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First-order systems of linear partial differential equations: normal forms, canonical systems, transform methods

Abstract. In this paper we consider first-order systems with constant coefficients for two real-valued functions of two real variables. This is both a problem in itself, as well as an alternative view of the classical linear partial differential equations of second order with constant coefficients. The classification of the systems is done using elementary methods of linear algebra. Each type presents its special canonical form in the associated characteristic coordinate system. Then you can formulate initial value problems in appropriate basic areas, and you can try to achieve a solution of these problems by means of transform methods.

1. Introduction

We consider first-order linear systems of partial differential equations

$$Aw_\xi + Bw_\eta = Qw + \phi(\xi, \eta). \tag{1}$$

Given A, B, Q as real, constant $(2,2)$-matrices, A, B both not singular, and functions

$$(\phi_1(\xi,\eta), \phi_2(\xi,\eta))^T = \phi(\xi,\eta), \qquad \phi_i \in \mathbf{C}(\mathbb{R}_+ \times \mathbb{R}_+),\ i = 1, 2,$$

we are looking for functions

$$(w_1(\xi,\eta), w_2(\xi,\eta))^T = w(\xi,\eta), \qquad w_i \in \mathbf{C}^\mathbf{1}(\mathbb{R}_+ \times \mathbb{R}_+),\ i = 1, 2$$

satisfying the differential equation (1). From (1) we derive the corresponding *normal form*, which is essentially determined by the eigenvalues of AB^{-1}. From the respective normal form we derive the *canonical systems* with respect to (1) by means of the eigenvalues and the eigenvectors. In the hyperbolic and elliptic case these canonical systems have long been known, regardless of their relationship to

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(1) and there are theorems concerning the existence of a solution, see e.g. [1], [11], [12], [13], [16], [23], [28], [29].

We will write down the steps in the way that one can track the impact to the canonical system of each individual coefficient from A, B, Q. Here, there arise aspects, especially in the case of parabolic and elliptic systems, which lead to a more consistent view of the characteristics (this also affects the consideration of the classical heat equation).

Once the canonical systems are present we can formulate initial value problems. These tasks are handled with transform methods (Laplace transform, Fourier transform). In the hyperbolic case you can specify the solution of the initial value problem in a axially parallel rectangle completely.

2. Normal forms and canonical systems

2.1. Normal forms

Lets start with (1) and assume w.l.o.g. that B is not singular and that the first columns of A and B are linearly independent. We multiply both sides of equation (1) on the left by a matrix T, which we will determine straightaway

$$TAw_\xi + TBw_\eta = TQw + T\phi(\xi,\eta). \tag{2}$$

Let λ_1, λ_2, where $\lambda_1 \neq \lambda_2$, denote the eigenvalues of AB^{-1} and let D be a diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2)$. We impose on T the requirement

$$TA = DTB \quad \text{and thus} \quad TAB^{-1}T^{-1} = D, \tag{3}$$

so T is chosen as the matrix (line by line) of the left-hand eigenvectors of AB^{-1} and TA is eliminated in (2).

Denote $\det(B) = |B|$, $2b = (b_{11}a_{22} + b_{22}a_{11} - a_{12}b_{21} - a_{21}b_{12})$. Then as a solution of $|A - \lambda B| = \lambda^2|B| - 2b\lambda + |A| = 0$ we have

$$\lambda_{1,2} = \frac{1}{|B|}\big\{b \pm \sqrt{b^2 - |A|\cdot|B|}\big\}.$$

We now provide the left-hand eigenvectors t^i, $i = 1, 2$, as a function of the elements of the matrices. Using the first formula in (3), we get

$$\begin{aligned}&\begin{bmatrix} t_{11}a_{11} + t_{12}a_{21} & t_{11}a_{12} + t_{12}a_{22} \\ t_{21}a_{11} + t_{22}a_{21} & t_{21}a_{12} + t_{22}a_{22}\end{bmatrix} \\ &\qquad = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix} t_{11}b_{11} + t_{12}b_{21} & t_{11}b_{12} + t_{12}b_{22} \\ t_{21}b_{11} + t_{22}b_{21} & t_{21}b_{12} + t_{22}b_{22}\end{bmatrix}.\end{aligned} \tag{4}$$

If we equate the corresponding elements of the first line in the matrix equation (4) we obtain a homogeneous linear system for $[t_{11}, t_{12}]$. Solving it we obtain

$$t^1 = [t_{11}, t_{12}] = [a_{21} - \lambda_1 b_{21}, -a_{11} + \lambda_1 b_{11}]$$

and accordingly

$$t^2 = [t_{21}, t_{22}] = [a_{21} - \lambda_2 b_{21}, -a_{11} + \lambda_2 b_{11}].$$

