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The class B_p for weighted generalized Fourier transform inequalities

Abstract. In the present paper, we prove weighted inequalities for the Dunkl transform (which generalizes the Fourier transform) when the weights belong to the well-known class B_p. As application, we obtain the Pitt's inequality for power weights.

1. Introduction

A key tool in the study of special functions with reflection symmetries are Dunkl operators. The basic ingredient in the theory of these operators are root systems and finite reflection groups, acting on $\mathbb{R}^d$. The Dunkl operators are commuting differential-difference operators T_i, $1 \leq i \leq d$ associated to an arbitrary finite reflection group W on $\mathbb{R}^d$ (see [7]). These operators attached with a root system R can be considered as perturbations of the usual partial derivatives by reflection parts. These reflection parts are coupled by parameters, which are given in terms of a non-negative multiplicity function k.

Dunkl theory was further developed by several mathematicians (see [6, 14]) and later was applied and generalized in different ways by many authors (see [1, 2]). The Dunkl kernel E_k has been introduced by C.F. Dunkl in [8]. For a family of weight functions w_k invariant under a reflection group W, we use the Dunkl kernel and the measure $w_k(x)dx$ to define the generalized Fourier transform $\mathcal{F}_k$, called the Dunkl transform, which enjoys properties similar to those of the classical Fourier transform. If the parameter $k \equiv 0$, then $w_k(x) = 1$, so that $\mathcal{F}_k$ becomes the classical Fourier transform and the T_i, $1 \leq i \leq d$ reduce to the corresponding

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partial derivatives $\frac{\partial}{\partial x_i}$, $1 \leq i \leq d$. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis (see next section, Remark 2.1).

Let μ be a non-negative locally integrable function on $(0,+\infty)$. We say that $\mu \in B_p$, $1 < p < +\infty$ if there is a constant $b_p > 0$ such that for all $s > 0$,

$$\int_s^{+\infty} \frac{\mu(t)}{t^p}\,dt \leq b_p \frac{1}{s^p} \int_0^s \mu(t)\,dt. \tag{1}$$

In the particular case when μ is non-increasing, one has $\mu \in B_p$.

The weighted Hardy inequality [16] (see also [9, 13]) states that if μ and ϑ are locally integrable weight functions on $(0,+\infty)$ and $1 < p \leq q < +\infty$, then there is a constant $c > 0$ such that for all non-increasing, non-negative Lebesgue measurable function f on $(0,+\infty)$, the inequality

$$\Big(\int_0^{+\infty} \Big(\frac{1}{t}\int_0^t f(s)\,ds\Big)^q \mu(t)\,dt\Big)^{\frac{1}{q}} \leq c\Big(\int_0^{+\infty} (f(t))^p \vartheta(t)\,dt\Big)^{\frac{1}{p}} \tag{2}$$

is satisfied if and only if

$$\sup_{s>0}\Big(\int_0^s \mu(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^s (\vartheta(t))\,dt\Big)^{-\frac{1}{p}} < +\infty \tag{3}$$

and

$$\sup_{s>0}\Big(\int_s^{+\infty} \frac{\mu(t)}{t^q}\,dt\Big)^{\frac{1}{q}}\Big(\int_0^s \Big(\frac{1}{t}\int_0^t \vartheta(l)\,dl\Big)^{-p'} \vartheta(t)\,dt\Big)^{\frac{1}{p'}} < +\infty. \tag{4}$$

Hardy's result still remains to be an important one as it is closely related to the Hardy-Littlewood maximal functions in harmonic analysis [17].

The aim of this paper is to prove under the B_p condition (1) and using the weight characterization of the Hardy operator, weighted Dunkl transform inequalities for general non-negative locally integrable functions u, v on $\mathbb{R}^d$,

$$\Big(\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^q u(x)\,d\nu_k(x)\Big)^{\frac{1}{q}} \leq c\Big(\int_{\mathbb{R}^d} |f(x)|^p v(x)\,d\nu_k(x)\Big)^{\frac{1}{p}},$$

where $1 < p \leq 2 \leq q < +\infty$ and $f \in L^p_{k,v}(\mathbb{R}^d)$. $L^p_{k,v}(\mathbb{R}^d)$ denote the space $L^p(\mathbb{R}^d, v(x)d\nu_k(x))$ with ν_k the weighted measure associated to the Dunkl operators defined by

$$d\nu_k(x) := w_k(x)dx \quad \text{with density } w_k(x) = \prod_{\xi\in R_+} |\langle \xi, x\rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.$$

