Acta Univ. Sapientiae, Informatica **10,** 2 (2018) 137– 157

DOI: 10.2478/ausi-2018-0008

Estimating clique size by coloring the nodes of auxiliary graphs

Sándor SZABÓ
University of Pécs, Pécs, Hungary
email: sszabo7@hotmail.com

Abstract. It is a common practice to find upper bound for clique number via legal coloring of the nodes of the graph. We will point out that with a little extra work we may lower this bound. Applying this procedure to a suitably constructed auxiliary graph one may further improve the clique size estimate of the original graph.

1 Introduction

A graph is called a finite simple graph if it has finitely many nodes and edges and in addition it does not have any loop or double edge. Let $G = (V, E)$ be a finite simple graph. A subgraph Δ of G is called a clique if each two distinct nodes of Δ are adjacent. If the clique Δ has k nodes we call it a k-clique of G. For each finite simple graph G there is a well defined integer k such that G contains a k-clique but G does not contain any $(k+1)$-clique. This k is called the clique number of G and it is denoted by $\omega(G)$. Each k-clique in G is called a maximum clique of G. (For more background information and applications of the clique problem the reader should consult with [2], [4], [6], [12].)

We color the nodes of G such that each node has exactly one color and adjacent nodes cannot receive the same color. This type of coloring of the nodes of G is called legal coloring. For each finite simple graph G there is a

Computing Classification System 1998: G.2.2
Mathematics Subject Classification 2010: 05C15, 05B45, 52C22
Key words and phrases: clique number, chromatic number, maximum clique legal coloring of the nodes, greedy coloring

well defined integer k such that the nodes of G can be legally colored using k colors but the nodes of G cannot be legally colored using $k-1$ colors. This k is called the chromatic number of G and it is denoted by $\chi(G)$.

It is well-known that the problems of determining $\omega(G)$ or $\chi(G)$ belong to the NP hard complexity class. (See [5].)

Many clique solver algorithms used in practice employ clique size upper estimates to curtail the size of the search space. (See [1], [7], [8], [11], [13], [15], [16].) Since $\omega(G) \leq \chi(G)$ holds it is a common practice to use a greedy coloring procedure to locate a legal coloring of the nodes of G and use the number of colors as an upper estimate for $\omega(G)$. We will point out that with a little more extra work one can reduce this upper bound.

Using the given graph G we construct an auxiliary graph Γ such that an upper estimate for $\omega(\Gamma)$ yields an upper estimate for $\omega(G)$. The new estimate is typically better but it comes for a computationally higher price. We will present two particular instances of such auxiliary graphs.

2 The basic procedure

In this section first we describe a procedure to estimate the clique size of a finite simple graph $G = (V, E)$. For the sake of easier reference we will call the proposed procedure as the method of profiles. As a starting point we legally color the nodes of G. We may use any coloring algorithm. (See [9], [3].) We do not assume that the number of colors we use is optimal. Let $C_1, \ldots, C_\gamma$ be the color classes of the nodes. Set

$$U = C_1 \cup \cdots \cup C_p \quad \text{and} \quad W = C_{p+1} \cup \cdots \cup C_\gamma, \tag{1}$$

where $p = \lfloor \gamma/2 \rfloor$. Let H and K be the subgraphs of G induced by the sets U and W, respectively.

To a node $u \in U$ we assign a quantity $\mathrm{cdeg}(u)$ called the clique degree of u. We form the subgraph L_u induced in G by the subset $N(u) \cap W$ of the nodes of G. Here $N(u)$ is the set of neighbors of u in G. We would prefer to set $\mathrm{cdeg}(u)$ to be $\omega(L_u)$. But computing $\omega(L_u)$ maybe overly time consuming. So we settle for an upper estimate of $\omega(L_u)$. We may use our favorite procedure to find an upper estimate for $\omega(L_u)$.

Analogously, to a node $w \in W$ we assign a clique degree $\mathrm{cdeg}(w)$. We consider the subgraph L_w induced by the set $N(w) \cap U$ and $\mathrm{cdeg}(w)$ is an upper estimate of $\omega(L_w)$.

For the remaining part of the description of the algorithm we assume that the clique degrees of the nodes of G are at our disposal. We define a profile for the graph H which is a sequence of numbers $\alpha'_1, \dots, \alpha'_p$. We set

$$\alpha_i = \max\{\mathrm{cdeg}(v) : v \in C_i\}, \quad 1 \le i \le p.$$

Then we arrange the numbers $\alpha_1, \dots, \alpha_p$ into a non-increasing order to get the profile $\alpha'_1, \dots, \alpha'_p$ of H. In a similar fashion we construct a profile $\beta'_1, \dots, \beta'_q$ for the graph K, where $q = \gamma - p$. We set

$$\beta_i = \max\{\mathrm{cdeg}(v) : v \in C_i\}, \quad p+1 \le i \le \gamma.$$

Finally we list the numbers $\beta_{p+1}, \dots, \beta_\gamma$ in a non-increasing order to get the profile $\beta'_1, \dots, \beta'_q$ of K.

