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Construction of a common element for the set of solutions of fixed point problems and generalized equilibrium problems in Hilbert spaces

Abstract. In this paper, we propose and analyse an iterative algorithm for the approximation of a common solution for a finite family of k-strict pseudo-contractions and two finite families of generalized equilibrium problems in the setting of Hilbert spaces. Strong convergence results of the proposed iterative algorithm together with some applications to solve the variational inequality problems are established in such setting. Our results generalize and improve various existing results in the current literature.

1. Introduction

Throughout this paper, we work in the setting of a real Hilbert space H equipped with the inner product $\langle \cdot\ ,\cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty subset of a real Hilbert space H and let $T\colon C \to C$ be a mapping. The set of fixed points of the mapping T is defined and denoted as $F(T) = \{x \in C :\ T(x) = x\}$. The mapping T is said to be

- (i) a contraction, if $\|Tx - Ty\| \le \alpha\|x - y\|$ for all $x, y \in C$ and $\alpha \in (0, 1)$;
- (ii) nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;
- (iii) Lipschitzian, if $\|Tx-Ty\| \le L\|x-y\|$ for some $L > 0$ and for all $x, y \in C$;
- (iv) firmly nonexpansive, if

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle \qquad \text{for all } x, y \in C; \tag{1}$$

- (v) a pseudo-contraction, if

$$\langle x - y\ , Tx - Ty\rangle \le \|x - y\|^2 \qquad \text{for all } x, y \in C; \tag{2}$$

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(vi) a strict pseudo-contraction, if there exists $0 < \lambda \leq \frac{1}{2}$, such that for all $x, y \in C$,

$$\langle x - y\ , Tx - Ty \rangle \leq \|x - y\|^2 - \lambda \|(I - T)x - (I - T)y\|^2; \tag{3}$$

(vii) a strong pseudo-contraction, if there exists a positive constant $\delta \in (0, 1)$ such that $T - \delta I$ is a pseudo-contractive mapping.

It is remarked that the inequalities (1) and (2) are equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2,$$

respectively. Note that, if we set $k = 1 - 2\lambda \in [0, 1)$, then (3) takes the form

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2.$$

Throughout the manuscript, we use the above formulation of the k-strict pseudo-contraction in the terminology of Browder and Petryshyn [3] which satisfies the following Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|.$$

Moreover, if we set $A := I - T$, where I is the identity mapping, then (3) is equivalent to

$$\langle x - y\ , Ax - Ay \rangle \geq \lambda \|Ax - Ay\|^2. \tag{4}$$

For such a case, the mapping A is then referred as a λ-inverse strongly monotone mapping.

Metric fixed point theory has its roots in the celebrated Banach Contraction Principle (BCP), which asserts that a contraction on a complete metric space has a unique fixed point. Besides this, the BCP also provides a constructive procedure for the approximation of such unique fixed points. Moreover, BCP has valuable applications to various nonlinear problems such as Fredholm and Volterra integral equations, ordinary differential equations, partial differential equations and image processing. Since then, metric fixed point theory has emerged as a powerful tool to solve various nonlinear problems arising in different branches of mathematical sciences. As a consequence, the BCP is a highly cited result in the whole theory of analysis.

It is worth mentioning that the BCP dominates fixed point theory only for the class of contraction mappings. The essential ingredient of the BCP such as completeness of the metric space, uniqueness of the fixed point, and the sequence of successive approximation whose order of convergence is of geometric progression, are no longer true for the class of nonexpansive mappings. In fact, fixed point theory of nonexpansive mappings depends on the geometrical structures of the underlying space. Fixed points of nonexpansive mappings have a diverse range of applications to solve problems such as variational inequality problem, convex minimization problem of a function and zeros of a monotone operator. It is therefore, natural to extend such powerful results of the class of nonexpansive mappings to more general class of mappings such as asymptotically nonexpansive mappings, pseudo-contractions, strict pseudo-contractions and others.

In 1967, Browder and Petryshyn [3] introduced the class of strict pseudo-contractions as an important and significant generalization of the class of nonexpansive mappings. It is evident from the definition of a k-strict pseudo-contraction T that

(i) nonexpansive mappings are 0-strict pseudo-contraction, whereas firmly nonexpansive mappings are -1-strict pseudo-contraction, and therefore the class of k-strict pseudo-contractions contains both firmly nonexpansive mappings and nonexpansive mappings;

(ii) the class of k-strict pseudo-contractions falls into the one between the classes of nonexpansive mappings and pseudo-contractions;

(iii) the class of strong pseudo-contractions is independent of the class of k-strict pseudo-contractions.

In view of the above the discussion, the class of k-strict pseudo-contractions is prominent among various classes of nonlinear mappings in the current literature. The iterative construction of fixed points of nonexpansive mappings have been extensively studied in the current literature. Although strict pseudo-contractions have more powerful applications than the nonexpansive mappings for solving inverse problems [19], iterative construction of fixed points of strict pseudo-contractions are far less developed, because of the second term in the definition, than those for nonexpansive mappings. On the other hand, various existing iterative algorithms for k-strict pseudo-contractions possess only weakly convergent characteristics. It is therefore, natural to propose and analyse iterative algorithms for the construction of fixed points of k-strict pseudo-contractions and established strong convergence results under suitable conditions.

Equilibrium problem theory provides a unified approach to address a variety of problems arising in various disciplines of science such as physics, optimization and economics. The existence result of an equilibrium problem can be found in the seminal work of Blum and Oettli [2]. From the computational view point, Combettes and Hirstoaga [5] introduced an iterative algorithm in Hilbert space for the best approximation of the equilibrium point assuming the set of solutions of equilibrium points is nonempty. It is worth to mention that the KKM lemma of Ky Fan [7] not only plays a key role in the development of a classical existence result of equilibrium problem theory but still in practice to prove the existence of equilibrium points for various generalized versions of classical equilibrium problems.