R_+ being a positive root system and $\langle .,. \rangle$ the standard Euclidean scalar product on $\mathbb{R}^d$ (see next section). As application, we make a study of power weights in this context. This all leads to the Pitt´s inequality:
for $1 < p \leq 2 \leq q < +\infty$, $-(2\gamma + d) < \alpha < 0$, $0 < \beta < (2\gamma + d)(p-1)$ and $f \in L^p_{k,v}(\mathbb{R}^d)$, one has

$$\Big(\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^q \|x\|^\alpha\,d\nu_k(x)\Big)^{\frac{1}{q}} \leq c\Big(\int_{\mathbb{R}^d} |f(x)|^p \|x\|^\beta\,d\nu_k(x)\Big)^{\frac{1}{p}},$$

with the index constraint $\frac{1}{2\gamma+d}(\frac{\alpha}{q}+\frac{\beta}{p}) = 1-\frac{1}{p}-\frac{1}{q}$, where $\gamma = \sum_{\xi\in R_+} k(\xi)$. This extend to the Dunkl analysis some results obtained for the classical Fourier analysis in [4].

The contents of this paper are as follows. In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

The section 3 is devoted to the proofs of the weighted Dunkl transform inequalities when the weights belong to the class B_p. As application, we obtain for power weights the Pitt´s inequality.

Along this paper we use c to denote a suitable positive constant which is not necessarily the same in each occurrence and we write for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x\rangle}$. Furthermore, we denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$,
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives,
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.

2. Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the surveys [15].

Let W be a finite reflection group on $\mathbb{R}^d$, associated with a root system R. For $\alpha \in R$, we denote by $\mathbb{H}_\alpha$ the hyperplane orthogonal to α. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha\in R} \mathbb{H}_\alpha$, we fix a positive subsystem $R_+ = \{\alpha \in R : \langle \alpha, \beta\rangle > 0\}$. We denote by k a non-negative multiplicity function defined on R with the property that k is W-invariant. We associate with k the index

$$\gamma = \sum_{\xi\in R_+} k(\xi) \geq 0$$

and a weighted measure ν_k given by

$$d\nu_k(x) := w_k(x)dx, \quad \text{where } w_k(x) = \prod_{\xi\in R_+} |\langle \xi, x\rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.$$

Further, we introduce the Mehta-type constant c_k by

$$c_k = \Big(\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x)\, dx\Big)^{-1}.$$

For every $1 \leq p \leq +\infty$, we denote respectively by $L^p_k(\mathbb{R}^d)$, $L^p_{k,u}(\mathbb{R}^d)$, $L^p_{k,v}(\mathbb{R}^d)$ the spaces $L^p(\mathbb{R}^d, d\nu_k(x))$, $L^p(\mathbb{R}^d, u(x)d\nu_k(x))$, $L^p(\mathbb{R}^d, v(x)d\nu_k(x))$ and $L^p_k(\mathbb{R}^d)^{rad}$ the subspace of those $f \in L^p_k(\mathbb{R}^d)$ that are radial. We use respectively $\| \ \|_{p,k}$, $\| \ \|_{p,k,u}$, $\| \ \|_{p,k,v}$ as a shorthand for $\| \ \|_{L^p_k(\mathbb{R}^d)}$, $\| \ \|_{L^p_{k,u}(\mathbb{R}^d)}$, $\| \ \|_{L^p_{k,v}(\mathbb{R}^d)}$.