After this phase of the algorithm the profiles of the graphs H and K are available. We call an ordered pair

$$(r, s), \quad 0 \le r \le p, \quad 0 \le s \le q \tag{2}$$

qualifying if each of the following inequalities

$$\alpha'_1 \ge s, \dots, \alpha'_r \ge s \tag{3}$$

$$\beta'_1 \ge r, \dots, \beta'_s \ge r \tag{4}$$

holds. We do not exclude the $r = 0$ possibility. In the $r = 0$ case the inequalities (4) clearly hold and the condition (3) vacuously satisfied. Similarly, the $s = 0$ possibility is not excluded. When $s = 0$, the inequalities (4) obviously hold and the requirement (4) vacuously satisfied.

We inspect the $(p+1)(q+1)$ ordered pairs (r, s) in (2) in the order

$$(p-i, q), (p-i+1, q-1), \dots, (p, q-i), \quad 0 \le i \le p+q$$

to find the quantity

$$t = \max\{r + s : (r, s) \text{ is qualifying}\}. \tag{5}$$

We claim that $\omega(G) \le t$. We state and prove this result more formally.

Lemma 1 *Let* $G = (V, E)$ *be a finite simple graph having at least one node. The quantity defined in (5) is an upper bound of the clique number of* G.

Proof. Set $k = \omega(G)$. Clearly G must contain a k-clique Δ. Let U' be the set of nodes of Δ that are in U and let W' be the set of nodes of Δ that are in

W. Here U and W are the subsets of V defined in (1). Obviously, $U' \cap W' = \emptyset$ and $|U'| + |W'| = k$. We distinguish four cases.

Case 1	$U' = \emptyset$	$W' = \emptyset$
Case 2	$U' = \emptyset$	$W' \neq \emptyset$
Case 3	$U' \neq \emptyset$	$W' = \emptyset$
Case 4	$U' \neq \emptyset$	$W' \neq \emptyset$

Since G has at least one node it must have a 1-clique. Thus $k \geq 1$ holds and so case 1 is not possible.

If $U' = \emptyset$, then Δ is a clique in the subgraph K of G induced by W. The nodes of K are legally colored with q colors and so $k \leq q$. Note that $p = 0$ and the ordered pair $(0, q)$ is a qualifying pair. It follows that $q \leq t$. Thus $k \leq q$ as required. This settles case 2. Case 3 can be sorted out in a similar way.

In case 4 the set of nodes of Δ is equal to $U' \cup W'$. This means that the unordered pair $\{u, w\}$ is an edge of G for each $u \in U'$, $w \in W'$. Let $r = |U'|$ and $s = |W'|$. The subgraph L_u of G induced by $N(u) \cap W$ must contain an s-clique. There are r choices for the node $u \in U'$. These choices show that the inequalities (3) hold. Similarly, the subgraph L_w of G induced by $N(w) \cap U$ must contain an r-clique. There are s choices for the node $w \in W'$. These choices show that the inequalities (4) hold. Therefore the ordered pair (r, s) is a qualifying pair. The inequality $k \leq r + s$ holds for each qualifying pair (r, s). Thus $k \leq t$, as required. □

3 A small size example

In this section we work out a small example in details to illustrate the method of profiles.

Example 2 *Let us consider the finite simple graph* $G = (V, E)$ *given by its adjacency matrix in Table 1. A geometric representation of* G *is depicted in Figure 1. The graph has* 16 *vertices and* 39 *edges.*

Using the simplest greedy sequential coloring procedure we colored the nodes of G legally. The procedure is presented in Table 2. The first column contains the nodes of the graph G. The last column holds the colors of the nodes. A column between the first and the last represents a partial coloring of the nodes of G. The "←" symbol points to the pivot node. The node to which we are assigning color at this phase. The "]" symbol after a color indicates that the

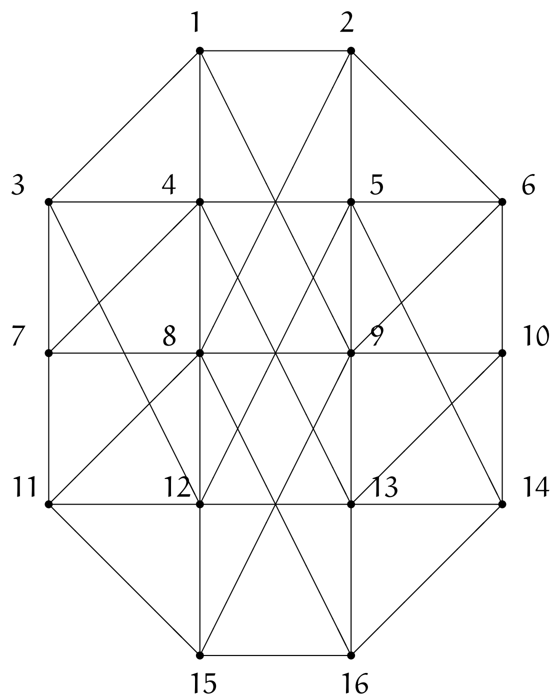

Figure 1: A geometric representation of the graph G in Example 2.

pivot node is adjacent to this node and the marked color cannot be assigned to the pivot node.

