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On scores in tournaments

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Abstract. A tournament is an orientation of a complete simple graph. The score of a vertex in a tournament is the outdegree of the vertex. In this paper, we obtain various results on the scores in tournaments.

1 Introduction

A tournament is an orientation of a complete simple graph. Let T be a tournament with vertex set $\{v_1, v_2, \ldots, v_n\}$. The score of a vertex v_i is defined as the outdegree of v_i and is denoted by s_{v_i}(or simply by s_i). Clearly $0 \leq s_i \leq n-1$ for all i, $1 \leq i \leq n$. The sequence $[s_1, s_2, \ldots, s_n]$ in non-decreasing order is called the score sequence of the tournament T. Several results on tournament scores can be seen in [21, 23]. The concept of scores in tournaments was extended to oriented graphs by Avery [1] and many results on oriented graph scores can be found in [19, 21, 22]. Pirzada et al. generalized score structure to other classes of digraphs and details can be seen in [17, 18]. Further score structure has been extended to hypertournaments, a generalization of tournaments [4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 24].

The following result [6] gives necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament and this result is also known as Landau's theorem.

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Theorem 1 (Landau [6]) *A sequence* $[s_1, s_2, \ldots, s_n]$ *of non-negative integers in non-decreasing order is a score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}, \quad (1)$$

for $1 \leq k \leq n$ *with equality when* $k = n$.

More work for scores in tournaments can be found in [2, 3, 7, 16].

For any two distinct vertices u and v of a tournament T, we have one of the following possibilities.
(i) An arc directed from u to v, denoted by $u(1-0)v$.
(ii) An arc directed form v to u, denoted by $u(0-1)v$.

2 Main Results

Now, we obtain the following results.

Theorem 2 *Let* $[s_1, s_2, \ldots, s_n]$ *be the score sequence of a tournament. Then the lowest score of the tournament is zero if* $\sum_{i=1}^{n} s_i^2$ *is maximum.*

Proof. Let v_1 be the vertex of the tournament with lowest score s_1. We shall show that $s_1 = 0$.
Suppose on contrary $s_1 > 0$. Then there exists a vertex v_p with score s_p such that $v_1(1-0)v_p$. Since $s_p \geq s_1$, therefore there exists another vertex v_q with score s_q such that $v_p(1-0)v_q$.
Now, by changing the arcs $v_1(1-0)v_p$ and $v_p(1-0)v_q$ to $v_1(0-1)v_p$ and $v_p(0-1)v_q$ respectively we get a new score sequence $[t_1, t_2, \ldots, t_n]$ where $t_1 = s_1 - 1, t_q = s_q + 1, t_r = s_r$ for all $r,\ 2 \leq r \leq n$ with $r \neq q$. Then

$$\begin{aligned}\sum_{i=1}^{n} t_i^2 &= \sum_{i=2, i\neq q}^{n} t_i^2 + t_1^2 + t_q^2 = \sum_{i=2, i\neq q}^{n} s_i^2 + (s_1 - 1)^2 + (s_q + 1)^2 \\ &= \sum_{i=2, i\neq q}^{n} s_i^2 + s_1^2 + 1 - 2s_1 + s_q^2 + 1 + 2s_q = \sum_{i=1}^{n} s_i^2 + 2(s_q - s_1 + 1) \\ &> \sum_{i=1}^{n} s_i^2,\end{aligned}$$

since $s_q \geq s_1$. This is a contradiction as $\sum\limits_{i=1}^{n} s_i^2$ was assumed to be maximum. Hence the result follows. □

Theorem 3 *Let* $[s_1, s_2, \ldots, s_n]$ *be the score sequence of a tournament. Then the highest score of the tournament is* $n-1$ *if* $\sum\limits_{i=1}^{n} s_i^2$ *is maximum.*

Proof. Let v_n be the vertex of the tournament with highest score s_n. We shall show that $s_n = n-1$. Suppose on contrary $s_n < n-1$. Then there exits a vertex v_p with score s_p such that $v_p(1-0)v_n$. Since $s_n \geq s_p$, therefore there exists another vertex v_q with score s_q such that $v_q(1-0)v_p$ and $v_q(0-1)v_n$.

