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Coloring the nodes of a directed graph

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Abstract. It is an empirical fact that coloring the nodes of a graph can be used to speed up clique search algorithms. In directed graphs transitive subtournaments can play the role of cliques. In order to speed up algorithms to locate large transitive tournaments we propose a scheme for coloring the nodes of a directed graph. The main result of the paper is that in practically interesting situations determining the optimal number of colors in the proposed coloring is an NP-hard problem. A possible conclusion to draw from this result is that for practical transitive tournament search algorithms we have to develop approximate greedy coloring algorithms.

1 Introduction

Let $G = (V, E)$ be a finite simple graph, that is, G has finitely many nodes and G does not have any loop or double edge. A subgraph D is a clique in G if each two distinct nodes of D are connected in G. If the clique D has k nodes, then we say that D is a k-clique in G. The number of nodes of a clique sometimes referred as the size of the clique. A k-clique in G is a maximum clique if G does not have any $(k + 1)$-clique. The graph G may have several maximum

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cliques but their common size is a well defined number. This number is called the *clique number* of the graph G and it is denoted by $\omega(G)$. The problem of determining the clique number of a given graph is an important problem in many areas of applied discrete mathematics. (For a list of applications see [1].) It is known that the problem is NP-complete. (For proofs see [5] or [12].) The most commonly used clique search algorithms employ coloring of the nodes of a graph to speed up the computations. (See [2, 8, 9, 11, 14].) The coloring of the nodes of the graph G with k colors assigns exactly one color to each node of the graph such that adjacent nodes never receive the same color. This type of coloring of the nodes sometimes referred as legal or well coloring of the nodes. The minimum number of colors with which the nodes of G can be legally colored is a well defined number. It is called the *chromatic number* of G and it is denoted by $\chi(G)$. Determining the chromatic number of a given graph is another important problem in the applied discrete mathematics with many applications. It is known that the problem of deciding if a given graph can be colored with k color is NP-complete for any fixed k, where $k \geq 3$. (See [5] or [12].) The problem for $k = 2$ belongs to the P (polynomial) complexity class.

Let $G = (V, E)$ be a finite simple directed graph. This means that G has finitely many nodes and G does not have any loop or double edges directed in parallel manner. In the particular case when there is exactly one directed edge between any two distinct nodes, then G is called a *tournament*. Tournaments can look back to a venerable history. (See [13, 3, 4, 10].) The directed graph G is transitive if $(x, y), (y, z) \in E$ implies $(x, z) \in E$. Motivated by applications in information theory [7] introduced the problem of determining the size of a maximum subtournament in a given finite simple directed graph. Since coloring proved to be advantageous in improving the efficiency of clique search algorithms we address the problem if coloring can be exploited in the algorithms locating maximum transitive tournaments in a given graph. We propose a type of colorings of the nodes of a finite simple directed graph. We will show that this coloring leads to an NP-hard problem. A practical implication of the result is that for coloring we should rely on approximate greedy coloring procedures instead of trying to compute the optimal number of colors.

Colorings are employed in at least two ways in clique search algorithms. One can color the nodes of a graph G before the clique search starts. In these cases the coloring is used as a possible preprocessing or preconditioning tool. On the other hand if in the course of the clique search algorithm one recolors the nodes of the subgraphs of G under consideration, then we call it an on-line coloring.

It must be ample clear that the requirements for preconditioning or for on-line coloring algorithms are not necessarily the same. In the case of preconditioning the graph coloring is a well separated phase of the computation. We may afford to use more time and memory space. While in the case of on-line coloring we have to trade speed for the quality of coloring.

By the main result of the paper determining the optimal number of colors is not a practically recommended option. In the same time there is a real need to introduce, implement and test various greedy coloring algorithms. We hope that our paper will stimulate this activity.

2 Coloring the nodes

Let T be a tournament whose nodes are $a_1, \ldots, a_k$, where $k \geq 2$.

Lemma 1 *If T is a transitive tournament, then there is a permutation $b_1, \ldots, b_k$ of $a_1, \ldots, a_k$ such that*

(1) $(b_i, b_{i+1}), \ldots, (b_i, b_k)$ *are edges of T for each i, $1 \leq i < k$.*

(2) The listed $n(n-1)/2$ edges are all the edges of T.