The color classes of the nodes are the following

$$C_1 = \{1, 5, 7, 10, 15\}, C_2 = \{2, 3, 9, 11, 14\}, C_3 = \{4, 6, 12, 16\}, C_4 = \{8, 13\}.$$

The coloring of the nodes gives that $\omega(G) \leq 4$. We try to reduce this upper estimate. We set

$$U = C_1 \cup C_2, \quad W = C_3 \cup C_4.$$

We computed the clique degrees of the nodes and the profiles of the graphs H, K. The results are summarized in the first three arrays of Table 3. An inspection of the qualifying pairs (r, s) reveals that $\omega(G) \leq 3$.

The inspection to decide if a given ordered pair (r, s) is qualifying or not is summarized in the last array of Table 3. We assume that there is a complete bipartite graph with independent sets A and B whose cardinalities are r and

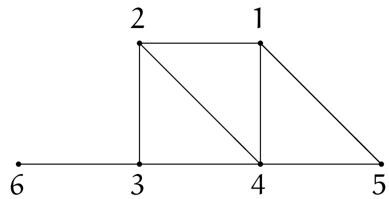

Figure 2: A graphical representation of the graph G in Examples 3 and 6.

s respectively and the graph of course has rs edges. Each of the r nodes of A needs to have a clique degree at least r and each of the s nodes of B needs to have clique degree at least r. These requirements are listed in a row labeled by the word "needed". The available clique degrees are listed in a row labeled by the word "found". Comparing these rows we can spot if the pair (r, s) is not qualifying. We used a "+" sign to indicate when the needed and the found clique degrees do not meet with the requirement.

We would like to emphasize that the method of profiles can produce an upper estimate for $\omega(G)$ which is below $\chi(G)$. (Such an estimate is termed as infra chromatic in the literature.) In order to exhibit such an example we note that $\chi(G) = 4$. Let us suppose on the contrary that $\chi(G) = 3$. Let us order the nodes of G as listed in the first column in the second array in Table 2. The nodes 1, 3, 4 are the nodes of a 3-clique in G. We may color these nodes by colors 1, 2, 3. After these choices the greedy coloring procedure will color the nodes up to node 10 uniquely. For node 13 we must use an additional color. The indirect assumption $\chi(G) = 3$ leads to a contradiction.

4 The first auxiliary graph

Let $G = (V, E)$ be a finite simple graph. Using G we construct a new graph $\Gamma_1 = (W, F)$. We call Γ_1 the first auxiliary graph associated with G. The nodes of Γ_1 are the ordered pairs (v, a), $v \in V$, $1 \leq a \leq 2$. If the unordered pair $\{v_1, v_2\}$ is an edge of G, then the four pair-wise distinct distinct nodes $(v_1, 1)$, $(v_1, 2)$, $(v_2, 1)$, $(v_2, 2)$ of Γ_1 are the nodes of a 4-clique in Γ_1. In other words if $\{v_1, v_2\} \in E$, then $\{w_1, w_2\} \in F$ for each distinct $w_1, w_2 \in \{(v_1, 1), (v_1, 2), (v_2, 1), (v_2, 2)\}$.

We illustrate the construction of the auxiliary graph in connection with a very small size toy example.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	×	•	•	•					•							
2	•	×			•	•		•								
3	•		×	•			•					•				
4	•		•	×	•		•	•					•			
5		•		•	×	•			•			•		•		
6		•			•	×			•	•						
7			•	•			×	•			•					
8		•		•			•	×	•		•	•				•
9	•				•	•		•	×	•			•		•	
10						•			•	×			•	•		
11							•	•			×	•			•	
12			•		•			•			•	×	•		•	
13				•					•	•		•	×	•		•
14					•					•			•	×		•
15									•		•	•			×	•
16								•					•	•	•	×

Table 1: The adjacency matrix of the graph G in Example 2.

Example 3 *Let us consider the finite simple graph* $G = (V, E)$ *given by its adjacency matrix in Table 4. A geometric representation of* G *is depicted in Figure 2. The graph has* 6 *vertices and* 8 *edges.*

A geometric representation of the auxiliary graph Γ_1 can be seen in Figure 3. The adjacency matrix of Γ_1 is given in Table 5. In fact two versions of the adjacency matrix are given. The nodes of Γ_1 are listed in different ways.

The clique numbers of the graphs G and Γ_1 are related. This is the content of the next lemma.

Lemma 4 *Let* G *be a finite simple graph and let* Γ_1 *be the associated auxiliary graph. Then* $2\omega(G) \leq \omega(\Gamma_1)$*.*

Proof. Set $k = \omega(G)$. The graph G contains a k-clique Δ. Let U be the set of nodes of Δ. Let $T = \{(u, a) : \ u \in U, \ 1 \leq a \leq 2\}$. Clearly $|T| = 2|U| = 2k$. Note that two distinct nodes (u_1, a_1), (u_2, a_2) in T are always adjacent nodes in Γ_1. □

Lemma 4 tells us that if t is an upper bound for $\omega(\Gamma_1)$, then $t/2$ is an upper bound for $\omega(G)$.

