 \leq j \leq s.$$

Certainly, f *is an induced graphoidal labeling of* $K_{r,s}$ *with* $\psi_f = \{(x_1, y_i, x_2) : 1 \leq i \leq s\}$ *and hence* $\eta_{il}(K_{r,s}) \leq |\psi_f| = s$*. As for any induced graphoidal labeling* f *of* $K_{2,s}$*, the induced graphoidal decomposition* ψ_f *will consist of just edges and paths of length two, which implies that* $\eta_{il}(K_{2,s}) \geq s$*. Thus we have*

$$\eta_{il}(K_{r,s}) = \begin{cases} s & \text{if } r = 2 \\ rs & \text{else} \end{cases}$$

(v) *As any path in a tree* T *is an induced path, from Theorem 6 we have* $\eta_{il}(T) = \eta_l(T) = m - b$*, where* b *denotes the number of vertices of degree 2 in* T.

Remark 22 *It is possible to have two different labelings giving rise to the same minimum induced label graphoidal decomposition. For example, consider the graph* G *given in Figure 5(a) and the labeling* f_1 *and* f_2 *of* G *as given in Figures 5(b) and 5(c) respectively. Certainly* $|\psi_{f_1}| = |\psi_{f_2}| = 7 = \eta_{il}(G)$.

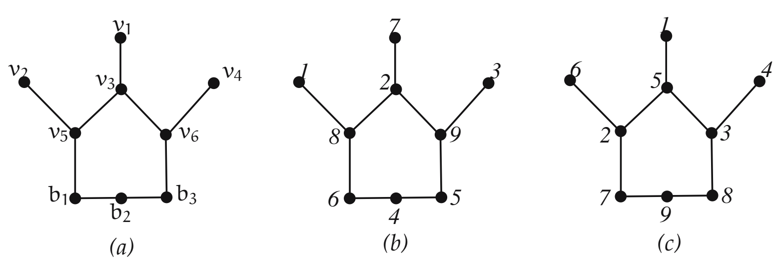

Figure 5

Let us now proceed to obtain some bounds for η_{il}. As seen earlier, by *internal vertices* of a path P, we mean the vertices of P other than its end vertices. For a graphoidal decomposition ψ of G, a vertex v is said to be *interior* to ψ if v is an internal vertex of an element of ψ and is called *exterior to* ψ otherwise.

Suppose G is an induced label graphoidal graph and let b denote number of vertices of degree 2. If f is an induced graphoidal labeling of G, we use b'_f to denote the number of vertices of degree 2 which are exterior to ψ_f. Let $b' = \min_f\{b'_f\}$ where the minimum is taken over all the induced graphoidal labeling f of G.

The following theorem which is useful to estimate the value of η_{il} for a given graph is analogous to a result for η_l given in [5].

Theorem 23 *Let* G *be an induced label graphoidal graph. Then* $\eta_{il}(G) = m - b + b'$.

Proof. Consider a labeling f of G with respect to which G has an induced graphoidal decomposition. Let ψ_f be the corresponding induced graphoidal decomposition. Then obviously, the interior vertices of all the elements of ψ_f are of degree 2. Therefore we have

$$\begin{aligned} m &= \sum_{P\in\psi_f} |E(P)| \\ &= \sum_{P\in\psi_f} (1 + \text{Number of vertices of degree 2 in } \psi_f) \\ &= |\psi_f| + b - b' \end{aligned}$$

so that we get $\eta_{il}(G) = m - b + b'$. □

Corollary 24 *For any induced graphoidal graph* G*,* $\eta_{il}(G) \geq m - b$*. Further, equality is obtained if and only if there exists an induced graphoidal labeling* f *of* G *such that every vertex of* G *with degree 2 is interior to* ψ_f*.*

Proof. The inequality follows from Theorem 23 as $b' \geq 0$. The rest follows from the fact that $\eta_{il} = m - b$ if and only if $b' = 0$. □

Although A graph g is 23 gives a formula to determine the value of η_{il} in terms of m and b', it is to be noted that determination of b' for a graph in general is not easy.

The following theorem gives a necessary condition for a graph G with $\eta_{il}(G) = m - b$.

Theorem 25 *Let* G *be an induced label graphoidal graph which is not a cycle. Then* $\eta_{il}(G) = m - b$ *implies that* G *is graph such that every cycle of* G *contains an even number of vertices of degree* ≥ 3 *of which at least one pair of vertices are non-adjacent.*

Proof. Suppose G is an induced label graphoidal graph with $\eta_{il} = m - b$. Then by virtue of Corollary 24, there exists an induced graphoidal labeling f such that every vertex of degree 2 is interior to ψ_f and of course $|\psi_f| = m - b$.

If G does not contain any cycle, then G is a tree and the result is trivial. Suppose G is not a tree and let C be a cycle in G. As any vertex of degree ≥ 3 is either a sink or a source in ψ_f and all the vertices of degree 2 are interior to it, it follows that the vertices on C with degree ≥ 3 must alternatively be sinks and sources. Hence C contains an even number of vertices of degree ≥ 3. Now, what we need to verify is that at least one pair of vertices on C with degree ≥ 3 are non-adjacent. If not, every pair of vertices on C with degree ≥ 3 are adjacent. As G is induced label graphoidal, it contains no triangles. Therefore, C contains exactly two vertices of degree ≥ 3, say u and v. Obviously, one of them, say u will be a source and the other is a sink. Then both the $u - v$ sections of C belong to ψ_f and at least one of them will not be induced which is a contradiction and hence the result. □

Lemma 26 *If G is an induced label graphoidal graph, there exists an induced graphoidal labeling f such that every vertex of degree 2 not lying on any cycle of G is interior to ψ_f.*

Proof. Let G be an induced label graphoidal graph. Let v_b be a vertex of degree 2 not lying on any cycle in G. If G is a tree, then the result follows from Remark 3.9 and Corollary 3.6. Suppose G is not a tree. Let w_1 and w_2 be vertices with degree ≥ 3 and nearest to v_b. Since the distance between w_1 and w_2 is > 1, it is possible to obtain a labeling f for which w_1 is a source and w_2 is a sink so that v_b is internal to ψ_f. □

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