Let C be a nonempty subset of a real Hilbert space H, let $A\colon C \to H$ be a nonlinear mapping and let $f\colon C \times C \to \mathbb{R}$ (the set of reals) be a bifunction. A generalized equilibrium problem, is to find $x \in C$ such that

$$f(x, y) + \langle Ax, y - x\rangle \geq 0 \qquad \text{for all } y \in C. \tag{5}$$

We denote the solution set of a generalized equilibrium problem by $GEP(f, A)$.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (c.f. [2] and [5])

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4) $f(x, \cdot)$ is convex and lower semi-continuous for all $x \in C$.

Note that, if $A \equiv 0$ then the problem (5) reduces to the classical equilibrium problem $EP(f)$. That is, to find $x \in C$ such that $f(x, y) \geq 0$. Moreover, if $f(x, y) \equiv 0$ then the problem (5) reduces to the classical variational inequality problem $VI(C, A)$. That is, to find $x \in C$ such that $\langle Ax, y - x\rangle \geq 0$.

We remark that the generalized equilibrium problem includes - as a special case - fixed point problems, the Nash equilibrium problem in noncooperative games and variational inequality problem. Moreover, as a direct consequence of the variational inequality problem, the generalized equilibrium problem can also be used to solve image recovery problems, inverse problems, transportation problems, optimization problems, minimax problems and others; see for instance [2, 5, 12, 15, 20, 17, 18].

It is worth mentioning that the problem of finding a common element in the set of solutions of an equilibrium problem and the set of fixed points of a nonlinear mapping is a fascinating field of research. Therefore, numerous iterative algorithms have been proposed and analyzed to solve such problems, see, for example, [4, 8, 10, 11, 13, 21, 22, 23] and the references cited therein. Approximation of a common solution for a pair of equilibrium problems has relevant important applications in various branches of applied mathematics. In fact, a common solution of the system of equilibrium problems solves the corresponding system of problems as mentioned above. In particular, a common solution of two or more equilibrium problems can be used to solve various forms of feasibility problems, see, for example, [1, 6] and the references cited therein.

In 2010, Kim et al. [14] approximated a common element of the set of common solutions of two generalized equilibrium problems and the set of fixed points of a strict pseudo-contraction in a Hilbert space. They proved the following result.

Theorem KCQ ([14])
Let C be a nonempty closed convex subset of a real Hilbert space H and let $T\colon C \to C$ be a k-strict pseudo-contraction. Let $F_1, F_2\colon C \times C \to \mathbb{R}$ be two bifunctions satisfying (A1)–(A4) *and let $A, B\colon C \to H$ be η-inverse-strongly monotone and ζ-inverse-strongly monotone mappings, respectively. Let $\{r_n\}$ and $\{s_n\}$ be two positive real sequences. Assume that $\mathbb{F} := F(T) \cap GEP(F_1, A) \cap GEP(F_2, B) \neq \emptyset$, and the sequence $\{x_n\}$ is generated by*

$$\begin{aligned}
x_1 &\in C_1 = C,\\
F_1(u_n,u) &+ \langle Ax_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, \quad \forall u \in C,\\
F_2(v_n,v) &+ \langle Bx_n, v-v_n\rangle + \frac{1}{s_n}\langle v-v_n, v_n-x_n\rangle \geq 0, \quad \forall v \in C,\\
z_n &= \gamma_n u_n + (1-\gamma_n)v_n,\\
y_n &= \alpha_n x_n + (1-\alpha_n)(\beta_n z_n + (1-\beta_n)Tz_n),\\
C_{n+1} &= \{z \in C_n : \ \|y_n - z\| \leq \|x_n - z\|\},\\
x_{n+1} &= P_{C_{n+1}}x_1, \quad n \geq 1,
\end{aligned} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$ *and* $\{s_n\}$ *satisfy the following restrictions:*

(*i*) $0 \leq \alpha_n < a < 1$;

(*ii*) $0 \leq k \leq \beta_n < b < 1$;

(*iii*) $0 \leq c \leq \gamma_n < d < 1$;

(*iv*) $0 < e \leq r_n \leq f < 2\eta$ *and* $0 < e^{'} \leq s_n \leq f^{'} < 2\zeta$.

Then the sequence $\{x_n\}$ *generated by* (6) *converges strongly to* $x = P_{\mathbb{F}}x_1$, *where* $P_{\mathbb{F}}$ *is the metric projection of* H *onto* $\mathbb{F}$.

In order to reduce the computational cost of the iteration (6), Kangtunyakarn [9] proposed an efficient and simple hybrid iteration - wherein the y_n component is modified - for the following strong convergence result.

Theorem K ([9])
Let C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $T\colon C \to C$ *be a* k*-strict pseudo-contraction. Let* $F, G\colon C \times C \to \mathbb{R}$ *be two bifunctions satisfying* (A1)–(A4) *and let* $A, B\colon C \to H$ *be* η*-inverse-strongly monotone and* ζ*-inverse-strongly monotone mappings, respectively. Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real sequences. Assume that* $\mathbb{F} := F(T) \cap GEP(F,A) \cap GEP(G,B) \neq \emptyset$, *and the sequence* $\{x_n\}$ *is generated by*

$$\begin{aligned}
x_1 &\in C_1 = C,\\
F(u_n,u) &+ \langle Ax_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, \quad \forall u \in C,\\
G(v_n,v) &+ \langle Bx_n, v-v_n\rangle + \frac{1}{s_n}\langle v-v_n, v_n-x_n\rangle \geq 0, \quad \forall v \in C,\\
z_n &= \gamma_n u_n + (1-\gamma_n)v_n,\\
y_n &= \alpha_n z_n + (1-\alpha_n)Tz_n,\\
C_{n+1} &= \{z \in C_n : \ \|y_n - z\| \leq \|x_n - z\|\},\\
x_{n+1} &= P_{C_{n+1}}x_1, \quad n \geq 1,
\end{aligned} \tag{7}$$

where $\{\alpha_n\}$ *is a sequences in* $[0,1]$, $r_n \in [a,b] \subset (0,2\eta)$ *and* $s_n \in [c,d] \subset (0,2\zeta)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$*;*

(ii) $0 \le k \le \alpha_n < 1$ *for all* $n \ge 1$*.*

Then the sequence $\{x_n\}$ *generated by* (7) *converges strongly to* $x = P_{\mathbb{F}} x_1$*.*

Inspired and motivated by the works of Kim et al. [14] and Kangtunyakarn [9], we propose and analyze an iterative algorithm to find a common element in the set of common solutions of two finite families of generalized equilibrium problems and the set of common fixed points of a finite family of k-strict pseudo-contraction in the setting of Hilbert spaces. We establish strong convergence results under some mild conditions on the control sequences of parameters and consequently refine and improve various results announced in the current literature.