1	1]	1]	1]	1	1	1	1	1]	1	1	1	1	1	1	1	1
2	←	2	2	2]	2]	2	2]	2	2	2	2	2	2	2	2	2
3		←	2]	2	2	2]	2	2	2	2	2]	2	2	2	2	2
4			←	3]	3	3]	3]	3	3	3	3	3]	3	3	3	3
5				←	1]	1	1	1]	1	1	1]	1	1]	1	1	1
6					←	3	3	3]	3]	3	3	3	3	3	3	3
7						←	1]	1	1	1]	1	1	1	1	1	1
8							←	4]	4	4]	4]	4	4	4	4]	4
9								←	2]	2	2	2]	2	2]	2	2
10									←	1	1	1]	1]	1	1	1
11										←	2]	2	2	2]	2	2
12											←	3]	3	3]	3	3
13												←	4]	4	4]	4
14													←	2	2]	2
15														←	1]	1
16															←	3

1	1	1	1	1	1	1]	1]	1	1	1	1	1
3	2]	2	2	2]	2	2	2	2	2	2	2	2
4	3]	3]	3	3	3	3	3	3]	3	3	3]	3
7	←	1]	1]	1	1	1	1	1	1	1	1	1
8		←	2]	2]	2	2]	2]	2	2	2	2	2
11			←	3]	3]	3	3	3	3	3	3	3
12				←	1]	1	1	1]	1	1	1]	1
15					←	2	2]	2	2	2	2	2
2						←	3	3]	3]	3	3	3
9							←	3]	3]	3]	3]	3
5								←	2]	2	2	2
6									←	1]	1	1
10										←	2]	2
13											←	4
14												
16												

Table 2: The greedy coloring of the nodes in Example 2.

	C_1					C_2				
node	1	5	7	10	15	2	3	9	11	14
clique degree	1	1	1	1	1	1	1	1	2	2
maximum	1					2				

	C_3				C_4	
node	4	6	12	16	8	13
clique degree	2	2	2	1	2	2
maximum	2				2	

profile of H	2	1
profile of K	2	2

$r = 2, s = 2$				
needed	2	2	2	2
found	2	1	2	2
		+		
$r = 1, s = 2$				
needed	2		1	1
found	2		2	2

Table 3: The nodes with clique degrees and the profiles of H and K in Example 2.

	1	2	3	4	5	6
1	×	•		•	•	
2	•	×	•	•		
3		•	×	•		•
4	•	•	•	×	•	
5	•			•	×	
6			•			×

Table 4: The adjacency matrix of the graph G in Examples 3 and 6.

	1,1	2,1	3,1	4,1	5,1	6,1	1,2	2,2	3,2	4,2	5,2	6,2
1,1	×	•		•	•		•	•		•	•	
2,1	•	×	•	•			•	•	•	•		
3,1		•	×	•		•		•	•	•		•
4,1	•	•	•	×	•		•	•	•	•	•	
5,1	•			•	×		•			•	•	
6,1			•			×			•			•
1,2	•	•		•	•		×	•		•	•	
2,2	•	•	•	•			•	×	•	•		
3,2		•	•	•		•		•	×	•		•
4,2	•	•	•	•	•		•	•	•	×	•	
5,2	•			•	•		•			•	×	
6,2			•			•			•			×

	1,1	1,2	2,1	2,2	3,1	3,2	4,1	4,2	5,1	5,2	6,1	6,2
1,1	×	•	•	•			•	•	•	•		
1,2	•	×	•	•			•	•	•	•		
2,1	•	•	×	•	•	•	•	•				
2,2	•	•	•	×	•	•	•	•				
3,1			•	•	×	•	•	•			•	•
3,2			•	•	•	×	•	•			•	•
4,1	•	•	•	•	•	•	×	•	•	•		
4,2	•	•	•	•	•	•	•	×	•	•		
5,1	•	•					•	•	×	•		
5,2	•	•					•	•	•	×		
6,1					•	•					×	•
6,2					•	•					•	×

Table 5: The adjacency matrix of the auxiliary graph Γ_1 in Example 3.

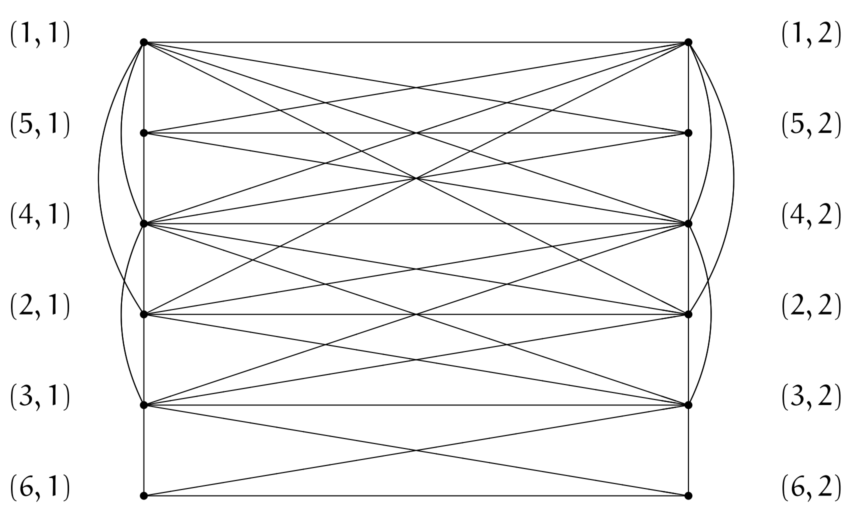

Figure 3: A graphical representation of the auxiliary graph Γ_1 in Example 3.