(3) Each subgraph of T is a transitive tournament.

Proof. Statement (1) clearly holds for $k = 2$. We assume that $k \geq 3$ and start an induction on k.

If T has a vertex, say a_1, such that each edge incident to a_1 goes out of a_1, then a_1 can be identified with b_1 and the inductive assumption is applicable to the graph whose nodes are $a_2, \ldots, a_k$.

If T has a vertex, say a_k such that each edge incident to a_k goes into a_k, then a_k can be identified with b_k and the inductive assumption is applicable to the graph whose nodes are $a_1, \ldots, a_{k-1}$.

For the remaining part of the proof of statement (1) we may assume that each vertex of T has an incident edge going in and has an incident edge going out. In this case T contains a directed cycle. On the other hand a transitive tournament cannot contain a directed cycle.

The reason why statement (2) holds is that a transitive tournament has $n(n-1)/2$ directed edges and we listed all of them in statement (1).

Statement (3) follows from the definition of the transitive tournament and statement (1), as any subset of the nodes can be ordered the same way. $\square$

A finite simple directed graph $G = (V, E)$ can be represented by an $|V|$ by $|V|$ adjacency matrix. The rows and the columns are labeled by the nodes of G. If the ordered pair (u, v) is an edge of G, then we put a bullet into the cell at the intersection of row u and column v. The adjacency matrix of course has 0, 1 entries when stored in a computer. The practice of using bullets instead of 1's is taken from [6]. It seems that it has a good visual effect and the computations are less prone to clerical errors when carried out using paper and pencil.

By Lemma 1, a tournament T is a transitive tournament if and only if the rows and the columns of its adjacency matrix can be permuted such that the adjacency matrix became an upper triangular matrix. A simple directed graph H with r nodes has a transitive tournament of r nodes if the adjacency matrix of H can be rearranged such that the upper triangular part is filled with bullets.

Let $G = (V, E)$ be a finite simple directed graph. Let U be a subset of V and let s be an integer such that $U \neq \emptyset$ and $s \geq 3$. The subset U of V is called an s-free subset if U does not contain any transitive tournament with s nodes. A partition of V into the subsets $V_1, \ldots, V_k$ is called an s-free partition of V if V_i is an s-free set for each i, $1 \leq i \leq k$.

A coloring of the nodes of a finite simple directed graph $G = (V, E)$ can be described by means of an onto function $f : V \to \{1, \ldots, k\}$. Here the numbers $1, \ldots, k$ are used as colors and node v receives color $f(v)$. The c-level set V_c of f is defined to be $V_c = \{v : f(v) = c, v \in V\}$.

The coloring $f : V \to \{1, \ldots, k\}$ of the nodes of the finite simple directed graph $G = (V, E)$ is called an s-free coloring if the level sets $V_1, \ldots, V_k$ form an s-free partition of V. The name intends to express the fact that color classes cannot contain any transitive tournament of size s. In other words color classes are free of tournaments of size s.

In the $s = 2$ special case a color class of an s-free coloring cannot contain any edge. ¿From this reason we will mainly deal with the $s \geq 3$ case.

The number of the color classes of an s-free coloring of the graph G can be used to establish an upper estimate of the size of a maximum transitive tournament in G.

Lemma 2 *If the finite simple directed graph* G *admits an* s*-free coloring with* k *colors and* G *has a transitive tournament of size* r*, then* $r \leq k(s-1)$.

Proof. Suppose G has a transitive tournament T of size r. Let $V_1, \ldots, V_k$ be the color classes of the s-free coloring and let W be the set of nodes of T. By Lemma 1, the subgraph of T with set of nodes $V_i \cap W$ is a transitive tournament. Since the coloring is s-free, it follows that $|V_i \cap W| \leq s-1$. We get that

$$r = |W| = |V_1 \cap W| + \cdots + |V_k \cap W| \leq k(s-1),$$

as required. □

From Lemma 2 we can see that the smaller is k, the better is the upper estimate of the size of the maximum transitive tournament in G. The following problem comes to mind naturally.