Thus overall

$$T = \begin{bmatrix} a_{21} - \lambda_1 b_{21} & -a_{11} + \lambda_1 b_{11} \\ a_{21} - \lambda_2 b_{21} & -a_{11} + \lambda_2 b_{11} \end{bmatrix}.$$

Therefore with $TB = B^* = [b^*_{ij}]$, $TQ = Q^* = [q^*_{ij}]$ and $T\phi = \phi^*$, $i, j = 1, 2$ we have

$$B^* = \begin{bmatrix} a_{21}b_{11} - a_{11}b_{21} & a_{21}b_{12} - a_{11}b_{22} + \lambda_1|B| \\ a_{21}b_{11} - a_{11}b_{21} & a_{21}b_{12} - a_{11}b_{22} + \lambda_2|B| \end{bmatrix} =: \begin{bmatrix} b^*_{11} & c^*_{12} + \lambda_1|B| \\ b^*_{11} & c^*_{12} + \lambda_2|B| \end{bmatrix}, \tag{5}$$

$$Q^* = \begin{bmatrix} (a_{21} - \lambda_1 b_{21})q_{11} + (-a_{11} + \lambda_1 b_{11})q_{21} & (a_{21} - \lambda_1 b_{21})q_{12} + (-a_{11} + \lambda_1 b_{11})q_{22} \\ (a_{21} - \lambda_2 b_{21})q_{11} + (-a_{11} + \lambda_2 b_{11})q_{21} & (a_{21} - \lambda_2 b_{21})q_{12} + (-a_{11} + \lambda_2 b_{11})q_{22} \end{bmatrix}$$

and so we obtain as the normal form of (1) in the case $\lambda_1 \neq \lambda_2$ the following formula

$$DB^* w_\xi + B^* w_\eta = Q^* w + \phi^*,$$

or in component wise notation the formula

$$\begin{aligned} b^*_{11}(\lambda_1 w_{1,\xi} + w_{1,\eta}) + b^*_{12}(\lambda_1 w_{2,\xi} + w_{2,\eta}) &= q^*_{11} w_1 + q^*_{12} w_2 + \phi^*_1, \\ b^*_{11}(\lambda_2 w_{1,\xi} + w_{1,\eta}) + b^*_{22}(\lambda_2 w_{2,\xi} + w_{2,\eta}) &= q^*_{21} w_1 + q^*_{22} w_2 + \phi^*_2. \end{aligned} \tag{NFHE}$$

Note the directional derivatives of w_1 and w_2 which are now present in (NFHE).

This normal form is valid for

$$\begin{aligned} &\lambda_1 \neq \lambda_2, && \lambda_i \in \mathbb{R},\ i = 1, 2, && \text{(hyperbolic case)} \\ &\lambda_1 = \mu + i\nu,\ \lambda_2 = \mu - i\nu, && \mu, \nu \in \mathbb{R},\ \nu \neq 0. && \text{(elliptic case)} \end{aligned}$$

The special case $\lambda_1 = \lambda_2 = \lambda$ indicates that there is only one direction of differentiation in the (ξ, η)-plane. It leads to a system of ordinary differential equations and this system should not be automatically associated to the parabolic case, but see, among others, [30], [13].

For the elliptic case we provide ϕ^* element wise

$$T\phi(\xi, \eta) = \phi^*(\xi, \eta) = \begin{bmatrix} (a_{21} - \mu b_{21})\phi_1 + (-a_{11} + \mu b_{11})\phi_2 + i\nu[b_{11}\phi_2 - b_{21}\phi_1] \\ (a_{21} - \mu b_{21})\phi_1 + (-a_{11} + \mu b_{11})\phi_2 - i\nu[b_{11}\phi_2 - b_{21}\phi_1] \end{bmatrix}.$$

There remains the case that $\lambda_1 = \lambda_2 = \lambda$ and AB^{-1} in (3) is not diagonalizable. We will then set up the matrix $T = T_p$ in (3) so that T_p enforces the Jordan normal form

$$T_p A B^{-1} T_p^{-1} = J, \qquad J = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}. \tag{6}$$

Instead (4) we have now

$$\begin{bmatrix} t_{11}a_{11} + t_{12}a_{21} & t_{11}a_{12} + t_{12}a_{22} \\ t_{21}a_{11} + t_{22}a_{21} & t_{21}a_{12} + t_{22}a_{22} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} t_{11}b_{11} + t_{12}b_{21} & t_{11}b_{12} + t_{12}b_{22} \\ t_{21}b_{11} + t_{22}b_{21} & t_{21}b_{12} + t_{22}b_{22} \end{bmatrix},$$

and calculate once again the eigenvector

$$t^1 = [t_{11}, t_{12}] = [a_{21} - \lambda b_{21}, -a_{11} + \lambda b_{11}].$$