Now, by changing the arcs $v_p(1-0)v_n$ and $v_q(1-0)v_p$ to $v_p(0-1)v_n$ and $v_q(0-1)v_n$ respectively we get a new score sequence $[t_1, t_2, \ldots, t_n]$ where $t_q = s_q - 1, t_n = s_n + 1, t_r = s_r$ for all r, $1 \leq r \leq n-1$ with $r \neq q$. Then

$$\begin{aligned}\sum_{i=1}^{n} t_i^2 &= \sum_{i=1, i\neq q}^{n-1} t_i^2 + t_q^2 + t_n^2 = \sum_{i=1, i\neq q}^{n-1} s_i^2 + (s_q-1)^2 + (s_n+1)^2 \\ &= \sum_{i=1, i\neq q}^{n-1} s_i^2 + s_q^2 + 1 - 2s_q + s_n^2 + 1 + 2s_n = \sum_{i=1}^{n} s_i^2 + 2(s_n - s_q + 1) \\ &> \sum_{i=1}^{n} s_i^2 \ \text{ since } \ s_n \geq s_q,\end{aligned}$$

which is a contradiction, since $\sum\limits_{i=1}^{n} s_i^2$ was assumed to be maximum. Hence the result follows. □

Theorem 4 *Let* $[s_1, s_2, \cdots, s_n]$ *be the score sequence of a tournament with vertex set* V *and let* m_i *be the average of the scores of the vertices* v_j *such that* $v_i(1-0)v_j$. *Then*

$$\max\{s_j + m_j : v_j \in V\} \leq \frac{3n-4}{2}, \tag{2}$$

with equality if and only if $s_i = n-1$ *where* $i = n$.

Proof. Let v_i be the vertex of a tournament where $s_i + m_i$ is maximum and let S be the sum of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then

$$\max\{s_j + m_j : v_j \in V\} = s_i + m_i = s_i + \frac{S}{s_i}.$$

Again, let g_i be the average of the scores of the vertices v_k such that $v_k(1{-}0)v_i$. Then

$$\begin{aligned} \frac{n(n-1)}{2} &= s_i + S + (n - s_i - 1)g_i, \quad \text{(by (1))} \\ \text{or} \quad \frac{n}{2} + n - 2 &= \frac{s_i + S + (n - s_i - 1)g_i}{n-1} + n - 2, \\ \text{or} \quad \frac{3n-4}{2} &= \frac{s_i + S + (n - s_i - 1)g_i}{n-1} + n - 2. \end{aligned}$$

So, we have to prove that

$$\begin{aligned} s_i + \frac{S}{s_i} &\leq \frac{s_i + S + (n - s_i - 1)g_i}{n-1} + n - 2, \\ \text{or} \quad (n-1)\left(s_i + \frac{S}{s_i}\right) &\leq s_i + S + (n - s_i - 1)g_i + (n-1)(n-2), \\ \text{or} \quad (n-1)\left(n - 2 - s_i - \frac{S}{s_i}\right) + s_i + S + (n - s_i - 1)g_i &\geq 0, \\ \text{or} \quad (n-1)\left(n - 2 - \frac{S}{s_i}\right) - (n-1)s_i + s_i + S + (n - s_i - 1)g_i &\geq 0, \\ \text{or} \quad (n-1)\left(n - 2 - \frac{S}{s_i}\right) - s_i\left(n - 2 - \frac{S}{s_i}\right) + (n - s_i - 1)g_i &\geq 0, \\ \text{or} \quad (n - 1 - s_i)\left(n - 2 - \frac{S}{s_i}\right) + (n - s_i - 1)g_i &\geq 0, \\ \text{or} \quad (n - s_i - 1)\left(n - 2 + g_i - \frac{S}{s_i}\right) &\geq 0. \qquad (3) \end{aligned}$$

If $s_i = n - 1$, then (3) holds. Now, if $s_i \leq n - 2$, then there is at least one vertex v_k such that $v_k(1-0)v_i$, so that $g_i \geq 1$. Also $\frac{S}{s_i} \leq n - 1$. Therefore (3) holds.

This completes the proof of first part.

Now assume that equality holds in (2). Then from (3), we have

$$(n - s_i - 1)\left(n - 2 + g_i - \frac{S}{s_i}\right) = 0,$$

which gives (a) $s_i = n - 1$ or (b) $\frac{S}{s_i} - g_i = n - 2$.

Case (a). $s_i = n{-}1$. This is possible only when $i = n$, that is, when $s_n = n{-}1$.