Problem 3 *Given a finite simple directed graph* $G = (V, E)$. *Further given integers* r, s *such that* $r \geq 3$, $s \geq 3$. *Decide if* G *has an* s*-free coloring with* r *colors.*

When we deal with coloring the nodes of a graph G we inevitably have to deal with incomplete or partial colorings, where each of the nodes of G receives at most one of the colors $1, \ldots, r$ but some of the nodes of G are left uncolored. Allocating color 0 for the uncolored nodes we can incorporate the incomplete colorings into the family of complete colorings.

3 Two auxiliary graphs

In this section we describe two finite simple directed graphs. They will play the roles of building blocks or switching devices in further constructions. Let r, s be fixed integers such that $r \geq 3$, $s \geq 3$. Set

$$h = (s-1)(r-1) + (s-2) + 2.$$

Let us consider the directed simple graph $H = (V, E)$, where $V = \{1, \ldots, h\}$. Set $W = \{2, \ldots, h-1\}$. We draw directed edges between the nodes in W such that the subgraph of H whose set of nodes is W forms a transitive tournament. From the node 1 we direct edges towards each node of W. Similarly, from the node h we direct edges towards each node of W.

For the sake of the illustration we worked out the special case $r = 3$, $s = 3$ in details. The adjacency matrix of H is in Table 1. A geometric representation of H is depicted in Figure 1.

We spell out the properties of the graph H we will use later as a Lemma.

	1	2	3	4	5	6	7
1		•	•	•	•	•	
2			•	•	•	•	
3				•	•	•	
4					•	•	
5						•	
6							
7		•	•	•	•	•	

Table 1: The adjacency matrix of the graph H in the special case $r = s = 3$.

Lemma 4 *(1) The nodes of the graph* H *have an* s*-free coloring with* r *colors.*

(2) In each s*-free coloring of the nodes of* H *with* r *colors the nodes* 1 *and* h *must receive the same colors.*

(3) Each partial coloring of the nodes of H*, where nodes* 1*,* h *receive the same color (and the remainig nodes of of* H *are left uncolored) can be extended to an* s*-free coloring of the nodes of* H *using* r *colors.*

Proof. In order to prove statement (1) let us consider the subsets $C_1, \ldots, C_r$, C_{r+1} of V such that these subsets are pair-wise disjoint and

$$|C_1| = \cdots = |C_{r-1}| = s - 1, \quad |C_r| = s - 2, \quad C_{r+1} = \{1, h\}.$$

Set $W = \{2, \ldots, h - 1\}$. Clearly, $C_1, \ldots, C_r$ form a partition of W. We use $C_1, \ldots, C_r$ as color classes to define a coloring of the subgraph L of H whose set of nodes is W.

By Lemma 1, the subgraph of H whose set of nodes is C_i is a transitive tournament for each i, $1 \leq i \leq r$. As $|C_i| \leq s - 1$, it follows that this graph does not contain a transitive tournament with s nodes. Therefore the coloring of L is an s-free coloring. The subgraph of H whose set of nodes is $C_i \cup \{1\}$ is a transitive tournament with s nodes. Consequently, the node 1 cannot receive color i for each i, $1 \leq i \leq r - 1$. On the other hand node 1 can receive color r since the subgraph of H whose set of nodes is $C_r \cup \{1, h\}$ is not an obstruction. Similarly node h may receive color r. This completes the proof of statement (1).

We can use the coloring constructed in the previous part and combine it with the fact that the colors in an s-free coloring of the nodes of H can be permuted among each other freely to settle statement (3).