2. Preliminaries

Throughout this paper, we write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate strong convergence (resp. weak convergence) of a sequence $\{x_n\}_{n=1}^{\infty}$. Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. For each $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \qquad \text{for all } y \in C.$$

Such a mapping $P_C \colon H \to C$ is known as a metric projection or a nearest point projection of H onto C. Moreover, P_C is characterized as the nonexpansive mapping in a Hilbert space that satisfies $\langle x - P_C x, P_C x - y\rangle \ge 0$ for all $x, y \in C$. Recall that a nonlinear mapping $A \colon C \to H$ is λ-inverse strongly monotone if it satisfies (4). Note that, if $A := I - T$ is a λ-inverse strongly monotone mapping, then

(i) A is $(\frac{1}{\lambda})$-Lipschitz continuous mapping;

(ii) if T is a nonexpansive mapping, then A is $(\frac{1}{2})$-inverse strongly monotone mapping;

(iii) if T is a k-strict pseudo-contraction, then A is $(\frac{1-k}{2})$-inverse strongly monotone mapping;

(iv) if $\eta \in (0, 2\lambda]$, then $I - \eta A$ is a nonexpansive mapping.

The nonexpansivity of $I - \eta A$ can be inferred from the following estimate

$$\begin{aligned} \|(I-\eta A)x - (I-\eta A)y\|^2 &= \|x - y - \eta(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\eta\langle x - y, Ax - Ay\rangle \\ &\quad + \eta^2\|Ax - Ay\|^2 \\ &\le \|x-y\|^2 + \eta(\eta - 2\lambda)\|Ax - Ay\|^2 \\ &\le \|x-y\|^2. \end{aligned} \tag{8}$$

Hence, when $\eta \le 2\lambda$, then $I - \eta A$ is nonexpansive.

The following result is crucial for the best approximation of equilibrium points and can be found in [2].

Lemma 2.1
Let C be a closed convex subset of a real Hilbert space H and let $f\colon C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)–(A4). *For $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0 \qquad \textit{for all } y\in C.$$

Moreover, define a mapping $V_r\colon H\to C$ by

$$V_r(x)=\Big\{z\in C:\ f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0\ \ \textit{for all } y\in C\Big\}$$

for all $x\in H$. Then, the following hold

(1) *$EP(f)$ is closed and convex;*

(2) *V_r is single-valued;*

(3) *V_r is firmly nonexpansive-type mapping, i.e.*

$$\|V_rx-V_ry\|^2\leq\langle V_rx-V_ry,x-y\rangle \qquad \textit{for all } x,y\in H,$$

(4) *$F(V_r)=EP(f)$.*

The following results collect some of the characterizations of a k-strict pseudo-contraction T and the set of fixed points $F(T)$ in Hilbert spaces.

Lemma 2.2 ([16, Proposition 2.1 (iii)])
Let C be a nonempty closed convex subset of a real Hilbert space H. If $T\colon C\to H$ is a k-strict pseudo-contraction, then the fixed point set $F(T)$ is closed and convex so that the projection $P_{F(T)}$ is well defined.

Lemma 2.3 ([16, Proposition 2.1 (ii)])
Let C be a nonempty closed convex subset of a real Hilbert space H and $T\colon C\to C$ a k-strict pseudo-contraction. Then $(I-T)$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C with $x_n\rightharpoonup x$ and $x_n-Tx_n\to 0$, then $x\in F(T)$.

The following identity in the form of a lemma is well-known in the context of a real Hilbert space.

Lemma 2.4
Let H be a real Hilbert space, then the following identity holds

$$\|\alpha x+(1-\alpha)y\|^2=\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2.$$

3. Strong convergence results

We first set some of the notions required in the sequel for our main result of this section. For a nonempty subset C of a real Hilbert space H, we assume that

(i) $T_i(\mathrm{mod}\,N)\colon C\to C$ be a finite family of k-strict pseudo-contractions such that $k=\max\{k_i:\ 1\leq i\leq N\}$;

(ii) $f_i(\mathrm{mod}\,N)\colon C\times C\to\mathbb{R}$ be a finite family of bifunctions;

(*iii*) $A_i(\mathrm{mod}\, N) \colon C \to H$ be a finite family of η-inverse-strongly monotone mappings such that $\eta = \max\{\eta_i : \ 1 \leq i \leq N\}$ and

(*iv*) $B_i(\mathrm{mod}\, N) \colon C \to H$ be a finite family of ζ-inverse-strongly monotone mappings such that $\zeta = \max\{\zeta_i : \ 1 \leq i \leq N\}$.

We are now in a position to prove our main result of this section.