Case (b). $\dfrac{S}{s_i} - g_i = n - 2$. Since $s_n \geq \dfrac{S}{s_i}$, therefore

$$s_n \geq n - 2 + g_i. \qquad (4)$$

Also $g_i \geq 0$ and $s_n \leq n-1$. Then from (4), we have $0 \leq g_i \leq 1$. If $g_i = 0$, then $s_n = n-1$. Again if $0 < g_i \leq 1$, then there is at least one vertex v_k such that $v_k(1-0)v_i$. Therefore $g_i \geq 1$. Hence $g_i = 1$. Thus from (4), we have $s_n \geq n-1$. Since $s_n \leq n-1$, therefore $s_n = n-1$.

Conversely, let $s_n = n-1$. Then $s_k \leq n-2$ for all k, $1 \leq k < n$. Now

$$\begin{aligned} s_k + m_k &= s_k + \frac{1}{s_k}\sum_{j=1}^{n}\{s_j : v_k(1-0)v_j\} \\ &\leq s_k + \frac{1}{s_k}\left\{\frac{s_k(s_k-1)}{2} + s_k(n-2-s_k)\right\} \\ &= s_k + \frac{s_k-1}{2} + n-2-s_k \\ &\leq \frac{n-2-1}{2} + n-2 = \frac{3n-7}{2} \end{aligned}$$

and

$$\begin{aligned} s_n + m_n &= s_n + \frac{1}{s_n}\sum_{j=1}^{n}\{s_j : v_n(1-0)v_j\} \\ &= n-1+\frac{1}{n-1}\sum_{i=1}^{n-1} s_i \\ &= n-1+\frac{1}{n-1}\left\{\sum_{i=1}^{n} s_i - s_n\right\} \\ &= n-1+\frac{1}{n-1}\left\{\frac{n(n-1)}{2} - (n-1)\right\} \quad \text{(by (1))} \\ &= \frac{3n-4}{2}. \end{aligned}$$

Hence, $max\{s_j + m_j : v_j \in V\} = \dfrac{3n-4}{2}$, completing the proof. □

Theorem 5 *Let* $[s_1, s_2, \ldots, s_n]$ *be the score sequence of a tournament and let* m_i *be the average of the scores of the vertices* v_j *such that* $v_i(1-0)v_j$. *Then*

$$s_i + m_i \leq \frac{n}{2} + \frac{n-2}{n-1}s_i + (s_n - s_1)\left(1 - \frac{s_i}{n-1}\right), \tag{5}$$

holds for each i. *Further, the equality holds if and only of* $s_i = n-1$ *where* $i = n$ *or the vertex* v_i *is such that* $v_i(1-0)v_j$ *for the* s_n *score vertices* v_j *and* $v_i(0-1)v_k$ *for the* s_1 *score vertices* v_k.

Proof. Let v_i be the vertex of score s_i in the tournament T. We consider two cases: (a) $s_i = n-1$ (b) $s_i < n-1$.
Case (a). $s_i = n-1$. Then $i = n$, so that $s_n = n-1$. Therefore

$$\begin{aligned}
s_n + m_n &= n-1+\frac{1}{s_n}\sum_{j=1}^{n}\{s_j : v_n(1-0)v_j\} = n-1+\frac{1}{n-1}\sum_{j=1}^{n-1}s_j \\
&= n-1+\frac{1}{n-1}\left\{\sum_{j=1}^{n}s_j - s_n\right\} \\
&= n-1+\frac{1}{n-1}\left\{\frac{n(n-1)}{2}-(n-1)\right\} \quad \text{(by (1))} \\
&= \frac{3n-4}{2}.
\end{aligned}$$

Hence (5) holds.
Case (b). $s_i < n-1$. Change the orientation of the arcs $v_k(1-0)v_i$, if any, to $v_i(1-0)v_k$. Suppose this new tournament is T_1 and let $max\{s_j + m_j : v_j \in V\}$ occurs at the vertex v_i and let it be $s'_i + m'_i$.
Now for T_1, we have

$$\begin{aligned}
s'_i + m'_i &= n-1+\frac{1}{s'_i}\left\{\sum_{j=1}^{n}s'_j : v_i(1-0)v_j\right\} \\
&= n-1+\frac{1}{n-1}\left\{\sum_{j=1}^{n}s'_j - s'_i\right\} \\
&= n-1+\frac{1}{n-1}\left\{\frac{n(n-1)}{2}-(n-1)\right\} \quad \text{(by (1))} \\
&= \frac{3n-4}{2}.
\end{aligned} \tag{6}$$