By using the homogeneity of degree 2γ of w_k, it is shown in [14] that for a radial function f in $L^1_k(\mathbb{R}^d)$, there exists a function F on $[0, +\infty)$ such that

$f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function F is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\, d\nu_k(x) &= \int_0^{+\infty} \Big(\int_{S^{d-1}} f(ry) w_k(ry)\, d\sigma(y) \Big) r^{d-1} dr \\ &= \int_0^{+\infty} \Big(\int_{S^{d-1}} w_k(ry)\, d\sigma(y) \Big) F(r) r^{d-1} dr \qquad (5) \\ &= d_k \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr,\end{aligned}$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$d_k = \int_{S^{d-1}} w_k(x)\, d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}. \qquad (6)$$

The Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where ρ_α is the reflection on the hyperplane $\mathbb{H}_\alpha$ and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, \dots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

Remark 2.1

In the case $k \equiv 0$, the weighted function $w_k \equiv 1$ and the measure ν_k associated to the Dunkl operators coincide with the Lebesgue measure. The T_j reduce to the corresponding partial derivatives. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.

For $y \in \mathbb{C}^d$, the system

$$\begin{cases} T_j u(x,y) = y_j u(x,y), & 1 \leq j \leq d, \\ u(0,y) = 1 \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x,y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z,z') = E_k(z',z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and for $x, y \in \mathbb{R}^d$, $|E_k(x, iy)| \leq 1$.

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y)\, d\nu_k(y), \qquad x \in \mathbb{R}^d.$$

We list some known properties of this transform:

i) The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ has the following basic property
$$\|\mathcal{F}_k(f)\|_\infty \leq \|f\|_{1,k}.$$

ii) The Dunkl transform is an automorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

iii) When both f and $\mathcal{F}_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula
$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y)\, d\nu_k(y), \qquad x \in \mathbb{R}^d.$$

iv) (Plancherel's theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric automorphism on $L^2_k(\mathbb{R}^d)$.

Since the Dunkl transform $\mathcal{F}_k(f)$ is of strong-type $(1,\infty)$ and $(2,2)$, then by interpolation, we get for $f \in L^p_k(\mathbb{R}^d)$ with $1 \leq p \leq 2$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, the Hausdorff-Young inequality
$$\|\mathcal{F}_k(f)\|_{p',k} \leq c\|f\|_{p,k}.$$

The Dunkl transform of a function in $L^1_k(\mathbb{R}^d)^{rad}$ is also radial. More precisely, according to ([14, proposition 2.4]), we have for $x \in \mathbb{R}$, the following results:
$$\int_{S^{d-1}} E_k(ix, y) w_k(y)\, d\sigma(y) = d_k j_{\gamma+\frac{d}{2}-1}(\|x\|),$$
and for f be in $L^1_k(\mathbb{R}^d)^{rad}$,
$$\begin{aligned} \mathcal{F}_k(f)(x) &= \int_0^{+\infty} \Big(\int_{S^{d-1}} E_k(-irx, y) w_k(y)\, d\sigma(y) \Big) F(r) r^{2\gamma+d-1} dr \\ &= d_k \int_0^{+\infty} j_{\gamma+\frac{d}{2}-1}(r\|x\|) F(r) r^{2\gamma+d-1} dr, \end{aligned} \tag{7}$$
where F is the function defined on $[0,+\infty)$ by $F(\|x\|) = f(x)$ and $j_{\gamma+\frac{d}{2}-1}$ the normalized Bessel function of the first kind and order $\gamma + \frac{d}{2} - 1$ given by
$$j_{\gamma+\frac{d}{2}-1}(\lambda x) = \begin{cases} 2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2}) \frac{J_{\gamma+\frac{d}{2}-1}(\lambda x)}{(\lambda x)^{\gamma+\frac{d}{2}-1}} & \text{if } \lambda x \neq 0, \\ 1 & \text{if } \lambda x = 0, \end{cases}$$
$\lambda \in \mathbb{C}$. Here $J_{\gamma+\frac{d}{2}-1}$ is the Bessel function of first kind,
$$\begin{aligned} J_{\gamma+\frac{d}{2}-1}(t) &= \frac{(\frac{t}{2})^{\gamma+\frac{d}{2}-1}}{\sqrt{\pi}\,\Gamma(\gamma+\frac{d}{2}-\frac{1}{2})} \int_0^\pi \cos(t\cos\theta)(\sin\theta)^{2\gamma+d-2}\, d\theta \\ &= C_\gamma t^{\gamma+\frac{d}{2}-1} \int_0^{\frac{\pi}{2}} \cos(t\cos\theta)(\sin\theta)^{2\gamma+d-2}\, d\theta, \end{aligned} \tag{8}$$
where $C_\gamma = \frac{1}{\sqrt{\pi}\, 2^{\gamma+\frac{d}{2}-2}\Gamma(\gamma+\frac{d}{2}-\frac{1}{2})}$.