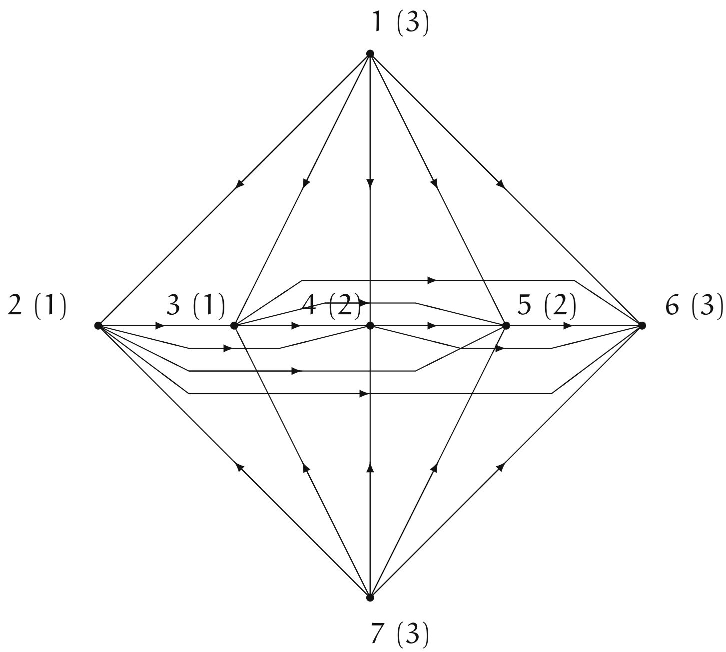

Figure 1: The auxiliary graph H in the special case $r = s = 3$. The numbers in parentheses are the colors of the nodes.

To prove statement (2) let us suppose that $f : V \to \{1, \dots, r\}$ is an s-free coloring of the nodes of H. Let us consider the subgraph L of H whose set of nodes is $W = \{2, \dots, h-1\}$. The restriction of f to W is an s-free coloring of the nodes of L. Let $C_1, \dots, C_r$ be the colors classes of this coloring. We may assume that $|C_1| \geq \dots \geq |C_r|$ since this is only a matter of exchanging the colors $1, \dots, r$ among each other.

By Lemma 1, the subgraph of H whose set of nodes is C_i is a transitive tournament for each i, $1 \leq i \leq r$. It follows that $|C_i| \leq s - 1$. Using

$$|C_1| + \dots + |C_r| = (s-1)(r-1) + (s-2)$$

we get that $|C_1| = \dots = |C_{r-1}| = s - 1$ and $|C_r| = s - 2$. The subgraph of H whose set of nodes is $C_i \cup \{1\}$ is a transitive tournament with s nodes for each i, $1 \leq i \leq r - 1$. We get that the node 1 cannot receive color i and so node 1

must receive color r. A similar reasoning gives that node h must receive color r too. This completes the proof of statement (2). $\square$

Let $r, s \geq 3$ be fixed integers. We construct a new auxiliary directed graph K. Let T be a transitive tournament with nodes $1, \ldots, s$. We consider $s-1$ isomorphic copies $H_1, \ldots, H_{s-1}$ of H. We choose the notation such that $H = (V, E)$ with $V = \{1, \ldots, h\}$ and $H_i = (V_i, E_i)$ with $V_i = \{(1,i), \ldots, (h,i)\}$. The correspondence

$$1 \longleftrightarrow (1,i), \ldots, h \longleftrightarrow (h,i)$$

defines the isomorphism between H and H_i.

Let us consider the nodes $(1,1), \ldots, (1, s-1)$ of $H_1, \ldots, H_{s-1}$, respectively and solder these nodes together to form a node u of K. Let us consider the nodes $(h,1), \ldots, (h, s-1)$ of $H_1, \ldots, H_{s-1}$, respectively and solder these nodes together with the nodes $1, \ldots, s-1$ of the tournament T, respectively. We rename node s of T to be v.

The graph K has

$$1 + \underbrace{(h-2) + \cdots + (h-2)}_{(s-1)\ \text{times}} + s = 1 + (s-1)(h-2) + s$$

nodes. We set $k = 1 + (s-1)(h-2) + s$ and rename the nodes of K by the numbers $1, \ldots, k$ such that $1 = u$ and $k = v$. We illustrated the construction in the special cases $s = 3$ and $s = 4$. The geometric versions of K can be seen in Figures 2 and 3.

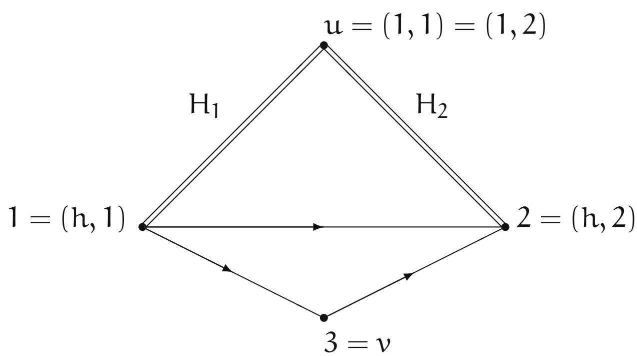

Figure 2: The auxiliary graph K in the special case $s = 3$. The double lines represent isomorphic copies of H.