Let S be the sum of the scores of the vertices v_j such that $v_i(1-0)v_j$ in the tournament T. Then $s_i + m_i = s_i + \frac{S}{s_i}$. Now,

$$\begin{aligned}
(s_i' + m_i') - (s_i + m_i) &= n - 1 + \frac{1}{s_i'} \sum_{j=1}^{n} \left\{ s_j' : v_i(1-0)v_j \right\} - \left(s_i + \frac{S}{s_i} \right) \\
&= n - s_i - 1 + \frac{1}{n-1} \{S + (n - s_i - 1)g_i - (n - s_i - 1)\} - \frac{S}{s_i} \\
&= n - s_i - 1 + \frac{1}{n-1} \{S + (n - s_i - 1)(g_i - 1)\} - \frac{S}{s_i},
\end{aligned}$$

(where g_i is the average score of the vertices v_k such that $v_k(1-0)v_i$ in T), that is,

$$\begin{aligned}
s_i + m_i &= s_i' + m_i' - (n - s_i - 1) - \frac{1}{n-1} \{S + (n - s_i - 1)(g_i - 1)\} + \frac{S}{s_i} \\
&= \frac{3n-4}{2} - (n - s_i - 1) - \frac{1}{n-1} \{S + (n - s_i - 1)(g_i - 1)\} + \frac{S}{s_i} \quad \text{(by (6))} \\
&= \frac{3n-4}{2} - (n - s_i - 1) - \frac{S}{n-1} - \frac{1}{n-1} \{(n - s_i - 1)(g_i - 1)\} + \frac{S}{s_i} \\
&= \frac{3n-4}{2} - \frac{(n - s_i - 1)(n - 1 + g_i - 1)}{n-1} + \frac{S}{s_i} - \frac{S}{n-1} \\
&= \frac{3n-4}{2} - \left(1 - \frac{s_i}{n-1}\right)(n - 2 + g_i) + \frac{S}{s_i}\left(1 - \frac{s_i}{n-1}\right) \\
&= \frac{3n-4}{2} - \left(1 - \frac{s_i}{n-1}\right)\left(n - 2 + g_i - \frac{S}{s_i}\right) \\
&= \frac{3n-4}{2} - \left(1 - \frac{s_i}{n-1}\right)(n-2) - \left(1 - \frac{s_i}{n-1}\right)\left(g_i - \frac{S}{s_i}\right) \\
&= \frac{3n-4}{2} - \left(n - 2 - \frac{(n-2)s_i}{n-1}\right) - \left(1 - \frac{s_i}{n-1}\right)\left(g_i - \frac{S}{s_i}\right) \\
&= \frac{n}{2} + \frac{n-2}{n-1}s_i - \left(1 - \frac{s_i}{n-1}\right)\left(g_i - \frac{S}{s_i}\right). \qquad (7)
\end{aligned}$$

Clearly $\frac{S}{s_i} \le s_n$, that is, $\frac{S}{s_i} - s_n \le 0$ and $g_i \ge s_1$, that is $g_i - s_1 \ge 0$. Therefore $g_i - s_1 \ge \frac{S}{s_i} - s_n$, that is, $g_i - \frac{S}{s_i} \ge s_1 - s_n$. Using this in (7), we have

$$s_i + m_i \ge \frac{n}{2} + \frac{n-2}{n-1}s_i - \left(1 - \frac{s_i}{n-1}\right)(s_1 - s_n),$$

that is, $s_i + m_i \geq \frac{n}{2} + \frac{n-2}{n-1}s_i + (s_n - s_1)\left(1 - \frac{s_i}{n-1}\right)$. This completes the proof of first part.