3. Weighted Dunkl transform inequalities

In this section, we denote by p' and q' respectively the conjugates of p and q for $1 < p \leq q < +\infty$. The proof requires a useful well-known facts which we shall now state in the following.

Proposition 3.1 (see [16])
Let $1 < p < +\infty$ and v be a non-negative function on $(0, +\infty)$. The following are equivalent:

i) $v \in B_p$,

ii) *there is a positive constant c such that for all $s > 0$,*

$$\Big(\int_0^s v(t)\,dt\Big)^{\frac{1}{p}}\Big(\int_0^s \Big(\frac{1}{t}\int_0^t v(l)\,dl\Big)^{1-p'} dt\Big)^{\frac{1}{p'}} \leq cs. \tag{9}$$

Remark 3.2
1/ (see [[5]) (Hardy's Lemma) Let f and g be non-negative Lebesgue measurable functions on $(0, +\infty)$, and assume

$$\int_0^t f(s)\,ds \leq \int_0^t g(s)\,ds$$

for all $t \geq 0$. If φ is a non-negative and decreasing function on $(0, +\infty)$, then

$$\int_0^{+\infty} f(s)\varphi(s)\,ds \leq \int_0^{+\infty} g(s)\varphi(s)\,ds. \tag{10}$$

2/ Let f be a measurable function on $\mathbb{R}^d$. The distribution function D_f of f is defined for all $s \geq 0$ by

$$D_f(s) = \nu_k(\{x \in \mathbb{R}^d : \ |f(x)| > s\}).$$

The decreasing rearrangement of f is the function f^* given for all $t \geq 0$ by

$$f^*(t) = \inf\{s \geq 0 : \ D_f(s) \leq t\}.$$

We have the following results:

i) Let $f \in L^p_k(\mathbb{R}^d)$, $1 \leq p < +\infty$, then

$$\int_{\mathbb{R}^d} |f(x)|^p\,d\nu_k(x) = p\int_0^{+\infty} s^{p-1} D_f(s)\,ds = \int_0^{+\infty} (f^*(t))^p\,dt. \tag{11}$$

ii) (see [12, Theorems 4.6 and 4.7]) Let $q \geq 2$, then there exists a constant $c > 0$ such that, for all $f \in L^1_k(\mathbb{R}^d) + L^2_k(\mathbb{R}^d)$ and $s \geq 0$,

$$\int_0^s (\mathcal{F}_k(f)^*(t))^q\,dt \leq c\int_0^s \Big(\int_0^{\frac{1}{t}} f^*(y)\,dy\Big)^q dt. \tag{12}$$

iii) (see [5, 10, 11]) (Hardy-Littlewood rearrangement inequality)
Let f and ϑ be non-negative measurable functions on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} f(x)\vartheta(x)\,d\nu_k(x) \leq \int_0^{+\infty} f^*(t)\vartheta^*(t)\,dt \tag{13}$$

and

$$\int_0^{+\infty} f^*(t)\Big[\Big(\frac{1}{\vartheta}\Big)^*(t)\Big]^{-1} dt \leq \int_{\mathbb{R}^d} f(x)\vartheta(x)\,d\nu_k(x). \tag{14}$$

Now, we begin with the proof of the following proposition which gives a necessary condition.

Proposition 3.3
Let u, v be non-negative ν_k-locally integrable functions on $\mathbb{R}^d$ and $1 < p \leq 2 \leq q < +\infty$. If there exists a constant $c > 0$ such that for all $f \in L_k^p(\mathbb{R}^d)$,

$$\Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} \leq c\Big(\int_0^{+\infty} \big(f^*(t)\big)^p \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{1}{p}}, \tag{15}$$

then it is necessary that

$$\sup_{s>0} s\Big(\int_0^{\frac{1}{s}} u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^s \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{-1}{p}} < +\infty. \tag{16}$$