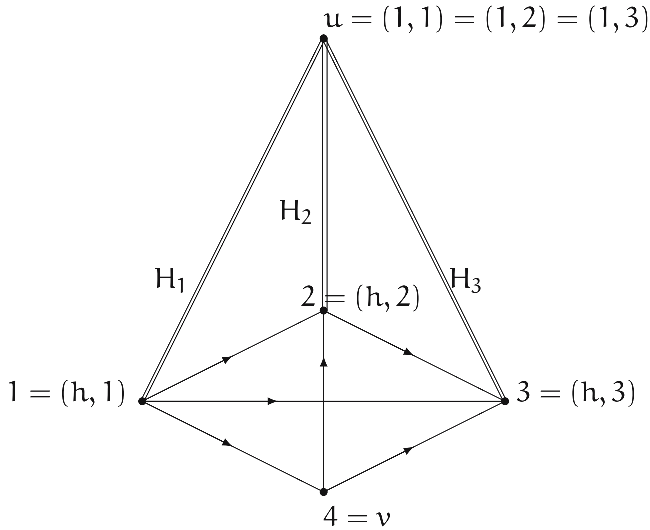

Figure 3: The auxiliary graph K in the special case $s = 4$. The double lines represent isomorphic copies of H.

The next lemma summarizes the essential properties of the graph K what we need later.

Lemma 5 *(1) The nodes of the graph K admit an s-free coloring with r colors.*

(2) In each s-free coloring of the nodes of the graph K with r colors the nodes 1 and k cannot receive the same colors.

(3) Each partial coloring of the nodes of K, where the nodes 1 and k are colored with distinct colors (and the other nodes of K are left uncolored) can be extended to an s-free coloring of the nodes of K using r colors.

Proof. By Lemma 4, the nodes of the graph H admit and s-free coloring with r colors. Consequently, the nodes of each of the graphs $H_1, \dots, H_{s-1}$ admit an s-free coloring with r colors. These colorings provide the same fixed color for $s-1$ nodes of the tournament T. The uncolored node v of T can be colored with any of the remaining $r-1$ colors. This proves statement (1).

By Lemma 4, the node V cannot receive the same color as node u. This settles statement (2).

By Lemma 4, each partial coloring of the nodes of H can be extended to an s-free coloring of the nodes of H using r colors. It follows that each partial coloring of the nodes of H_i can be extended to an s-free coloring of the nodes of H_i using r colors for each i, $1 \leq i \leq s-1$. This provides a partial coloring of the nodes of the tournament T. In this partial coloring of the nodes of T each node except node v receives the same color. Namely the color of node u. The last uncolored node v clearly can be colored with any of the remaining $r-1$ colors. This proves statement (3). □

4 The main result

The main result of this paper is the following theorem

Theorem 6 *Problem 3 is NP-hard for each* $r, s \geq 3$.

Proof. Let $r, s \geq 3$ be fixed integers. Assume on the contrary that Problem 3 is not NP-hard, that is, there is an "efficient" (polynomial running time) algorithm that solves Problem 3. Let $G = (V, E)$ be a finite simple graph with undirected edges. Using G and the auxiliary graphs H, K described in the previous chapter we construct a finite simple directed graph $G' = (V', E')$ such that the following conditions hold.

(1) If the nodes of G' have an s-free coloring with r colors, then the nodes of G have a legal coloring with r colors.

(2) If the nodes of G have a legal coloring with r colors, then the nodes of G' have an s-free coloring with r colors.

(3) The number of nodes G' can be upper bounded by a polynomial of the number of the nodes of G.

Thus for each legal (edge free) coloring problem we can construct a directed s-free coloring problem. If the second can be solved in polynomial time, it means that the first one can be solved in polynomial time as well.