Theorem 3.1

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i \colon C \to C$ be a finite family of k-strict pseudo-contractions. Let $f_i, g_i \colon C \times C \to \mathbb{R}$ be two finite families of bifunctions satisfying (A1)–(A4) *and let $A_i, B_i \colon C \to H$ be two finite families of η-inverse-strongly monotone and ζ-inverse-strongly monotone mappings, respectively. Assume that*

$$\mathbb{F} := \left[\bigcap_{i=1}^{N} F(T_i)\right] \cap \left[\bigcap_{i=1}^{N} GEP(f_i, A_i)\right] \cap \left[\bigcap_{i=1}^{N} GEP(g_i, B_i)\right] \neq \emptyset$$

and the sequence $\{x_n\}$ is generated by

$$\begin{aligned}
&x_1 \in C_1 = C, \\
&f_i(u_{n,i}, u) + \langle A_i x_n, u - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle u - u_{n,i}, u_{n,i} - x_n\rangle \geq 0, \quad \forall u \in C, \\
&g_i(v_{n,i}, v) + \langle B_i x_n, v - v_{n,i}\rangle + \frac{1}{s_{n,i}}\langle v - v_{n,i}, v_{n,i} - x_n\rangle \geq 0, \quad \forall v \in C, \\
&z_{n,i} = \alpha_{n,i} u_{n,i} + (1-\alpha_{n,i}) v_{n,i}, \\
&y_{n,i} = \beta_{n,i} z_{n,i} + (1-\beta_{n,i}) T_i z_{n,i}, \\
&C_{n+1} = \Big\{ z \in C_n : \ \sup_{i \geq 1} \|y_{n,i} - z\| \leq \|x_n - z\| \Big\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1,
\end{aligned} \tag{9}$$

where $\{r_{n,i}\}$, $\{s_{n,i}\}$ are two positive real sequences and $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ are sequences in $(0,1)$. Assume that $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ satisfy the following restrictions:

($C1$) $0 \leq k < a \leq \alpha_{n,i}, \ \beta_{n,i} \leq b < 1$*;*

($C2$) $0 < c \leq r_{n,i} \leq d < 2\eta$ *and* $0 < c' \leq s_{n,i} \leq d' < 2\zeta$ *for all* $n \geq 1$ *and for all* $i \in I$*.*

Then the sequence $\{x_n\}$ generated by (9) *converges strongly to $x = P_{\mathbb{F}} x_1$.*

Proof. Let $p \in \mathbb{F}$ is a common element, therefore $p = T_i p$, for each $1 \leq i \leq N$, represents that p is a common fixed point of the finite family of strict pseudo-contractions T_i. Moreover,

$$p = \bigcap_{i=1}^{N} (V_{r_{n,i}}(p - r_{n,i} A_i p)) = \bigcap_{i=1}^{N} (V_{s_{n,i}}(p - s_{n,i} B_i p)),$$

as mentioned in Lemma 2.1, represents that p is a common solution of two finite families of generalized equilibrium problems. Furthermore, Lemma 2.1 implies that $u_{n,i}$ can be written as $u_{n,i} = V_{r_{n,i}}(x_n - r_{n,i}A_i x_n)$ for all $n \geq 1$. It follows from the estimate (8), that $I - r_{n,i}A_i$ is nonexpansive. Similarly, we can establish the nonexpansivity of $I - s_{n,i}B_i$. This further implies that $V_{r_{n,i}}(I - r_{n,i}A_i)$ and $V_{s_{n,i}}(I - s_{n,i}B_i)$ are nonexpansive as well. Hence we have

$$\|u_{n,i} - p\| \leq \|x_n - p\| \quad \text{and} \quad \|v_{n,i} - p\| \leq \|x_n - p\|. \tag{10}$$

As a direct consequence of (10) we get

$$\|z_{n,i} - p\| \leq \|x_n - p\|. \tag{11}$$

It is now easy to show that C_n is closed and convex for every $n \in \mathbb{N}$. It follows from the definition that C_{n+1} is closed. Next, we show by induction that C_{n+1} is convex. Since $C_1 = C$ is convex, we assume that C_k is convex for some $k \geq 2$. For any $z \in C_k$, the inequality $\|y_{n,i} - z\| \leq \|x_n - z\|$ is equivalent to

$$\|y_{n,i}\|^2 - \|x_n\|^2 - 2\langle z, y_{n,i} - x_n, \rangle \geq 0.$$

This implies that C_{k+1} is convex and hence we conclude that C_{n+1} is convex for each $n \geq 1$. It remains to show that $\mathbb{F} \subset C_{n+1}$ for all $n \geq 1$. Obviously, $\mathbb{F} \subset C_1 = C$. Let $p \in \mathbb{F}$. Utilizing (10)–(11) and Lemma 2.4, we have the following estimate

$$\begin{aligned} \|y_{n,i} - p\|^2 &= \|\beta_{n,i}(z_{n,i} - p) + (1 - \beta_{n,i})(T_i z_{n,i} - p)\|^2 \\ &= \beta_{n,i}\|z_{n,i} - p\|^2 + (1 - \beta_{n,i})\|T_i z_{n,i} - p\|^2 \\ &\quad - \beta_{n,i}(1 - \beta_{n,i})\|z_{n,i} - T_i z_{n,i}\|^2 \\ &\leq \beta_{n,i}\|z_{n,i} - p\|^2 \\ &\quad + (1 - \beta_{n,i})(\|z_{n,i} - p\|^2 + k\|z_{n,i} - T_i z_{n,i}\|^2) \\ &\quad - \beta_{n,i}(1 - \beta_{n,i})\|z_{n,i} - T_i z_{n,i}\|^2 \\ &\leq \|z_{n,i} - p\|^2 - (1 - \beta_{n,i})(\beta_{n,i} - k)\|z_{n,i} - T_i z_{n,i}\|^2. \end{aligned} \tag{12}$$

Since $\beta_{n,i} - k \geq 0$ (by $(C1)$), therefore the above estimate (12) yields

$$\|y_{n,i} - p\| \leq \|z_{n,i} - p\| \leq \|x_n - p\|. \tag{13}$$

This implies that $p \in C_{n+1}$ for all $n \geq 1$ and hence the iteration (9) is well-defined.

We now only compute estimates which we subsequently use in the sequel.