Proof. Put for any fixed $r > 0$,

$$R = \Big(r\frac{\nu_k(B(0,1))}{1+(\nu_k(B(0,1)))^2}\Big)^{\frac{1}{2\gamma+d}},$$

and take $f = \chi_{(0,R)}$ in (15), where $\chi_{(0,R)}$ is the characteristic function of the interval $(0,R)$. For $s \geq 0$ and by (5) and (6), the distribution function of f is

$$\begin{aligned} D_f(s) &= \nu_k(\{x \in \mathbb{R}^d :\ \chi_{(0,R)}(\|x\|) > s\}) \\ &= \frac{d_k}{2\gamma+d} R^{2\gamma+d}\chi_{(0,1)}(s) \\ &= \nu_k(B(0,1))R^{2\gamma+d}\chi_{(0,1)}(s) \\ &= r'\chi_{(0,1)}(s), \end{aligned}$$

where

$$r' = \nu_k(B(0,1))R^{2\gamma+d} = r\frac{(\nu_k(B(0,1)))^2}{1+(\nu_k(B(0,1)))^2}. \tag{17}$$

This yields for $t \geq 0$,

$$f^*(t) = \inf\{s \geq 0 :\ D_f(s) \leq t\} = \chi_{(0,r')}(t).$$

Observe that $r' < r$, hence we have

$$\begin{aligned} \Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} &\leq c\Big(\int_0^{r'} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{1}{p}} \\ &\leq c\Big(\int_0^{r} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{1}{p}}. \end{aligned} \tag{18}$$

According to (7), for $x \in \mathbb{R}^d$, we can assert that

$$\begin{aligned} \mathcal{F}_k(f)(x) &= c_k^{-1} \int_0^R j_{\gamma+\frac{d}{2}-1}(\|x\|t) \frac{t^{2\gamma+d-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}\, dt \\ &= c_k^{-1} \|x\|^{\frac{2-2\gamma-d}{2}} \int_0^R J_{\gamma+\frac{d}{2}-1}(\|x\|t) t^{\frac{2\gamma+d}{2}}\, dt. \end{aligned} \tag{19}$$

Since $\cos(t\|x\|\cos\theta) \geq \cos 1 > \frac{1}{2}$, for $t \in (0, R)$, $\|x\| \in (0, \frac{1}{R})$ and $\theta \in (0, \frac{\pi}{2})$, then we obtain from (8), the estimate

$$\begin{aligned} J_{\gamma+\frac{d}{2}-1}(\|x\|t) &> \frac{1}{2} C_\gamma (\|x\|t)^{\gamma+\frac{d}{2}-1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\gamma+d-2}\, d\theta \\ &= \frac{1}{2} C_\gamma (\|x\|t)^{\gamma+\frac{d}{2}-1} \frac{\sqrt{\pi}\,\Gamma(\gamma+\frac{d}{2}-\frac{1}{2})}{2\Gamma(\gamma+\frac{d}{2})} \\ &= \frac{(\|x\|t)^{\frac{2\gamma+d-2}{2}}}{2^{\frac{2\gamma+d}{2}}\Gamma(\frac{2\gamma+d}{2})}, \end{aligned}$$

which gives by (5), (6), (17), (19) and for $\|x\| \in (0, \frac{1}{R})$

$$\begin{aligned} \mathcal{F}_k(f)(x) &> c_k^{-1} \|x\|^{\frac{2-2\gamma-d}{2}} \int_0^R \frac{(\|x\|t)^{\frac{2\gamma+d-2}{2}}}{2^{\frac{2\gamma+d}{2}}\Gamma(\frac{2\gamma+d}{2})} t^{\frac{2\gamma+d}{2}}\, dt \\ &= \frac{c_k^{-1}}{2^{\frac{2\gamma+d}{2}}\Gamma(\frac{2\gamma+d}{2})} \int_0^R t^{2\gamma+d-1}\, dt \\ &= \frac{r'}{2}. \end{aligned} \tag{20}$$