Let $v_1, \ldots, v_n$ be the edges of G. In other words let $V = \{v_1, \ldots, v_n\}$. We consider an isomorphic copy $K_{i,j} = (W_{i,j}, F_{i,j})$ of the auxiliary graph $K = (W, F)$ for each i, j, $1 \leq i < j \leq n$. We recall that the nodes of K are labeled by the numbers $1, \ldots, k$. The nodes of $K_{i,j}$ will be labeled by the ordered triples $(i, j, 1), \ldots, (i, j, k)$. Here the correspondence

$$1 \longleftrightarrow (i, j, 1), \ldots, k \longleftrightarrow (i, j, k)$$

defines the isomorphism between K and $K_{i,j}$.

With each of the nodes $v_1, \ldots, v_n$ we associate a node $v'_1, \ldots, v'_n$ of the graph G'. At this moment our only concern is that $v'_1, \ldots, v'_n$ are pair-wise distinct points and they are nodes of G'. But G' may have further nodes.

If the unordered pair $\{v_i, v_j\}$ is an edge of G, then we add additional $k-2$ nodes to G'. We identify the nodes $(i, j, 1)$, (i, j, k) of $K_{i,j}$ with the nodes v'_i, v'_j of G', respectively. Next, we add the remaining $k-2$ nodes of $K_{i,j}$ to the nodes of G'. Finally, we add all the edges of $K_{i,j}$ to the edges of G'.

If the unordered pair $\{v_i, v_j\}$ is not an edge of G, then we do not add any nodes and we do not add any edges to G'. Clearly, G' has directed edges and it has $|V|+|E|(k-2)$ nodes. Since r, s are fixed numbers, it follows that $k-2 = c$ is a constant and so $|V'|$ can be upper bounded by $n + cn(n-1)/2$ which is a second degree polynomial in terms of n. This observation shows that condition (3) is satisfied.

In order to show that condition (1) is satisfied let us assume that $f' : V' \to \{1, \ldots, r\}$ is an s-free coloring of the nodes of G'. Using f' we define a coloring $f : V \to \{1, \ldots, r\}$ of the nodes of G. We set $f(v_i)$ to be equal to $f'(v'_i)$.

We claim that $f(v_i) = f(v_j)$ implies that the unordered pair $\{v_i, v_j\}$ is not an edge of G.

To verify the claim we assume on the contrary that $f(v_i) = f(v_j)$ and $\{v_i, v_j\}$ is an edge of G. The restriction of f' to $W_{i,j}$ is an s-free coloring of the nodes of the graph $K_{i,j}$. By Lemma 5, the nodes $(i, j, 1)$ and (i, j, k) cannot receive the same color. Using $v'_i = (i, j, 1)$, $v'_j = (i, j, k)$ we get the

$$f(v_i) = f'(v'_i) \neq f'(v'_j) = f(v_j)$$

contradiction.

To demonstrate that condition (2) is satisfied let us suppose that $f : V \to \{1, \ldots, r\}$ is a legal coloring of the nodes of G. Using f we define a coloring $f' : V' \to \{1, \ldots, r\}$ of the nodes of G'. We set $f'(v'_i)$ to be equal to $f(v_i)$.

Let us consider two distinct nodes v'_i, v'_j of G'. If the unordered pair $\{v'_i, v'_j\}$ is an edge of G', then by the construction of G' the nodes v'_i, v'_j are identical with the nodes $(i, j, 1)$, (i, j, k) of $K_{i,j}$, respectively. Thus $v'_i = (i, j, 1)$, $v'_j = (i, j, k)$. Since f is a legal coloring of the nodes of G, it follows that $f(v_i) \neq f(v_j)$ and so by the definition of f', we get that $f'(v'_i) \neq f'(v'_j)$.

For the sake of definiteness let us suppose that $f'(v'_i) = 1$ and $f'(v'_j) = 2$. We have a partial coloring of the nodes of $K_{i,j}$. Namely, the nodes $(i, j, 1)$, (i, j, k)

are colored with colors $1, 2$, respectively. Other nodes of $K_{i,j}$ are left uncolored. By Lemma 5, this partial coloring of $K_{i,j}$ can be extended to an s-free coloring of $K_{i,j}$. Since this can be accomplished in connection with each adjacent nodes v'_i, v'_j of G', it follows that the nodes of G' have an s-free coloring with r colors. □

5 A second proof of the main result

In the proof of Theorem 6 we used only the auxiliary graph K. The auxiliary graph H made an appearance only in the proof of Lemma 5 when we established the key properties of the auxiliary graph K. In this section we present an informal new proof where the graph H plays a more direct role. The node edge incidence matrix M of a finite simple graph $G = (V, E)$ is a $|V|$ by $|E|$ matrix. The rows and the columns of M are labeled by the nodes and the edges of G, respectively. If $e = \{u, v\}$ is an edge of G, then we place two bullets into M. We put one bullet into the cell at the intersection of row u and column e. Then we put a bullet into the cell at the intersection of row v and column e.