Note that $x_{n+1} = P_{C_{n+1}}x_1$, therefore $\|x_{n+1} - x_1\| \leq \|x_0 - x_1\|$ for all $x_0 \in C_{n+1}$. In particular, we have $\|x_{n+1} - x_1\| \leq \|P_{\mathbb{F}}x_1 - x_1\|$. This implies that $\{x_n\}$ is bounded, so are $\{u_{n,i}\}$, $\{v_{n,i}\}$, $\{y_{n,i}\}$ and $\{z_{n,i}\}$. On the other hand, $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\leq -\|x_1 - x_{n1}\|^2 + \|x_{n+1} - x_1\|\|x_n - x_1\|. \end{aligned}$$

The above estimate implies that $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$. That is, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. This implies that the following limit exists

$$\lim_{n\to\infty} \|x_n - x_1\|. \tag{14}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1\rangle \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n + x_n - x_1\rangle \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n\rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\end{aligned}$$

Taking $\limsup$ on both sides of the above estimate and utilizing (14), we have $\limsup_{n\to\infty} \|x_{n+1} - x_n\|^2 = 0$. That is

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{15}$$

Since $x_{n+1} \in C_{n+1}$, we have $\|y_{n,i} - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. This implies that

$$\lim_{n\to\infty} \|y_{n,i} - x_{n+1}\| = 0. \tag{16}$$

Utilizing (15) and (16) and the following triangular inequality

$$\|y_{n,i} - x_n\| \leq \|y_{n,i} - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we get

$$\lim_{n\to\infty} \|y_{n,i} - x_n\| = 0 \qquad \text{for all } i \in I. \tag{17}$$

Since $u_{n,i} = V_{r_{n,i}}(x_n - r_{n,i}A_i x_n)$ and $v_{n,i} = V_{s_{n,i}}(x_n - r_{n,i}B_i x_n)$. Therefore, observe the following variant of the estimate (13)

$$\begin{aligned}
&\|y_{n,i} - p\|^2 \\
&\quad\leq \|z_{n,i} - p\|^2 \leq \alpha_{n,i}\|u_{n,i} - p\|^2 + (1-\alpha_{n,i})\|v_{n,i} - p\|^2 \\
&\quad= \alpha_{n,i}\|V_{r_{n,i}}(x_n - r_{n,i}A_i x_n) - V_{r_{n,i}}(p - r_{n,i}A_i p)\|^2 \\
&\qquad + (1-\alpha_{n,i})\|V_{s_{n,i}}(x_n - s_{n,i}B_i x_n) - V_{s_{n,i}}(p - s_{n,i}B_i p)\|^2 \\
&\quad\leq \alpha_{n,i}\|(x_n - p) - r_{n,i}(A_i x_n - A_i p)\|^2 \\
&\qquad + (1-\alpha_{n,i})\|(x_n - p) - s_{n,i}(B_i x_n - B_i p)\|^2 \\
&\quad= \alpha_{n,i}\big(\|x_n - p\|^2 - 2r_{n,i}\langle x_n - p, A_i x_n - A_i p\rangle + r_{n,i}^2\|A_i x_n - A_i p\|^2\big) \\
&\qquad + (1-\alpha_{n,i})\big(\|x_n - p\|^2 - 2s_{n,i}\langle x_n - p, B_i x - B_i p\rangle + s_{n,i}^2\|B_i x_n - B_i p\|^2\big) \\
&\quad\leq \|x_n - p\|^2 - r_{n,i}(2\eta - r_{n,i})\|A_i x_n - A_i p\|^2 - s_{n,i}(2\zeta - s_{n,i})\|B_i x_n - B_i p\|^2.
\end{aligned} \tag{18}$$

The following two consequences of the estimate (18) are crucial

$$\begin{aligned}
&r_{n,i}(2\eta - r_{n,i})\|A_i x_n - A_i p\|^2 \\
&\qquad\leq \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\
&\qquad= (\|x_n - p + y_{n,i} - p\|)(\|x_n - p - y_{n,i} + p\|) \\
&\qquad\leq (\|x_n - p\| + \|y_{n,i} - p\|)\|x_n - y_{n,i}\|
\end{aligned} \tag{19}$$

and

$$\begin{aligned} s_{n,i}(2\zeta - s_{n,i})\|B_i x_n - B_i p\|^2 & \\ &\leq \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\ &= (\|x_n - p + y_{n,i} - p\|)(\|x_n - p - y_{n,i} + p\|) \\ &\leq (\|x_n - p\| + \|y_{n,i} - p\|)\|x_n - y_{n,i}t\|. \end{aligned} \tag{20}$$

Utilizing the restriction $(C2)$ and estimate (17), it follows from (19) that

$$\lim_{n\to\infty} \|A_i x_n - A_i p\| = 0 \qquad \text{for all } i \in I. \tag{21}$$

Reasoning as above, the estimate (20) implies that,

$$\lim_{n\to\infty} \|B_i x_n - B_i p\| = 0 \qquad \text{for all } i \in I. \tag{22}$$

On the other hand, it follows from the firm nonexpansivity of $V_{r_{n,i}}$ that

$$\begin{aligned} \|u_{n,i} - p\|^2 &= \|V_{r_{n,i}}(I - r_{n,i}A_i)x_n - V_{r_{n,i}}(I - r_{n,i}A_i)p\|^2 \\ &\leq \langle (I - r_{n,i}A_i)x_n - (I - r_{n,i}A_i)p, u_{n,i} - p\rangle \\ &= \frac{1}{2}\big(\|(I - r_{n,i}A_i)x_n - (I - r_{n,i}A_i)p\|^2 + \|u_{n,i} - p\|^2 \\ &\quad - \|x_n - u_{n,i} - r_{n,i}(A_i x_n - A_i p)\|^2\big) \\ &= \frac{1}{2}\big(\|(I - r_{n,i}A_i)x_n - (I - r_{n,i}A_i)p\|^2 + \|u_{n,i} - p\|^2 \\ &\quad - (\|x_n - u_{n,i}\|^2 + r_{n,i}^2\|A_i x_n - A_i p\|^2 \\ &\quad - 2r_{n,i}\langle x_n - u_{n,i}, A_i x_n - A_i p\rangle)\big) \\ &\leq \frac{1}{2}\big(\|x_n - p\|^2 + \|u_{n,i} - p\|^2 - \|x_n - u_{n,i}\|^2 - r_{n,i}^2\|A_i x_n - A_i p\|^2 \\ &\quad + 2r_{n,i}\langle x_n - u_{n,i}, A_i x_n - A_i p\rangle\big). \end{aligned}$$