The chromatic numbers of the graphs G and Γ_1 are not independent of each other. This is the content of the next lemma.

Lemma 5 *Let G be a finite simple graph and let Γ_1 be the associated auxiliary graph. Then $\chi(\Gamma_1) \leq 2\chi(G)$.*

Proof. Set $k = \chi(G)$. The nodes of G have a legal coloring using k colors. The coloring of the nodes of G can be given by a function $f : V \to \{1, 2, \ldots, k\}$, where $f(v)$ is the color of node v of G. Let

$$D = \{(x, y) : 1 \leq x \leq k, 1 \leq y \leq 2\}.$$

Clearly D has $2k$ elements. Using f we construct a function $g : W \to D$ by setting $g((v, a)) = (f(v), a)$ for each $v \in V$, $a \in \{1, 2\}$. We would like to verify that g defines a legal coloring of the nodes of Γ_1.

Let $w_1 = (v_1, a_1)$ and $w_2 = (v_2, a_2)$ be two distinct nodes of Γ_1. Assume on the contrary that w_1, w_2 are adjacent nodes in Γ_1 and $g(w_1) = g(w_2)$. Note that $g(w_1) = g(w_2)$ implies $f(v_1) = f(v_2)$ and $a_1 = a_2$. As $a_1 = a_2$ it follows that v_1 and v_2 are adjacent nodes in G. In this situation $f(v_1) = f(v_2)$ cannot hold. This contradiction shows that g defines a legal coloring of the nodes of Γ_1. Thus $\chi(\Gamma_1) \leq 2k$, as required. □

Combining the results of Lemmas 4 and 5 gives that

$$2\omega(G) \leq \omega(\Gamma_1) \leq \chi(\Gamma_1) \leq 2\chi(G)$$

and so

$$\omega(G) \leq [\chi(\Gamma_1)]/2 \leq \chi(G).$$

Thus $[\chi(\Gamma_1)]/2$ gives a better estimate for the clique number of G than $\chi(G)$ does.

When G is a cycle of odd length, then $\chi(G) = 3$ and $\chi(\Gamma_1) = 5$. There are infinitely many cases with $\chi(\Gamma_1)/2 < \chi(G)$. In a typical application we do not compute chromatic numbers instead using a greedy coloring procedure we locate legal colorings for the nodes of G and Γ_1. The number of colors we find in this way are only upper estimates of the corresponding chromatic numbers. Lemma 5 says nothing about the relation of these upper bounds. On the other hand from the proof of Lemma 5 we can read off that if the nodes of G can be legally colored using k colors, then this coloring can be extended to a legal coloring of the nodes of Γ_1 using $2k$ colors.

When we use a computationally not demanding greedy coloring algorithm we may locate a legal coloring of the nodes of G and Γ_1. Then using the number of colors we establish two upper bounds for $\omega(G)$ and we can use the better one.

	1 2	1 4	1 5	2 3	2 4	3 4	3 6	4 5
1,2	×	•			•			
1,4	•	×	•		•			•
1,5		•	×					•
2,3				×	•	•		
2,4	•	•		•	×	•		
3,4				•	•	×		
3,6							×	
4,5		•	•					×

Table 6: The adjacency matrix of the auxiliary graph Γ_2 in Example 6.

5 The second auxiliary graph

Let $G = (V, E)$ be a finite simple graph. Using G we construct a new graph $\Gamma_2 = (W, F)$. We call this new graph the second auxiliary graph associated with G. The nodes of Γ_2 are the edges of G. Let $w_1 = \{u_1, v_1\}$, $w_2 = \{u_2, v_2\}$ be two distinct nodes of Γ_2. Set $U = \{u_1, v_1, u_2, v_2\}$. Note that as w_1 and w_2 are distinct edges in G, the cardinality of U is either 3 or 4. If the subgraph induced by the set U in G is a clique in G, then we connect the nodes w_1 and w_2 by an edge in Γ_2.

We work out the details of the construction of the auxiliary graph in connection with a small size graph.

Example 6 *Let us consider the finite simple graph* $G = (V, E)$ *given in Example 3.*

A geometric representation of the auxiliary graph Γ_2 is depicted in Figure 4 and Table 6 contains the adjacency matrix of Γ_2.

The clique numbers of the graphs G and Γ_2 are related. This is the content of the next lemma.

Lemma 7 *Let* G *be a finite simple graph and let* Γ_2 *be the associated auxiliary graph. Then* $[\omega(G)][\omega(G) - 1] \leq 2\omega(\Gamma_2)$.