If we equate the corresponding elements of the second line in the matrix equation (6) we obtain a homogeneous linear system for $[t_{21}, t_{22}]$,

$$\begin{aligned}(a_{11}-\lambda b_{11})t_{21}+(a_{21}-\lambda b_{21})t_{22} &= t_{11}b_{11}+t_{12}b_{21},\\ (a_{12}-\lambda b_{12})t_{21}+(a_{22}-\lambda b_{22})t_{22} &= t_{11}b_{12}+t_{12}b_{22},\end{aligned}$$

that is a linear inhomogeneous system with a rank-deficient coefficient matrix for determining the left-hand generalized eigenvector t^2 (see [31]),

$$(t_{21}, t_{22})(A-\lambda B) = (t_{11}, t_{12})B.$$

If we assume w.l.o.g. that $b_{11}-\lambda a_{11} \neq 0$, we can specify the solution in the form

$$\begin{aligned}\begin{bmatrix} t_{21}\\ t_{22}\end{bmatrix}^T = t^2 = c\begin{bmatrix} a_{21}-\lambda b_{21}\\ -a_{11}+\lambda b_{11}\end{bmatrix}^T&\\ +\begin{bmatrix} \dfrac{1}{a_{11}-\lambda b_{11}}[(a_{21}-\lambda b_{21})b_{11}+(-a_{11}+\lambda b_{11})b_{21}]\\ 0\end{bmatrix}^T&, \quad c\in\mathbb{R}.\end{aligned}$$

We choose $c = 1$. Thus we have

$$T_p = \begin{bmatrix} a_{21}-\lambda b_{21} & -a_{11}+\lambda b_{11}\\ \dfrac{a_{21}-\lambda b_{21}}{a_{11}-\lambda b_{11}}[a_{11}-\lambda b_{11}+b_{11}]-b_{21} & -a_{11}+\lambda b_{11}\end{bmatrix}.$$

Let us denote (now $\star$ instead of $*$ in (5)) $T_pB = B^\star = [b^\star_{ij}]$, $T_pQ = Q^\star = [q^\star_{ij}]$, $T_p\phi = \phi^\star$, $i, j = 1, 2$, and we use in the same time also the elements b^*_{ij} from (5) (with $\lambda_1 = \lambda_2 = \lambda$), so we obtain

$$\begin{aligned} B^\star &= \begin{bmatrix} b^\star_{11} & b^\star_{12}\\ b^\star_{21} & b^\star_{22}\end{bmatrix}\\ &= \begin{bmatrix} b^*_{11} & b^*_{12}\\ b^*_{11}+b^2_{11}\dfrac{a_{21}-\lambda b_{21}}{a_{11}-\lambda b_{11}}-b_{11}b_{21} & b^*_{12}+b_{11}b_{12}\dfrac{a_{21}-\lambda b_{21}}{a_{11}-\lambda b_{11}}-b_{12}b_{21}\end{bmatrix}\\ &=: \begin{bmatrix} b^*_{11} & b^*_{12}\\ b^*_{11}+B_{21} & b^*_{12}+B_{22}\end{bmatrix}.\end{aligned}$$

So we have in the case $\lambda_1 = \lambda_2$ as a normal form of (1) the formula

$$JB^\star w_\xi + B^\star w_\eta = Q^\star\ w + \phi^\star,$$

or component wise the formula

$$\begin{aligned} b^*_{11}(\lambda w_{1,\xi}+w_{1,\eta})+b^*_{12}(\lambda w_{2,\xi}+w_{2,\eta}) &= q^\star_{11}w_1+q^\star_{12}w_2+\phi^\star_1,\\ b^\star_{11}(w_{1,\xi}+w_{1,\eta})+(b^\star_{11}+B_{21})[\lambda w_{1,\xi}+w_{1,\eta}]&\\ +(b^\star_{12}+B_{22})[\lambda w_{2,\xi}+w_{2,\eta}]+b^\star_{12}w_{1,\eta} &= q^\star_{21}w_1+q^\star_{22}w_2+\phi^\star_2.\end{aligned} \tag{NFP}$$

Thus, the normal form in the parabolic case is therefore characterized by

1. $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$,

2. λ has an eigenvector and moreover a generalized eigenvector.

2.2. Canonical systems

2.2.1. The hyperbolic case

Our starting point in the hyperbolic case is (NFHE) and we consider in the (ξ, η)-coordinate-system two families of lines in the plane, these are the characteristics of the hyperbolic case

$$\begin{aligned} \frac{d\xi}{d\eta} &= \lambda_1, \qquad \xi - \xi_0 = \lambda_1(\eta - \eta_0), \qquad \xi - \lambda_1\eta = \xi_0 - \lambda_1\eta_0 =: x, \\ \frac{d\xi}{d\eta} &= \lambda_2, \qquad \xi - \xi_0 = \lambda_2(\eta - \eta_0), \qquad \xi - \lambda_2\eta = \xi_0 - \lambda_2\eta_0 =: y. \end{aligned} \tag{7}$$