That is

$$\begin{aligned} \|u_{n,i} - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_{n,i}\|^2 - r_{n,i}^2\|A_i x_n - A_i p\|^2 & \\ + 2r_{n,i}\|x_n - u_{n,i}\|\|A_i x_n - A_i p\|. & \end{aligned} \tag{23}$$

Similarly

$$\begin{aligned} \|v_{n,i} - p\|^2 \leq \|x_n - p\|^2 - \|x_n - v_{n,i}\|^2 - s_{n,i}^2\|B_i x_n - B_i p\|^2 & \\ + 2s_{n,i}\|x_n - v_{n,i}\|\|B_i x_n - B_i p\|. & \end{aligned} \tag{24}$$

Since $\|y_{n,i} - p\|^2 \leq \alpha_{n,i}\|u_{n,i} - p\|^2 + (1 - \alpha_{n,i})\|v_{n,i} - p\|^2$, therefore substituting (23) and (24), we get

$$\begin{aligned} \|y_{n,i} - p\|^2 \leq \alpha_{n,i}\big(&\|x_n - p\|^2 - \|x_n - u_{n,i}\|^2 - r_{n,i}^2\|A_i x_n - A_i p\|^2 \\ &+ 2r_{n,i}\|x_n - u_{n,i}\|\|A_i x_n - A_i p\|\big) \\ &+ (1 - \alpha_{n,i})\big(\|x_n - p\|^2 - \|x_n - v_{n,i}\|^2 - s_{n,i}^2\|B_i x_n - B_i p\|^2 \end{aligned}$$

$$+ 2s_{n,i}\|x_n - v_{n,i}\|\|B_i x_n - B_i p\|\big).$$

Simplifying the above estimate and utilizing $(C1)$, we obtain the following two estimates

$$\begin{aligned} a\|x_n - u_{n,i}\|^2 \le {} & (\|x_n - p\| + \|y_{n,i} - p\|)\|x_n - y_{n,i}\| \\ & + 2r_{n,i}\|x_n - u_{n,i}\|\|A_i x_n - A_i p\| \\ & + 2s_{n,i}\|x_n - v_{n,i}\|\|B_i x_n - B_i p\|, \end{aligned} \tag{25}$$

and

$$\begin{aligned} (1-b)\|x_n - v_{n,i}\|^2 \le {} & (\|x_n - p\| + \|y_{n,i} - p\|)\|x_n - y_{n,i}\| \\ & + 2r_{n,i}\|x_n - u_{n,i}\|\|A_i x_n - A_i p\| \\ & + 2s_{n,i}\|x_n - v_{n,i}\|\|B_i x_n - B_i p\|. \end{aligned} \tag{26}$$

Letting $n \to \infty$ in the estimates (25)–(26) respectively, and then utilizing (17), (21) and (22) we have

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0 = \lim_{n\to\infty} \|x_n - v_{n,i}\| \qquad \text{for all } i \in I. \tag{27}$$

Observe that $\|z_{n,i} - x_n\|^2 \le \alpha_{n,i}\|u_{n,i} - x_n\|^2 + (1-\alpha_{n,i})\|v_{n,i} - x_n\|^2$. Hence (27) implies that

$$\lim_{n\to\infty} \|z_{n,i} - x_n\| = 0 \qquad \text{for all } i \in I. \tag{28}$$

Also from (17) and (28), we get that

$$\lim_{n\to\infty} \|y_{n,i} - z_{n,i}\| = 0 \qquad \text{for all } i \in I. \tag{29}$$

Further $\|y_{n,i} - z_{n,i}\| = (1-\beta_{n,i})\|T_i z_{n,i} - z_{n,i}\|$. It follows from (29) and the fact that $\lambda < a \le \beta_{n,i} \le b < 1$, we get

$$\lim_{n\to\infty} \|T_i z_{n,i} - z_{n,i}\| = 0 \qquad \text{for all } i \ge 1. \tag{30}$$

Next we show that $\omega(x_n) \subset \mathbb{F}$, where $\omega(x_n)$ is the set of all weak ω-limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, therefore $\omega(x_n) \ne \emptyset$. Let $q \in \omega(x_n)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$. It follows from the first estimate of (27) that $u_{n_j,i} \rightharpoonup q$. We first show that $q \in EP(f_1, A_1)$, where $f_1 = f_{n_j}$ for some $j \ge 1$. Note that, for a finite family of generalized equilibrium problems, the indexing $f_1 = f_{n_j}$ results from the modulo function $j \equiv 1 (\mathrm{mod}\, N)$ whereas the corresponding term of the infinite sequence $\{x_n\}$ would then be $\{x_{n_j}\}$. Similarly, we can have $f_{n_k} = f_2$ for some $k \ge 1$. From $u_{n_j,i} = V_{r_{n_j,i}}(I - r_{n_j,i}A_i)x_n$ for all $n \ge 1$, we have

$$f_1(u_{n_j,i}, y) + \langle A_1 x_{n_j}, y - u_{n_j,i}\rangle + \frac{1}{r_{n_j,i}}\langle y - u_{n_j,i}, u_{n_j,i} - x_{n_j}\rangle \ge 0 \qquad \text{for all } y \in C.$$