Proof. Set $k = \omega(G)$. The graph G contains a k-clique Δ. Let U be the set of nodes of Δ. Let $T = \{\{u, v\} : u, v \in U, u \neq v\}$. Clearly $|T| = |U|(|U| - 1)/2 = k(k - 1)/2$. Note that any two distinct nodes $\{u_1, v_1\}$, $\{u_2, v_2\}$ in T are always adjacent in Γ_2. □

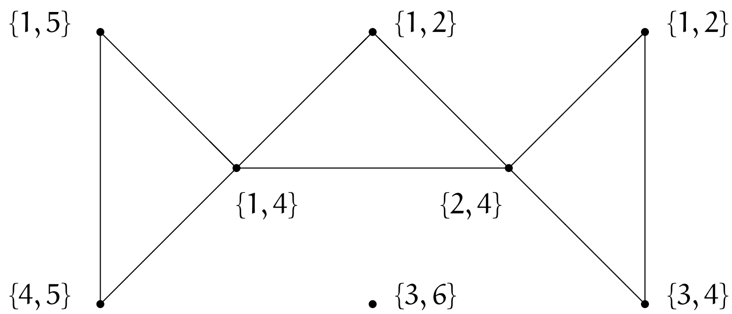

Figure 4: A graphical representation of the auxiliary graph Γ_2 in Example 6.

Using the method of profiles described in Section 2 we may establish that t is an upper bound of $\omega(\Gamma_2)$. By Lemma 7, $[\omega(G)][\omega(G)-1] \leq 2t$. Therefore if t' is the largest integer for which $t'(t'-1) \leq 2t$, then t' is an upper bound for $\omega(G)$.

The chromatic numbers of the graphs G and Γ_2 are not independent of each other. This is the content of the next lemma.

Lemma 8 *Let G be a finite simple graph and let Γ_2 be the associated auxiliary graph. Then $2\chi(\Gamma_2) \leq [\chi(G)][\chi(G)-1]$.*

Proof. Set $k = \chi(G)$. The nodes of G can be colored legally using k colors. The coloring can be given by a function $f : V \to \{1, 2, \ldots, k\}$, where $f(v)$ is the color of the node v of G. Set

$$D = \{\{x, y\} : 1 \leq x, y \leq k, x \neq y\}.$$

Obviously the cardinality of D is equal to $k(k-1)/2$. Using f we construct a function $g : W \to D$ defined by $g(\{u, v\}) = \{f(u), f(v)\}$.

We would like to show that g defines a legal coloring of the nodes of Γ_2. Let $w_1 = \{u_1, v_1\}$, $w_2 = \{u_2, v_2\}$ be two distinct adjacent nodes of Γ_2. Let $U = \{u_1, v_1, u_2, v_2\}$. Assume on the contrary that $g(w_1) = g(w_2)$.

Let us consider the case when the cardinality of the set U is four. In this case the nodes u_1, v_1, u_2, v_2 are pairwise distinct and they are nodes of a 4-clique in G. As $\{u_1, v_1\}$ is an edge of G and f is a legal coloring of the nodes of G it follows that $f(u_1) \neq f(v_1)$. Similarly $f(u_2) \neq f(v_2)$ must hold. From the assumption

$$g(w_1) = \{f(u_1), f(v_1)\} = \{f(u_2), f(v_2)\} = g(w_2)$$

we get that either

$$f(u_1) = f(u_2), \quad f(v_1) = f(v_2)$$

or

$$f(u_1) = f(v_2), \quad f(v_1) = f(u_2).$$

The unordered pairs

$$\{u_1, u_2\}, \{u_1, v_2\}, \{v_1, u_2\}, \{v_1, v_2\}$$

are edges of G. This violates the fact that f is a legal coloring.

Let us turn to the case when U has three elements. In this case we may assume that $u_1 = u_2$ and $v_1 \neq v_2$ since this is only a matter of renaming the nodes. In this situation $\{v_1, v_2\}$ is an edge of G. The $g(w_1) = g(w_2)$ assumption reduces to $\{f(v_1)\} = \{f(v_2)\}$ and we get the contradiction that the end nodes of the edge $\{v_1, v_2\}$ are not legally colored. □

Let t be the largest integer for which $t(t-1) \leq 2\chi(\Gamma_2)$. Combining the results of Lemmas 7 and 8 we get

$$[\omega(G)][\omega(G) - 1] \leq 2\omega(\Gamma_2) \leq 2\chi(\Gamma_2) \leq [\chi(G)][\chi(G) - 1]$$

and so $\omega(G) \leq t \leq \chi(G)$. This means that using $\chi(\Gamma_2)$ one gets a better estimate for $\omega(G)$ than using $\chi(G)$.

J. Mycielski [10] proved the following result. For each positive integer n there is a graph M_n such that $\omega(M_n) = 2$ and $\chi(M_n) = n$. Let G be M_n. In this case the auxiliary graph Γ_2 consists of isolated nodes. Thus $\chi(\Gamma_2) = 1$. Now $\chi(\Gamma_2)$ and the inequality $[\omega(G)][\omega(G)-1] \leq 2\chi(\Gamma_2)$ provide $\omega(G) \leq 2$. In other words using $\chi(\Gamma_2)$ we get the upper bound 2 for $\omega(G)$ while using $\chi(G)$ we get n as an upper bound for $\omega(G)$.

6 Numerical experiments

In order to test the practical utility and feasibility of the method of profiles we have carried out numerical experiments. In this section we describe the results of these experiments.