We introduce the family parameters x and y as new coordinates (x, y), also referred to as the characteristic or the canonical coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} -\lambda_2 & \lambda_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{8}$$

In the (x, y)-system the characteristics appear as families of lines parallel to the respective axes. In a change for coordinates, we note

$$w(\xi, \eta) = w(\xi(x, y), \eta(x, y)) =: \tilde{u}(x, y) = \tilde{u}(x(\xi, \eta), y(\xi, \eta)), \tag{9}$$

and thus $w_\xi(\xi, \eta) = \tilde{u}_x \cdot 1 + \tilde{u}_y \cdot 1$, and $w_\eta(\xi, \eta) = \tilde{u}_x \cdot (-\lambda_1) + \tilde{u}_y \cdot (-\lambda_2)$. After the transformation (8) and (9) the normal form (NFHE) appears as follows

$$\begin{aligned} b^*_{11}(\lambda_1 - \lambda_2)\tilde{u}_{1,y} + b^*_{12}(\lambda_1 - \lambda_2)\tilde{u}_{2,y} &= q^*_{11}\tilde{u}_1 + q^*_{12}\tilde{u}_2 + \tilde{\phi}^*_1, \\ b^*_{11}(\lambda_1 - \lambda_2)\tilde{u}_{1,x} + b^*_{22}(\lambda_1 - \lambda_2)\tilde{u}_{2,x} &= -q^*_{21}\tilde{u}_1 - q^*_{22}\tilde{u}_2 - \tilde{\phi}^*_2. \end{aligned}$$

We are combining linearly new functions

$$b^*_{11}\tilde{u}_1 + b^*_{22}\tilde{u}_2 =: -u(x, y), \qquad b^*_{11}\tilde{u}_1 + b^*_{12}\tilde{u}_2 =: v(x, y).$$

So we achieve

$$\begin{aligned} (\lambda_1 - \lambda_2) \begin{bmatrix} u_x \\ v_y \end{bmatrix} &= \begin{bmatrix} q^*_{21} & q^*_{22} \\ q^*_{11} & q^*_{12} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} + \begin{bmatrix} \tilde{\phi}^*_2 \\ \tilde{\phi}^*_1 \end{bmatrix} \\ &= \begin{bmatrix} q^*_{21} & q^*_{22} \\ q^*_{11} & q^*_{12} \end{bmatrix} \cdot \frac{1}{b^*_{11}(b^*_{12} - b^*_{22})} \cdot \begin{bmatrix} b^*_{12} & -b^*_{22} \\ -b^*_{11} & b^*_{11} \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} + \begin{bmatrix} \tilde{\phi}^*_2 \\ \tilde{\phi}^*_1 \end{bmatrix}, \end{aligned}$$

and we have

$$\begin{bmatrix} u_x \\ v_y \end{bmatrix} = \frac{1}{b^*_{11}(\lambda_1 - \lambda_2)^2|B|} \begin{bmatrix} b^*_{11}q^*_{22} - b^*_{12}q^*_{21} & b^*_{11}q^*_{22} - b^*_{22}q^*_{21} \\ b^*_{11}q^*_{12} - b^*_{12}q^*_{11} & b^*_{12}q^*_{12} - b^*_{22}q^*_{11} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \frac{1}{(\lambda_1 - \lambda_2)} \begin{bmatrix} \tilde{\phi}^*_2 \\ \tilde{\phi}^*_1 \end{bmatrix},$$

or in new, obvious designations

$$\begin{aligned} u_x &= h_{11}u(x, y) + h_{12}v(x, y) + f_1(x, y), \\ v_y &= h_{21}u(x, y) + h_{22}v(x, y) + f_2(x, y). \end{aligned} \tag{CHS}$$

Equations (CHS) are called the canonical normal form of (1) in the hyperbolic case or the canonical-hyperbolic system.

An initial value problem (Cauchy problem) for (CHS) is given, provided:

- u is given in an interval which is not a part of any x-characteristic, say on the interval $(0, b)$ of the y-axis (y-characteristic).
- v is given in an interval which is not a part of any y-characteristic, say on the interval $(0, a)$ of the x-axis (x-characteristic). See also chapter 3.3.

Such problems are often referred to in the literature as the characteristic initial value problems, this term is slightly misleading, see e.g. [23], [25].