Now assume that equality holds in (5). Then $s_i = n-1$ or $-\left(g_i - \frac{S}{s_i}\right) = s_n - s_1$, that is, $s_i = n-1$ where $i = n$ or $-g_i s_i + S = s_n s_i - s_1 s_i$. From $-g_i s_i + S = s_n s_i - s_1 s_i$, we have $\frac{-P}{n-s_i-1}s_i + s_1 s_i = s_n s_i - S$, (where P is the sum of the scores of the vertices v_k such that $v_k(1-0)v_i$ in T) or $s_1 - \frac{P}{n-s_i-1} = \frac{s_n s_i - S}{s_i}$, or $\frac{s_n s_i - S}{s_i} = \frac{(n-s_i-1)s_1 - P}{n-s_i-1}$ or $s_1 - \frac{P}{n-s_i-1} = \frac{s_n s_i - S}{s_i}$, or $\frac{(n-s_i-1)s_1 - P}{n-s_i-1} = \frac{s_n s_i - S}{s_i} \geq 0$, since $\frac{S}{s_i} \leq s_n$, that is, $(n-s_i-1)s_1 - P \geq 0$, or $P \leq (n-s_i-1)s_1$. But $P \geq (n-s_i-1)s_1$. Therefore $P = (n-s_i-1)s_1$. This means that all those vertices v_k with $v_k(1-0)v_i$ are of score s_1. Using this fact in

$$\frac{(n-s_i-1)s_1 - P}{n-s_i-1} = \frac{s_n s_i - S}{s_i},$$

we have

$$\frac{(n-s_i-1)s_1 - (n-s_i-1)s_1}{n-s_i-1} = \frac{s_n s_i - S}{s_i},$$

or $\frac{s_n s_i - S}{s_i} = 0$ or $S = s_n s_i$ or $\frac{S}{s_i} = s_n$. This means that all those vertices v_j with $v_i(1-0)v_j$ are of score s_n.

Conversely, let $s_i = n-1$, where $i = n$ or $v_i(1-0)v_j$ for the s_n score vertices v_j and $v_i(0-1)v_k$ for the s_1 score vertices v_k. For $s_i = n-1$, where $i = n$, the equality holds in (5) by using case (a). Now, if $v_i(1-0)v_j$ for the s_n score vertices v_j and $v_i(0-1)v_k$ for the s_1 score vertices v_k, then

$$s_i + m_i = s_i + \frac{s_n s_i}{s_i} = s_i + s_n$$

and

$$
\begin{aligned}
\frac{n}{2} + \frac{n-2}{n-1}s_i + &(s_n - s_1)\left(1 - \frac{s_i}{n-1}\right) \\
&= \frac{n(n-1)}{2}\frac{1}{n-1} + \frac{n-2}{n-1}s_i + \frac{(s_n - s_1)(n-1-s_i)}{n-1} \\
&= \frac{1}{n-1}\left\{\sum_{i=1}^{n} s_i + (n-2)s_i + (s_n - s_1)(n-1-s_i)\right\} \text{ by (1)} \\
&= \frac{1}{n-1}\{s_i + s_n s_i + s_1(n - s_i - 1) \\
&\quad + (n-2)s_i + s_n(n-1-s_i) - s_1(n-1-s_i)\} \\
&= \frac{1}{n-1}\{s_i + s_n s_i + n s_i - 2s_i + n s_n - s_n - s_n s_i\} \\
&= \frac{1}{n-1}\{(n-1)s_i + (n-1)s_n\} = s_i + s_n.
\end{aligned}
$$

Therefore, the equality holds in (5). □

Corollary 6 *Let* $[s_1, s_2, \ldots, s_n]$ *be the score sequence of a tournament and let* m_i *be the average of the scores of the vertices* v_j *such that* $v_i(1-0)v_j$. *Then*

$$s_i + m_i \leq \frac{n}{2} + \frac{n-2}{n-1}s_n + (s_n - s_1)\left(1 - \frac{s_n}{n-1}\right), \tag{8}$$

holds for each i. *Further, the equality holds if and only if* $s_i = n-1$ *where* $i = n$ *or the vertex* v_i*(score is* s_n*) is such that* $v_i(1-0)v_j$ *for the* s_n *score vertices* v_j *and* $v_i(0-1)v_k$ *for the* s_1 *score vertices* v_k.

Proof. Since (5) is true for each i and since $s_i \leq s_n$, therefore we get (8). Using Theorem 5, we conclude that the equality holds if and only if $s_i = n-1$ where $i = n$ or the vertex v_i(score is s_n) is such that $v_i(1-0)v_j$ for the s_n score vertices v_j and $v_i(0-1)v_k$ for the s_1 score vertices v_k. □

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