By the fact that

$$\Big\{t \in \Big(0, \frac{1}{r}\Big) : \ (\mathcal{F}_k(f))^*(t) > s\Big\} = \Big\{t \in (0, \frac{1}{r}) : \ D_{\mathcal{F}_k(f)}(s) > t\Big\},$$

we have from (11)

$$\begin{aligned} &\Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\, dt\Big)^{\frac{1}{q}} \\ &\qquad \geq \Big(\int_0^{\frac{1}{r}} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\, dt\Big)^{\frac{1}{q}} \\ &\qquad = \Big(q\int_0^{+\infty} s^{q-1}\Big(\int_{\{t\in(0,\frac{1}{r}),(\mathcal{F}_k(f))^*(t)>s\}} u^*(t)\, dt\Big)\, ds\Big)^{\frac{1}{q}} \\ &\qquad = \Big(q\int_0^{+\infty} s^{q-1}\Big(\int_0^{\min(D_{\mathcal{F}_k(f)}(s),\frac{1}{r})} u^*(t)\, dt\Big)\, ds\Big)^{\frac{1}{q}}. \end{aligned}$$

If $s < \frac{r'}{2}$, then by (20)

$$B\Big(0, \frac{1}{R}\Big) \subseteq \Big\{x \in \mathbb{R}^d : \ |\mathcal{F}_k(f)(x)| > \frac{r'}{2}\Big\} \subseteq \{x \in \mathbb{R}^d : \ |\mathcal{F}_k(f)(x)| > s\},$$

thus using (5) and (6), we have

$$\begin{aligned} D_{\mathcal{F}_k(f)}(s) &= \int_{\{x\in\mathbb{R}^d:\,|\mathcal{F}_k(f)(x)|>s\}} w_k(x)\,dx \\ &\geq d_k \int_0^{\frac{1}{R}} \rho^{2\gamma+d-1}\,d\rho = \frac{1}{r}\big(1+(\nu_k(B(0,1)))^2\big) \\ &> \frac{1}{r}, \end{aligned}$$

wich gives that

$$\begin{aligned} \Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} &\geq \Big(q\int_0^{\frac{r'}{2}} s^{q-1}\Big(\int_0^{\frac{1}{r}} u^*(t)\,dt\Big)\,ds\Big)^{\frac{1}{q}} \\ &= \Big(q\int_0^{\frac{r'}{2}} s^{q-1}\,ds\Big)^{\frac{1}{q}}\Big(\int_0^{\frac{1}{r}} u^*(t)\,dt\Big)^{\frac{1}{q}} \\ &= \frac{r'}{2}\Big(\int_0^{\frac{1}{r}} u^*(t)\,dt\Big)^{\frac{1}{q}}. \end{aligned}$$

According to (17) and (18), we deduce that

$$\begin{aligned} r\Big(\int_0^{\frac{1}{r}} u^*(t)\,dt\Big)^{\frac{1}{q}}&\Big(\int_0^{r} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{-\frac{1}{p}} \\ &\leq c\Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{r} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{-\frac{1}{p}} \\ &\leq c, \end{aligned}$$

which gives (16). This completes the proof.

Theorem 3.4
Let u, v be non-negative ν_k-locally integrable functions on $\mathbb{R}^d$ and $1<p\leq 2\leq q<+\infty$. Assume $\frac{1}{(\frac{1}{v})^*}\in B_p$ and

$$\sup_{s>0} s\Big(\int_0^{\frac{1}{s}} u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{s} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{\frac{-1}{p}} < +\infty, \tag{21}$$

then there exists a constant $c>0$ such that for all $f\in L^p_k(\mathbb{R}^d)$, we have

$$\Big(\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^q u(x)\,d\nu_k(x)\Big)^{\frac{1}{q}} \leq c\Big(\int_{\mathbb{R}^d} |f(x)|^p v(x)\,d\nu_k(x)\Big)^{\frac{1}{p}}. \tag{22}$$

Proof. In order to establish this result, we need to show that

$$\Big(\int_0^{+\infty} \big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} \leq c\Big(\int_0^{+\infty} \big(f^*(t)\big)^p\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{\frac{1}{p}}. \tag{23}$$

Take $f \in L^p_k(\mathbb{R}^d)$, then using (10) and (12), we obtain

$$\Big(\int_0^{+\infty}\big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} \le c\Big(\int_0^{+\infty}\Big(\int_0^{\frac{1}{t}} f^*(s)\,ds\Big)^q u^*(t)\,dt\Big)^{\frac{1}{q}}.$$