The graphs we used are belonging to three families. However all test graphs are coming from coding theory. Monotonic matrices are related to certain one

n	$\|V\|$	$\|E\|$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
3	27	189	6	6	6	5	6	5
4	64	1 296	12	10	12	10	12	10
5	125	5 500	20	18	20	17	20	18
6	216	17 550	30	27	30	26	30	27
7	343	46 305	42	37	42	38	42	39
8	512	106 624	56	50	56	51	56	52
9	729	221 616	72	66	72	67	72	68
10	1 000	425 250	90	83	90	84	90	85
11	1 331	765 325	110	103	110	103	110	105
12	1 728	1 306 800	132	124	132	124	132	126
13	2 197	2 135 484	156	145	156	147	156	150

Table 7: Monotonic matrices, simple greedy coloring, first auxiliary graph.

n	$\|V\|$	$\|E\|$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
3	8	9	2	2	2	2	2	2
4	16	57	4	4	4	4	4	4
5	32	305	8	8	7	7	6	6
6	64	1 473	14	14	13	12	12	11
7	128	6 657	26	26	23	22	22	20
8	256	28 801	50	50	45	44	40	39
9	512	121 089	101	98	88	86	79	75
10	1 024	499 713	199	194	170	165	146	143
11	2 048	2 037 761	395	386	329	325	278	274

Table 8: Deletion error detecting codes, simple greedy coloring, first auxiliary graph.

n	$\lvert V\rvert$	$\lvert E\rvert$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
6	15	45	4	4	4	3	4	3
7	35	385	10	9	10	8	10	8
8	70	1 855	20	19	20	17	20	18
9	126	6 615	35	33	35	31	35	32
10	210	19 425	56	53	56	52	56	52
11	330	49 665	84	81	84	78	84	79
12	495	114 345	120	116	120	114	120	114
13	715	242 385	165	162	165	159	165	158
14	1 001	480 480	220	216	220	214	220	212
15	1 365	900 900	286	282	286	277	286	278
16	1 820	1 611 610	364	358	364	355	364	354

Table 9: Johnson codes, simple greedy coloring, first auxiliary graph.

n	$\lvert V\rvert$	$\lvert E\rvert$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
3	27	189	6	6	5	5	5	5
4	64	1 296	11	11	10	9	10	8
5	125	5 500	17	17	16	15	16	14
6	216	17 550	26	26	22	21	24	21
7	343	46 305	36	34	34	32	32	30
8	512	106 624	47	46	43	41	43	41
9	729	221 616	58	58	56	55	53	51
10	1 000	425 250	74	74	69	67	67	65
11	1 331	765 325	90	90	85	84	86	84

Table 10: Monotonic matrices, dsatur coloring, first auxiliary graph.

n	$\lvert V\rvert$	$\lvert E\rvert$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
3	8	9	2	2	2	2	2	2
4	16	57	4	4	4	4	4	4
5	32	305	7	7	6	6	6	6
6	64	1 473	13	13	12	12	11	11
7	128	6 657	23	23	22	21	22	20
8	256	28 801	43	43	40	40	43	40
9	512	121 089	79	79	80	77	79	77
10	1 024	499 713	156	156	154	154	153	151

Table 11: Deletion error detecting codes, dsatur coloring, first auxiliary graph.

n	$\lvert V\rvert$	$\lvert E\rvert$	$\overline{\omega}_1$	$\widehat{\omega}_1$	$\overline{\omega}_2$	$\widehat{\omega}_2$	$\overline{\omega}_4$	$\widehat{\omega}_4$
6	15	45	4	4	4	3	3	3
7	35	385	9	9	8	8	9	8
8	70	1 855	17	17	16	15	15	14
9	126	6 615	29	27	28	26	28	28
10	210	19 425	46	46	44	43	43	42
11	330	49 665	67	67	63	63	62	62
12	495	114 345	99	99	90	89	87	85
13	715	242 385	132	132	121	120	122	121
14	1 001	480 480	172	172	153	153	160	160
15	1 365	900 900	221	221	201	201	206	205

Table 12: Johnson codes, dsatur coloring, first auxiliary graph.

n	$\lvert V\rvert$	$\lvert E\rvert$	$\overline{\chi}$	$\overline{\omega}$	$\widehat{\chi}$	$\widehat{\omega}$
3	27	189	10	5	10	5
4	64	1 296	37	9	32	8
5	125	5 500	113	15	103	14
6	216	17 550	273	23	257	23
7	343	46 305	565	34	542	33

Table 13: Monotonic matrices, simple greedy coloring, second auxiliary graph.

n	$\|V\|$	$\|E\|$	$\bar{\chi}$	$\bar{\omega}$	$\hat{\chi}$	$\hat{\omega}$
3	8	9	1	2	1	2
4	16	57	6	4	6	4
5	32	305	17	6	16	6
6	64	1 473	60	11	53	10
7	128	6 657	221	21	207	20
8	256	28 801	875	42	846	41

Table 14: Deletion error detecting codes, simple greedy coloring, second auxiliary graph.

n	$\|V\|$	$\|E\|$	$\bar{\chi}$	$\bar{\omega}$	$\hat{\chi}$	$\hat{\omega}$
6	15	45	3	3	3	3
7	35	385	23	7	23	7
8	70	1 855	107	15	98	14
9	126	6 615	391	28	372	27
10	210	19 425	1 131	48	1 098	48
11	330	49 665	2 754	74	2 703	73

Table 15: Johnson codes, simple greedy coloring, second auxiliary graph.