From (A2), we have

$$\langle A_1 x_{n_j}, y - u_{n_j,i}\rangle + \frac{1}{r_{n_j,i}}\langle y - u_{n_j,i}, u_{n_j,i} - y_{n_j,i}\rangle \ge f_1(y, u_{n_j,i}), \qquad y \in C. \tag{31}$$

Let $y_t = ty + (1-t)q$ for $0 < t < 1$ and $y \in C$. Since $q \in C$. this implies that $y_t \in C$. It follows from the estimate (31) that

$$\begin{aligned}\langle y_t - u_{n_j,i}, A_1 y_t\rangle &\geq \langle y_t - u_{n_j,i}, A_1 y_t\rangle - \langle A_1 x_{n_j}, y_t - u_{n_j,i}\rangle \\ &\quad - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j,i}}\right\rangle + f_1(y_t, u_{n_j,i}) \\ &= \langle y_t - u_{n_j,i}, A_1 y_t - A_1 u_{n_j,i}\rangle \\ &\quad + \langle y_t - u_{n_j,i}, A_1 u_{n_j,i} - A_1 x_{n_j}\rangle \\ &\quad - \left\langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j,i}}\right\rangle + f_1(y_t, u_{n_j,i}).\end{aligned} \tag{32}$$

Since $\lim_{n\to\infty}\|x_{n_j} - u_{n_j,i}\| = 0$, therefore $\lim_{n\to\infty}\|A_1 x_{n_j} - A_1 u_{n_j,i}\| = 0$. Moreover, it follows from the monotonicity of A_1 that $\langle y_t - u_{n_j,i}, A_1 y_t - A_1 u_{n_j,i}\rangle \geq 0$. Finally, (A4) implies that

$$\langle y_t - q, A_1 y_t\rangle \geq f_1(y_t, q). \tag{33}$$

Using (33), (A1) and (A4) the following estimate

$$\begin{aligned}0 = f_1(y_t, y_t) &\leq t f_1(y_t, y) + (1-t) f_1(y_t, q) \\ &\leq t f_1(y_t, y) + (1-t)\langle y_t - q, A_1 y_t\rangle \\ &= t f_1(y_t, y) + (1-t)t\langle y - q, A_1 y_t\rangle,\end{aligned}$$

implies that

$$f_1(y_t, y) + (1-t)\langle y - q, A_1 y_t\rangle \geq 0. \tag{34}$$

Letting $t \to 0$, we have $f_1(q, y) + \langle y - q, A_1 q\rangle \geq 0$ for all $y \in C$. Thus, $q \in GEP(f_1, A_1)$. In a similar fashion, we we can show that $q \in GEP(f_2, A_2)$, where $f_2 = f_{n_l}$ for some $l \geq 1$. Therefore, $q \in \bigcap_{i=1}^N GEP(f_i, A_i)$. Reasoning as above, we can also show that $q \in \bigcap_{i=1}^N GEP(g_i, B_i)$. Since $x_{n_j} \rightharpoonup q$, so it follows from (28) and Lemma 2.3 that $q \in \bigcap_{i=1}^N F(T_i)$. Hence $q \in \mathbb{F}$. Let $x = P_{\mathbb{F}} x_1$, since $\|x_{n+1} - x_1\| \leq \|x_0 - x_1\|$ for all $x_0 \in C_{n+1}$. It follows that

$$\|x_1 - x\| \leq \|x_1 - q\| \leq \liminf_{n\to\infty}\|x_1 - x_{n_j}\| \leq \limsup_{n\to\infty}\|x_1 - x_{n_j}\| \leq \|x_1 - x\|.$$

On the other hand,

$$\|x_1 - q\| = \lim_{j\to\infty}\|x_1 - x_{n_j}\| = \|x_1 - x\|.$$

This implies $x_{n_j} \to q = P_{\mathbb{F}} x_1$. From the arbitrariness of the subsequence $\{x_{n_j}\}$ of $\{x_n\}$, we conclude that $x_n \to x$ as $n \to \infty$. This completes the proof.

In particular, if T_i - in iteration (9) - is a finite family of nonexpansive mappings, then the following result holds

Corollary 3.1
Let C be a nonempty closed convex subset of a real Hilbert space H and let

$T_i\colon C \to C$ be a finite family of nonexpansive mappings. Let $f_i, g_i\colon C\times C \to \mathbb{R}$ be two finite families of bifunctions satisfying (A1)–(A4) *and let $A_i, B_i\colon C \to H$ be two finite families of η-inverse-strongly monotone and ζ-inverse-strongly monotone mappings, respectively. Let $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ be as in Theorem 3.1 and satisfy the following restrictions*

- $(C1)$ $0 \le k < a \le \alpha_{n,i}, \beta_{n,i} \le b < 1$;
- $(C2)$ $0 < c \le r_{n,i} \le d < 2\eta$ *and* $0 < c^{'} \le s_{n,i} \le d^{'} < 2\zeta$ *for all* $n \ge 1$ *and for all* $i \in I$.

Assume that

$$\mathbb{F} := \left[\bigcap_{i=1}^{N} F(T_i)\right] \cap \left[\bigcap_{i=1}^{N} GEP(f_i, A_i)\right] \cap \left[\bigcap_{i=1}^{N} GEP(g_i, B_i)\right] \ne \emptyset,$$

then the sequence $\{x_n\}$ generated by (9) *converges strongly to $x = P_{\mathbb{F}}x_1$.*

In order to solve the variational inequality problem together with the fixed point problem, we prove the following result.

Theorem 3.2

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i\colon C \to C$ be a finite family of k-strict pseudo-contractions. Let $A_i, B_i\colon C \to H$ be two finite families of η-inverse-strongly monotone and ζ-inverse-strongly monotone mappings, respectively. Let $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ be as in Theorem 3.1 and satisfy the following restrictions

- $(C1)$ $0 \le k < a \le \alpha_{n,i},\ \beta_{n,i} \le b < 1$;
- $(C2)$ $0 < c \le r_{n,i} \le d < 2\eta$ *and* $0 < c^{'} \le s_{n,i} \le d^{'} < 2\zeta$ *for all* $n \ge 1$ *and for all* $i \in I$.