If we make the change of variable $t = \frac{1}{s}$ on the right side, we get

$$\Big(\int_0^{+\infty}\big((\mathcal{F}_k(f))^*(t)\big)^q u^*(t)\,dt\Big)^{\frac{1}{q}} \le c\Big(\int_0^{+\infty}\Big(\frac{1}{s}\int_0^{s} f^*(t)\,dt\Big)^q \frac{u^*(\frac{1}{s})}{s^{2-q}}\,ds\Big)^{\frac{1}{q}},$$

which gives from (2), (3) and (4), that the inequality (23) is satisfied if and only if

$$\sup_{s>0}\Big(\int_0^{s}\frac{u^*(\frac{1}{t})}{t^{2-q}}\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{s}\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{-\frac{1}{p}} < +\infty$$

and

$$\sup_{s>0}\Big(\int_0^{+\infty}\frac{u^*(\frac{1}{t})}{t^2}\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{s}\Big(\frac{1}{t}\int_0^{t}\Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1}dl\Big)^{-p'}\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{\frac{1}{p'}} < +\infty.$$

In order to complete the proof, we must verify that (21) implies these two conditions between the weights u^* and $\frac{1}{(\frac{1}{v})^*}$. This follows closely the argumentations of [4]. More precisely, since u^* is non-increasing, then $u^* \in B_q$ and by (1), it yields

$$\int_0^{s} u^*\Big(\frac{1}{t}\Big)t^{q-2}\,dt = \int_{\frac{1}{s}}^{+\infty}\frac{u^*(t)}{t^q}\,dt \le b_q s^q\int_0^{\frac{1}{s}} u^*(t)\,dt.$$

Hence by (21), we get

$$\begin{aligned}
&\Big(\int_0^{s} u^*\Big(\frac{1}{t}\Big)t^{q-2}\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{s}\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{-\frac{1}{p}}\\
&\qquad \le b_q^{\frac{1}{q}}\, s\Big(\int_0^{\frac{1}{s}} u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{s}\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{-\frac{1}{p}}\\
&\qquad < +\infty,
\end{aligned}$$

and so we obtain the first condition.

To show that the second condition is satisfied, observe that by means of a change of variable, we have

$$\Big(\int_s^{+\infty}\frac{u^*(\frac{1}{t})}{t^2}\,dt\Big)^{\frac{1}{q}} = \Big(\int_0^{\frac{1}{s}} u^*(t)\,dt\Big)^{\frac{1}{q}}. \tag{24}$$

Now, define the function G by

$$G(s) = \Big(\int_0^{s}\Big(\frac{1}{t}\int_0^{t}\Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1}dl\Big)^{-p'}\Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1}dt\Big)^{\frac{1}{p'}},$$

then by integration by parts, we get

$$G(s) = \Big[p' G(s)^{p'} + s^{p'} \Big(\int_0^s \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{1-p'} - p' \int_0^s \Big(\frac{1}{t} \int_0^t \Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1} dl\Big)^{1-p'} dt\Big]^{\frac{1}{p'}},$$

which implies

$$(p'-1)G(s)^{p'} \leq p' \int_0^s \Big(\frac{1}{t} \int_0^t \Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1} dl\Big)^{1-p'} dt,$$

and so

$$G(s) \leq \Big(\frac{p'}{p'-1} \int_0^s \Big(\frac{1}{t} \int_0^t \Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1} dl\Big)^{1-p'} dt\Big)^{\frac{1}{p'}}.$$

Since $\frac{1}{(\frac{1}{v})^*} \in B_p$, we can invoke (9) and we obtain

$$\Big(\int_0^s \Big(\frac{1}{t} \int_0^t \Big[\Big(\frac{1}{v}\Big)^*(l)\Big]^{-1} dl\Big)^{-p'} \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{1}{p'}} \leq cs\Big(\int_0^s \Big[\Big(\frac{1}{v}\Big)^*(t)\Big]^{-1} dt\Big)^{\frac{-1}{p}}.$$

Combining this inequality and (24), we deduce (23).

Note that $(|f|^p)^* = (f^*)^p$ and $(|\mathcal{F}_k(f)|^q)^* = ((\mathcal{F}_k(f))^*)^q$, then applying (13) and (14) for the inequality (23), we obtain (22). This completes the proof.