Example

We consider in a (ξ, η)-coordinate-system the inhomogeneous wave equation

$$u_{\eta,\eta} - u_{\xi\xi} = -k^2 u + p(\xi, \eta).$$

Such a representation is often referred to in the literature as the canonical form of the wave equation, see [27]. We do not use this way of speaking, because we tie on the canonic form to the canonical coordinates. With [16], charcteristic coordinates (x, t) and (7), (8), (9) and $\lambda_1 = 1$, $\lambda_2 = -1$ we put

$$\begin{aligned} u(\xi, \eta) &=: U_1(\xi, \eta) = U_1(\xi(x, t), \eta(x, t)) =: U(x, t) = U(x(\xi, \eta), t(\xi, \eta)), \\ u_\xi + u_\eta &=: U_2(\xi, \eta) = U_2(\xi(x, t), \eta(x, t)) =: V(x, t) = V(x(\xi, \eta), t(\xi, \eta)), \end{aligned}$$

and so, we have firstly a system in the normal form (NFHE) in the (ξ, η)-coordinate-system

$$\begin{aligned} U_{1,\xi} + U_{1,\eta} &= U_2(\xi, \eta), \\ -U_{2,\xi} + U_{2,\eta} &= -k^2 U_1(\xi, \eta) + p(\xi, \eta), \end{aligned}$$

and after the transition to the characteristic coordinates and with $p(\xi, \eta) = \tilde{p}(x, t)$ the canonical-hyperbolic system

$$\begin{aligned} U_t &= \frac{1}{2} V, \\ V_x &= \frac{1}{2} k^2 U - \tilde{p}(x, t). \end{aligned}$$

2.2.2. The elliptic case

We are investigating the elliptic case in (NFHE), i.e. $\lambda_1 = \lambda = \mu + i\nu$, $\lambda_2 = \overline{\lambda} = \mu - i\nu$, $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$, and have in the (ξ, η)-coordinate-system:

$$\begin{aligned} \frac{d\xi}{d\eta} &= \lambda, \\ \xi - \xi_0 &= (\mu + i\nu)(\eta - \eta_0), \\ \xi - \mu\eta - i\nu\eta &= \xi_0 - \mu\eta_0 - i\nu\eta_0 =: x + iy, \\ \frac{d\xi}{d\eta} &= \overline{\lambda}, \\ \xi - \xi_0 &= (\mu - i\nu)(\eta - \eta_0), \\ \xi - \mu\eta + i\nu\eta &= \xi_0 - \mu\eta_0 + i\nu\eta_0 =: x - iy. \end{aligned} \tag{10}$$

Using the x and y again as parameters of the families of straight lines so we have, after fission of the right-hand sides of (10) into real part and imaginary part, two families of straight lines in the coordinates (ξ, η), which are called the characteristics in the elliptic case. The connection between the (ξ, η)- and the (x, y)-coordinate systems is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -\mu \\ 0 & -\nu \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{-\nu} \begin{bmatrix} -\nu & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{11}$$

In the (x, y)-system the characteristics appear as lines parallel to the respective axes.

At change of coordinates, we note (9), and thus

$$w_\xi(\xi, \eta) = \tilde{u}_x \cdot 1, \qquad w_\eta(\xi, \eta) = \tilde{u}_x \cdot (-\mu) + \tilde{u}_y \cdot (-\nu). \tag{12}$$

Using (5) and the transformation (11), (12) the normal form (NFHE) appears as follows: first line of (NFHE)

$$\begin{aligned} &b^*_{1,1}[i\nu\tilde{u}_{1,x} - \nu\tilde{u}_{1,y}] + [c^*_{1,2} + (\mu + i\nu)|B|][i\nu\tilde{u}_{2,x} - \nu\tilde{u}_{2,y}] \\ &\quad = [(a_{21} - (\mu + i\nu)b_{21})q_{11} + (-a_{11} + (\mu + i\nu)b_{11})q_{21}]\tilde{u}_1(x, y) \\ &\qquad + [(a_{21} - (\mu + i\nu)b_{21})q_{12} + (-a_{11} + (\mu + i\nu)b_{11})q_{22}]\tilde{u}_2(x, y) + \tilde{\phi}^*_1(x, y), \end{aligned} \tag{13}$$

and the second line of (NFHE)

$$\begin{aligned} &b^*_{1,1}[-i\nu\tilde{u}_{1,x} - \nu\tilde{u}_{1,y}] + [c^*_{1,2} + (\mu - i\nu)|B|][-i\nu\tilde{u}_{2,x} - \nu\tilde{u}_{2,y}] \\ &\quad = [(a_{21} - (\mu - i\nu)b_{21})q_{11} + (-a_{11} + (\mu - i\nu)b_{11})q_{21}]\tilde{u}_1(x, y) \\ &\qquad + [(a_{21} - (\mu - i\nu)b_{21})q_{12} + (-a_{11} + (\mu - i\nu)b_{11})q_{22}]\tilde{u}_2(x, y) + \tilde{\phi}^*_2(x, y). \end{aligned} \tag{14}$$