n	$\|V\|$	$\|E\|$	$\bar{\chi}$	$\bar{\omega}$	$\hat{\chi}$	$\hat{\omega}$
3	27	189	10	5	10	5
4	64	1 296	31	8	31	8
5	125	5 500	83	13	75	12

Table 16: Monotonic matrices, dsatur coloring, second auxiliary graph.

n	$\|V\|$	$\|E\|$	$\bar{\chi}$	$\bar{\omega}$	$\hat{\chi}$	$\hat{\omega}$
3	8	9	1	2	1	2
4	16	57	6	4	6	4
5	32	305	15	6	15	6
6	64	1 473	50	9	50	9
7	128	6 657	196	20	183	19

Table 17: Deletion error detecting codes, dsatur coloring, second auxiliary graph.

n	$\|V\|$	$\|E\|$	$\overline{\chi}$	$\overline{\omega}$	$\widehat{\chi}$	$\widehat{\omega}$
6	15	45	3	3	3	3
7	35	385	23	7	22	7
8	70	1 855	111	15	101	14
9	126	6 615	340	25	323	24

Table 18: Johnson codes, dsatur coloring, second auxiliary graph.

error correcting codes. (The reader can find further details in [14].) A deletion error occurs when a fixed length code word is loosing one letter during transmission. The deletion error correcting codes are connected to this phenomenon. Binary codes with fixed length code words with a specified number of zeros are the Johnson codes. The graphs associated with these codes are commonly used for testing clique search algorithm. (See for instance [6].)

The method of profiles is flexible in the sense that we are free to choose any node coloring algorithm to construct a legal coloring of the nodes of the original graph or the auxiliary graphs. In the numerical experiments we carried out only two greedy coloring algorithms were employed. One of them is the most commonly used simple greedy sequential coloring. The other one is the dsatur coloring algorithm described in [3].

In Table 7 the first column contains the parameter n of the graph. This parameter is related to the size of the alphabet over which the code is defined. The columns labeled by $|V|$ and $|E|$ hold the numbers of the nodes and the edges of the graph, respectively. The column headed by $\overline{\omega}_1$ holds the clique size estimate we get coloring the nodes of the original graph using the simple greedy coloring algorithm. Here the number of the colors is the upper estimate of the clique size. The column headed by $\widehat{\omega}_1$ holds the clique size estimate we get coloring the nodes of the original graph using the simple greedy coloring algorithm. This time the estimate of the clique size is the result of the method of profiles.

The column labeled by $\overline{\omega}_2$ refers to the clique size estimate we get coloring the nodes of the first auxiliary graph using the simple greedy coloring algorithm. Here half of the number of the colors is the upper estimate of the clique size. The column labeled by $\widehat{\omega}_2$ refers to the clique size estimate we get coloring the nodes of the first auxiliary graph using the simple greedy coloring algorithm. This time the estimate of the clique size is the result of the method of profiles.

The column labeled by $\overline{\omega}_4$ gives the clique size estimate we get coloring the nodes of the first auxiliary graph of the first auxiliary using the simple greedy coloring algorithm. Here quarter of the number of the colors is the upper estimate of the clique size. The column labeled by $\widehat{\omega}_4$ gives the clique size estimate we get coloring the nodes of the first auxiliary graph of the first auxiliary graph using the simple greedy coloring algorithm. This time the estimate of the clique size is the result of the method of profiles.

Note that the number of the nodes of the first auxiliary graph is the double of the number of the nodes of the original graph. We adopt the terminology that the original graph is a 1-fold version of itself, the first auxiliary graph is a 2-fold version of the original graph, and the first auxiliary graph of the first auxiliary graph is a 4-fold version of the original graph. Using this terminology we may say that the column labeled by $\overline{\omega}_b$ contains the clique size estimate based on the b-fold version of the original graph not using the proposed procedure. Further the column labeled by $\widehat{\omega}_b$ contain the clique size estimate based on the b-fold version of the original graph using the method of profiles.

Tables 8 and 9 exhibit analogous information as Table 7. In these tables the graphs associated with monotonic matrices are replaced by graphs associated with deletion error correcting and Johnson codes, respectively.

Tables 10, 11, 12 summarize similar results that are in Tables 7, 8, 9. The only difference is that at these occasions the simple greedy coloring algorithm is replaced by the dsatur coloring algorithm.

The first three columns in Table 13 are labeled by n, $|V|$, $|E|$ record the parameter, the number of the nodes, the number of the edges of the graph associated with a monotonic matrix. The columns labeled by $\overline{\chi}$ and $\overline{\omega}$ contain the number of colors produced by simple greedy coloring procedure applied to the second auxiliary graph and the clique size estimate derived from this number of colors, respectively. The last two columns labeled by $\widehat{\chi}$ and $\widehat{\omega}$ show the reduced number of colors the method of profiles gives and the derived clique size estimate, respectively.

Tables 14 and 15 present similar results as Table 13 the only thing which has changed is that the graph associated with monotonic matrices are replaced by graphs associated with deletion error correcting and Johnson codes.

Finally Tables 16, 17, 18 exhibit similar results as Tables 13, 14, 15 but this time the simple greedy coloring procedure is replaced by the dsatur coloring algorithm.

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Received: February 23, 2018 • Revised: September 10, 2018