Application 3.5 (Pitt's inequality)
Let $u(x) = \|x\|^\alpha$, $v(x) = \|x\|^\beta$, $x \in \mathbb{R}^d$ with $\alpha < 0$ and $\beta > 0$. Using (5) and (6), we have for $s \geq 0$

$$D_u(s) = \nu_k(\{x \in \mathbb{R}^d:\ \|x\|^\alpha > s\}) = \nu_k(B(0, s^{\frac{1}{\alpha}})) = \frac{d_k}{2\gamma+d} s^{\frac{2\gamma+d}{\alpha}},$$

which gives for $t \geq 0$

$$u^*(t) = \inf\{s \geq 0:\ D_u(s) \leq t\} = \Big(\frac{2\gamma+d}{d_k}\Big)^{\frac{\alpha}{2\gamma+d}} t^{\frac{\alpha}{2\gamma+d}}.$$

On the other hand, using (5) and (6) again, we have for $s \geq 0$,

$$D_{\frac{1}{\vartheta}}(s) = \nu_k(\{x \in \mathbb{R}^d:\ \|x\|^{-\beta} > s\}) = \nu_k(B(0, s^{-\frac{1}{\beta}})) = \frac{d_k}{2\gamma+d} s^{-\frac{2\gamma+d}{\beta}},$$

which gives for $t \geq 0$,

$$\Big(\frac{1}{\vartheta}\Big)^*(t) = \inf\{s \geq 0:\ D_{\frac{1}{\vartheta}}(s) \leq t\} = \Big(\frac{2\gamma+d}{d_k}\Big)^{-\frac{\beta}{2\gamma+d}} t^{-\frac{\beta}{2\gamma+d}}.$$

For these weights and $1 < p \leq 2 \leq q < +\infty$, the hypothesis of Theorem 3.4, gives respectively that the integrals in the B_p-inequality (1) for $\frac{1}{(\frac{1}{v})^*}$ are finite and the boundedness condition (21) is valid if and only if

$$0 < \beta < (2\gamma+d)(p-1) \qquad \text{and} \qquad \begin{cases} -(2\gamma+d) < \alpha < 0, \\ \frac{1}{2\gamma+d}\big(\frac{\alpha}{q} + \frac{\beta}{p}\big) = 1 - \frac{1}{p} - \frac{1}{q}. \end{cases}$$

Under these conditions and index constraints, we obtain from Theorem 3.4 and for $f \in L^p_{k,v}(\mathbb{R}^d)$, the Pitt's inequality

$$\Big(\int_{\mathbb{R}^d} \|x\|^{\alpha} |\mathcal{F}_k(f)(x)|^q \, d\nu_k(x)\Big)^{\frac{1}{q}} \leq c \Big(\int_{\mathbb{R}^d} \|x\|^{\beta} |f(x)|^p \, d\nu_k(x)\Big)^{\frac{1}{p}}.$$

In particular for $p = q = 2$ and $0 < \beta < 2\gamma + d$, we get

$$\Big(\int_{\mathbb{R}^d} \|x\|^{-\beta} |\mathcal{F}_k(f)(x)|^2 \, d\nu_k(x)\Big)^{\frac{1}{2}} \leq c \Big(\int_{\mathbb{R}^d} \|x\|^{\beta} |f(x)|^2 \, d\nu_k(x)\Big)^{\frac{1}{2}}.$$

In the classical Fourier analysis, this inequality plays an important role for which some uncertainty principles hold. One of them is the Beckner's logarithmic uncertainty principle (see [3]).

Remark 3.6

The limiting case $\beta = 0$, $\alpha = (2\gamma + d)(p - 2)$ and $1 < p = q \leq 2$ was obtained in ([1, Section 4, Lemma 1]) and gives the Hardy-Littlewood-Paley inequality

$$\Big(\int_{\mathbb{R}^d} \|x\|^{(2\gamma+d)(p-2)} |\mathcal{F}_k(f)(x)|^p \, d\nu_k(x)\Big)^{\frac{1}{p}} \leq c \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\nu_k(x)\Big)^{\frac{1}{p}}.$$

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