From (14) we have

$$\begin{aligned} &[-b^*_{11}\nu\tilde{u}_{1,y} - c^*_{12}\nu\tilde{u}_{2,y} - \nu\mu|B|\tilde{u}_{2,y} - \nu^2|B|\tilde{u}_{2,x}] \\ &\qquad + i \cdot [b^*_{11}\nu\tilde{u}_{1,x} + c^*_{12}\nu\tilde{u}_{2,x} + \nu\mu|B|\tilde{u}_{2,x} - \nu^2|B|\tilde{u}_{2,y}] \\ &\quad = \operatorname{Re}\{\text{right side of (14)}\} + i\operatorname{Im}\{\text{right side of (14)}\} \\ &\quad =: \mathfrak{A} + i\mathfrak{B} \end{aligned} \tag{15}$$

with

$$\begin{aligned} \mathfrak{A} &= [(a_{21} - \mu b_{21})q_{11} + (-a_{11} + \mu b_{11})q_{21}]\tilde{u}_1 + [(a_{21} - \mu b_{21})q_{12} \\ &\quad + (-a_{11} + \mu b_{11})q_{22}]\tilde{u}_2 + (a_{21} - \mu b_{21})\tilde{\phi}_1 + (-a_{11} + \mu b_{11})\tilde{\phi}_2, \\ \mathfrak{B} &= \nu\{(b_{11}q_{21} - b_{21}q_{11})\tilde{u}_1 + (b_{11}q_{22} - b_{21}q_{12})\tilde{u}_2 + (b_{11}\tilde{\phi}_2 - b_{21}\tilde{\phi}_1)\}. \end{aligned}$$

Analogous we have with (15)

$$\begin{aligned} &[-b^*_{11}\nu\tilde{u}_{1,y} - c^*_{12}\nu\tilde{u}_{2,y} - \nu\mu|B|\tilde{u}_{2,y} - \nu^2|B|\tilde{u}_{2,x}] \\ &\qquad + i \cdot [-b^*_{11}\nu\tilde{u}_{1,x} - c^*_{12}\nu\tilde{u}_{2,x} - \nu\mu|B|\tilde{u}_{2,x} + \nu^2|B|\tilde{u}_{2,y}] \\ &\quad = \operatorname{Re}\{\text{right side of (15)}\} + i\operatorname{Im}\{\text{right side of (15)}\} \\ &\quad = \mathfrak{A} - i\mathfrak{B}. \end{aligned} \tag{16}$$

We introduce new functions

$$u(x,y) := b_{11}^* \nu \tilde{u}_1 + c_{12}^* \nu \tilde{u}_2 + \nu\mu|B|\tilde{u}_2, \qquad v(x,y) := \nu^2|B|\tilde{u}_2, \tag{17}$$

and we obtain from (15) the following formulas for the real parts and for the imaginary parts

$$-u_y - v_x = \mathfrak{A}, \qquad u_x - v_y = \mathfrak{B}.$$

Similarly from (16) we get

$$-u_y - v_x = \mathfrak{A}, \qquad -u_x + v_y = -\mathfrak{B}.$$

We see that the corresponding splitting of the two different formulas (14), (15) into real part and imaginary part does not lead to a contradiction, but to the same result.

Now we express the functions $(\tilde{u}_1, \tilde{u}_2)$ in terms of (u,v),

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \frac{1}{\nu^3 b_{11}^* |B|} \begin{bmatrix} \nu^2|B| & -\nu b_{11}^* - \mu\nu|B| \\ 0 & \nu b_{11}^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

and summarize the coefficients at u, v in $\mathfrak{A}$, $\mathfrak{B}$ in the quantities e_{ij} as well as the inhomogeneities in f_i, $i = 1,2$. So we obtain the known system

$$\begin{aligned} u_y + v_x &= e_{11}u(x,y) + e_{12}v(x,y) + f_1(x,y), \\ u_x - v_y &= e_{21}u(x,y) + e_{22}v(x,y) + f_2(x,y). \end{aligned} \tag{CES}$$

(CES) is called the canonical normal form of (1) in the elliptic case or the canonical-elliptic system.

Remark 2.1
If we replace in (17) $u(x,y)$ by $-u(x,y)$, we obtain (CES) modified in a form which we will use later in (37).

2.2.3. The parabolic case

We start from the normal form (NFP) and put

$$\begin{aligned} b_{11}^* w_1 + b_{12}^* w_2 &=: U(\xi(x,y), \eta(x,y)) =: u(x,y) = u(x(\xi,\eta), y(\xi,\eta)), \\ B_{21} w_1 + B_{22} w_2 &=: V(\xi(x,y), \eta(x,y)) =: v(x,y) = v(x(\xi,\eta), y(\xi,\eta)). \end{aligned} \tag{18}$$

So that with $\Delta = b_{11}^* B_{22} - b_{12}^* B_{21}$ and (NFP) arises

$$\begin{bmatrix} \lambda U_\xi + U_\eta \\ U_\xi + \lambda U_\xi + \lambda V_\xi + U_\eta + V_\eta \end{bmatrix} = \frac{1}{\Delta} Q^* \begin{bmatrix} B_{22} & -b_{12}^* \\ -B_{21} & b_{11}^* \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} \phi_1^\star \\ \phi_2^\star \end{bmatrix}. \tag{19}$$

Note please also here the now present directional derivatives.