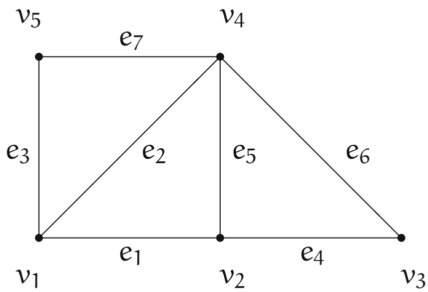

Figure 4: The toy example Γ.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	•	•	•				
v_2	•			•	•		
v_3				•		•	
v_4		•			•	•	•
v_5			•				•

Table 2: The node edge incidence matrix of the toy example Γ.

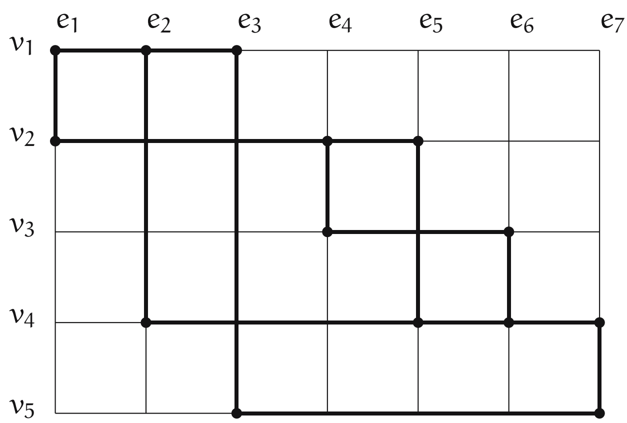

Figure 5: The graph Γ^* associated with the toy example Γ.

For the sake of illustration we included a toy example. The graph Γ can be seen in Figure 4. The node edge incidence matrix of Γ is in Table 2.

Suppose we are given a finite simple undirected graph $G = (V, E)$. Using G we construct a graph $G^* = (V^*, E^*)$. The construction is guided by the node edge incidence matrix of G. Let $V = \{v_1, \dots, v_n\}$, $E = \{e_1, \dots, e_m\}$. If $e_t = \{v_i, v_j\}$, then we add the ordered pairs (i, t), (j, t) to the set of nodes of G^*. Thus V^* is a set whose elements are ordered pairs. Clearly $|V^*| = 2|E| = 2m$.

We can form a mesh consisting of n horizontal and m vertical lines. The intersection of the horizontal and vertical lines form $(n)(m)$ mesh points. The nodes of G^* can be identified with some of these mesh points.

Two distinct nodes (i, t) and (j, t) of G^* on a vertical mesh line are connected with a vertical undirected edge in G^*. Two distinct nodes (i, x) and (i, z) of G^* on a horizontal line are connected with a horizontal undirected edge in G^* if there is no node in the form (i, y) such that $x < y < z$. Figure 5 depicts the graph Γ^* associated with the toy example Γ. The mesh lines are represented by thin lines. Bold lines represent the edges of Γ^*.

We replace each horizontal edge of G^* by an isomorphic copy of the auxiliary graph H. Next we replace each vertical edge of G^* by an isomorphic copy of the auxiliary graph K. After all possible replacements we get a finite simple directed graph G'.

Suppose that the nodes of G' have an s-free coloring with r colors. The isomorphic copies of the auxiliary graph H guarantee that the nodes of G' on a fixed horizontal line all receive the same color. The isomorphic copies of the auxiliary graph K make sure that the two nodes of G' on a fixed vertical line receive distinct colors. In this way we get a legal coloring of the nodes of G with r colors.

Next suppose that the nodes of G have a legal coloring with r colors. This coloring will provide partial colorings of the nodes of the isomorphic copies of the graphs H and K. One can extend these partial colorings to a complete s-free coloring of the nodes of G' with r colors.

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