Assume that

$$\mathbb{F} := \left[\bigcap_{i=1}^{N} F(T_i)\right] \cap \left[\bigcap_{i=1}^{N} VI(C, A_i)\right] \cap \left[\bigcap_{i=1}^{N} VI(C, B_i)\right] \ne \emptyset,$$

and the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_1 &\in C_1 = C,\\ \phi_{n,i} &= P_C(I - r_{n,i}A_i)x_n,\\ \varphi_{n,i} &= P_C(I - s_{n,i}B_i)x_n,\\ z_{n,i} &= \alpha_{n,i}\phi_{n,i} + (1-\alpha_{n,i})\varphi_{n,i},\\ y_{n,i} &= \beta_{n,i}z_{n,i} + (1-\beta_{n,i})T_i z_{n,i},\\ C_{n+1} &= \Big\{z \in C_n :\ \sup_{i\ge 1}\|y_{n,i} - z\| \le \|x_n - z\|\Big\},\\ x_{n+1} &= P_{C_{n+1}}x_1, \quad n \ge 1, \end{aligned}$$

converges strongly to $x = P_{\mathbb{F}}x_1$.

Proof. Set $f_i(x,y) = g_i(x,y) \equiv 0$ for each $i \geq 1$, then

$$\langle A_i x_n, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq 0$$

is equivalent to

$$\langle x_n - r_{n,i}A_i x_n - u_{n,i}, u_{n,i} - y\rangle \geq 0.$$

This implies that $u_{n,i} = P_C(x_n - r_{n,i}A_i x_n) = \phi_{n,i}$. Similarly, $v_{n,i} = P_C(x_n - s_{n,i}B_i x_n) = \varphi_{n,i}$. The desired result then follows from Theorem 3.1 immediately.

If we substitute $A_i = B_i \equiv 0$ - in iteration (9) - then we have the following result for the equilibrium problem together with the fixed point problem.

THEOREM 3.3
Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i\colon C \to C$ be a finite family of k-strict pseudo-contractions. Let $f_i, g_i\colon C \times C \to \mathbb{R}$ be two finite families of bifunctions satisfying (A1)–(A4). *Let $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ be as in Theorem 3.1 and satisfy the following restriction*

$(C1)$ $0 \leq k < a \leq \alpha_{n,i},\ \beta_{n,i} \leq b < 1.$

Assume that

$$\mathbb{F} := \left[\bigcap_{i=1}^{N} F(T_i)\right] \cap \left[\bigcap_{i=1}^{N} EP(f_i)\right] \cap \left[\bigcap_{i=1}^{N} EP(g_i)\right] \neq \emptyset,$$

then the sequence $\{x_n\}$ generated by

$$\begin{aligned}
x_1 &\in C_1 = C,\\
&f_i(u_{n,i}, u) + \frac{1}{r_{n,i}}\langle u - u_{n,i}, u_{n,i} - x_n\rangle \geq 0, \quad \forall u \in C,\\
&g_i(v_{n,i}, v) + \frac{1}{s_{n,i}}\langle v - v_{n,i}, v_{n,i} - x_n\rangle \geq 0, \quad \forall v \in C,\\
z_{n,i} &= \alpha_{n,i}u_{n,i} + (1 - \alpha_{n,i})v_{n,i},\\
y_{n,i} &= \beta_{n,i}z_{n,i} + (1 - \beta_{n,i})T_i z_{n,i},\\
C_{n+1} &= \Big\{z \in C_n : \sup_{i \geq 1} \Big\| y_{n,i} - z\| \leq \|x_n - z\|\Big\},\\
x_{n+1} &= P_{C_{n+1}}x_1, \quad n \geq 1,
\end{aligned}$$

converges strongly to $x = P_{\mathbb{F}}x_1$.

THEOREM 3.4
Let C be a nonempty closed convex subset of a real Hilbert space H and let $T_i\colon C \to C$ be a finite family of k-strict pseudo-contractions. Let $f_i\colon C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying (A1)–(A4) *and let $A_i\colon C \to H$ be a finite family of η-inverse-strongly monotone mappings. Let $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ and $\{r_{n,i}\}$ be as in Theorem 3.1 and satisfy the following restrictions*

$(C1)$ $0 \leq k < a \leq \alpha_{n,i},\ \beta_{n,i} \leq b < 1;$

(C2) $0 < c \leq r_{n,i} \leq d < 2\eta$ *for all* $n \geq 1$ *and for all* $i \in I$.

Assume that

$$\mathbb{F} := \left[\bigcap_{i=1}^{N} F(T_i)\right] \cap \left[\bigcap_{i=1}^{N} GEP(f_i, A_i)\right] \neq \emptyset,$$

then the sequence $\{x_n\}$ *generated by*

$$\begin{aligned}
&x_1 \in C_1 = C,\\
&f_i(u_{n,i}, u) + \langle A_i x_n, u - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle u - u_{n,i}, u_{n,i} - x_n\rangle \geq 0, \quad \forall u \in C,\\
y_{n,i} &= \beta_{n,i} u_{n,i} + (1-\beta_{n,i}) T_i u_{n,i},\\
C_{n+1} &= \left\{z \in C_n : \sup_{i\geq 1} \|y_{n,i} - z\| \leq \|x_n - z\|\right\},\\
x_{n+1} &= P_{C_{n+1}} x_1, \quad n \geq 1,
\end{aligned}$$

converges strongly to $x = P_{\mathbb{F}} x_1$.

Proof. The desired result is an immediate consequence of the proof of Theorem 3.1 by substituting $f_i = g_i$, $A_i = B_i$ and $u_{n,i} = v_{n,i}$ for all $i \in I$.

Remark 3.1
The results presented in this section improve and extend various results announced in the current literature, in particular established in [9] and [14]. It is of worth interest to establish such results in general Banach spaces.

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