Using x and y again as parameters of families of straight lines so we have (x,y) as characteristic coordinates, which are introduced by

$$\frac{d\xi}{d\eta} = \lambda, \qquad \xi - \xi_0 = \lambda(\eta - \eta_0), \qquad \xi - \lambda\eta = \xi_0 - \lambda\eta_0 =: x, \qquad \eta =: y, \tag{20}$$

and we have also in the parabolic case axially parallel families of lines as characteristics. So we obtain with (18), (19) and (20)

$$\begin{aligned}&\begin{bmatrix} u_y \\ u_x + v_y \end{bmatrix}\\ &\quad = \frac{1}{\Delta}\begin{bmatrix} q_{11}^* B_{22} - q_{12}^* B_{21} & q_{12}^* b_{11}^* - q_{11}^* b_{12}^* \\ (q_{21}^* - q_{11}^*) B_{22} + (q_{12}^* - q_{22}^*) B_{21} & (q_{22}^* - q_{12}^*) b_{11}^* + (q_{11}^* - q_{21}^*) b_{12}^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}\\ &\qquad + \begin{bmatrix} \tilde{\phi}_1^*(x,y) \\ \tilde{\phi}_2^*(x,y) - \tilde{\phi}_1^*(x,y) \end{bmatrix},\end{aligned}$$

or in new designations regarding the coefficients and the inhomogeneities

$$\begin{aligned} u_y &= p_{11} u(x,y) + p_{12} v(x,y) + f_1(x,y), \\ u_x + v_y &= p_{21} u(x,y) + p_{22} v(x,y) + f_2(x,y). \end{aligned} \tag{CPS}$$

This is the canonical normal form for (1) in the parabolic case or the canonic-parabolic system.

Remark 2.2
Use we slightly more general than (18) the substitution.

$$b_{11}^* w_1 + b_{12}^* w_2 =: \rho \cdot U(\xi(x,y), \eta(x,y)) =: \rho \cdot u(x,y) = \rho \cdot u(x(\xi,\eta), y(\xi,\eta))$$

for $\rho \in \{\mathbb{R}\backslash 0\}$, so we have now (CPS) for the pair $(\rho\, u, v)^T$, and particularly for $\rho = -1$ the formula (CPS) is modified on the left-hand side by a sign and on the right-hand side in the coefficients and in the inhomogeneities

$$\begin{aligned} u_y &= m_{11} u(x,y) + m_{12} v(x,y) + \check{f}_1(x,y), \\ u_x - v_y &= m_{21} u(x,y) + m_{22} v(x,y) + \check{f}_2(x,y). \end{aligned} \tag{CWS}$$

The formula (CWS) facilitates our connection to the usual formulation of the heat conduction problems (with: y as spatial coordinate, x as time coordinate), see [26].

Remark 2.3
Our treatment of (1) according to (2) and (3) is to some extent a division-free method. Alternatively, in (1) we can immediately with $w = S\bar{u}$ move to a new pair of functions $\bar{u}(\xi,\eta) = (\bar{u}_1(\xi,\eta), (\bar{u}_2(\xi,\eta))^T$. The matrix S is still undetermined. The resulting equation will be left-hand multiplied by $S^{-1}B^{-1}$ and we have

$$S^{-1}B^{-1}AS\bar{u}_\xi + S^{-1}B^{-1}BS\bar{u}_\eta = S^{-1}B^{-1}QS\bar{u}(\xi,\eta) + S^{-1}B^{-1}\phi(\xi,\eta). \tag{21}$$

Creates S the Jordan form of $B^{-1}A$, then we have immediately the analogue of (NFHE), respectively (NFP); however the right-hand sides now require more effort. In concrete practical cases (fixed values, possibly sparse matrices A, B, Q) formula (21) produces often faster the characteristic systems.

3. Transform methods

3.1. The canonic-parabolic system

3.1.1. Treatment with the Laplace transform

We start from (CWS), using its right-hand side in a new designation, and we investigate the following initial value problem $\mathbf{P^{11}}$ in the strip $S = (0,l) \times (0,\infty)$ ($\mathbf{P^{11}}$ means that the solution $(u,v)^T$) is calculated so that the first component u on the left border and on the right border of the strip in each case a given initial condition must fulfil, see the problems $\mathbf{P^{ij}}$ in [26]), so we consider

$$\begin{aligned} u_x &= q_{11}u + q_{12}v + f_1(x,t), \qquad u,v \in \mathbf{C^1}[(0,\infty)\times(0,\infty)],\\ u_t - v_x &= q_{21}u + q_{22}v + f_2(x,t), \qquad f_i \in \mathbf{C}[(0,\infty)\times(0,\infty)],\\ \lim_{x\to+0} u(x,t) &= u^0(t), \quad \lim_{x\to l-0} u(x,t) = u^l(t), \qquad u^0, u^l \in \mathbf{C}[(0,\infty)],\\ \lim_{t\to+0} u(x,t) &= f(x), \quad \text{where } f \in \mathbf{C}[(0,l)],\ u^0, u^l \in \mathbf{C}[(0,\infty)] \text{ are given,}\\ & q_{ij} \in \mathbb{R},\ i,j \in \{1,2\},\ q_{12} > 0,\ q_{21} \le 0,\ (\text{“heat typ”}). \end{aligned} \tag{22}$$

We will treat this problem (22) by means of the one-dimensional Laplace transform [7]. Let

$$L[u(x,t);t](s) = \int_0^\infty e^{-s\tau} u(x,\tau)\, d\tau = \omega_1(x,s),$$

which will be abbreviated as $u(x,t) \circ\!\to\!\bullet\, \omega_1(x,s)$. For the other parts in (22) we have

$$\begin{aligned} &v(x,t) \circ\!\to\!\bullet\, \omega_2(x,s), \quad v_x(x,t) \circ\!\to\!\bullet\, \omega_{2,x}(x,s), \quad u^0(t) \circ\!\to\!\bullet\, \omega_1^0(s),\\ &u_t(x,t) \circ\!\to\!\bullet\, s\cdot\omega_1(x,s) - f(x), \quad u_x(x,t) \circ\!\to\!\bullet\, \omega_{1,x}(x,s), \quad u^l(t) \circ\!\to\!\bullet\, \omega_1^l(s),\\ &f_i(x,t) \circ\!\to\!\bullet\, \varphi_i(x,s), \quad i = 1,2. \end{aligned}$$

We obtain in the (x,s)- image range of the Laplace transform a system of ordinary differential equations

$$\begin{aligned} &\begin{bmatrix} \omega_{1,x} \\ \omega_{2,x} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ s - q_{21} & -q_{22} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \varphi_1 \\ -f - \varphi_2 \end{bmatrix} \quad (\text{abbrev. } \omega_x = Q^\bullet \omega + r(x,s)),\\ &\omega_1(0,s) = \omega_1^0(s), \quad \omega_1(l,s) = \omega_1^l(s), \quad (\text{a parameter-dependent boundary value problem}). \end{aligned} \tag{23}$$

Notice that (23) is called $\mathbf{\Pi^{11}}$, the problem, which is associated to $\mathbf{P^{11}}$, see [26].

Using the two “border matrices” B_0^{11} und B_l^{11} we write on the border in problem (23)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1^0(s) \\ \omega_2^0(s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1^l(s) \\ \omega_2^l(s) \end{bmatrix} =: B_0^{11} \begin{bmatrix} \omega_1^0(s) \\ \omega_2^0(s) \end{bmatrix} + B_l^{11} \begin{bmatrix} \omega_1^l(s) \\ \omega_2^l(s) \end{bmatrix} = \begin{bmatrix} \omega_1^0(s) \\ w_1^l(s) \end{bmatrix}.$$

The matrix $Q^\bullet$ from (23) has the following eigenvalues

$$\lambda_{1,2} = \frac{q_{11} - q_{22}}{2} \pm \frac{1}{2}\sqrt{(q_{11} + q_{22})^2 - 4q_{12}q_{21} + 4q_{12}s},$$

and the eigenvectors

$$e^1 = \begin{bmatrix} q_{12} \\ \lambda_1 - q_{11} \end{bmatrix}, \qquad e^2 = \begin{bmatrix} -q_{12} \\ -\lambda_2 + q_{11} \end{bmatrix}.$$

With the abbreviations

$$q_{12}q_{21} =: q, \qquad \frac{q_{11}+q_{22}}{2} =: \mathfrak{s}, \qquad \frac{q_{11}-q_{22}}{2} =: d, \qquad \mathfrak{s}^2 - q =: \delta,$$

we write the general solution of the homogeneous system to (23) as follows

$$\begin{aligned} \omega_{1,hom}(x,s) &= c_1 q_{12} e^{dx} \cdot e^{\sqrt{\delta+q_{12}s}x} - c_2 q_{12} e^{dx} \cdot e^{-\sqrt{\delta+q_{12}s}x}, \\ \omega_{2,hom}(x,s) &= c_1(-\mathfrak{s} + \sqrt{\delta+q_{12}s})e^{dx} \cdot e^{\sqrt{\delta+q_{12}s}x} \\ &\qquad + c_2(\mathfrak{s} + \sqrt{\delta+q_{12}s})e^{dx} \cdot e^{-\sqrt{\delta+q_{12}s}x}, \qquad c_1, c_2 \in \mathbb{R}. \end{aligned}$$

We solve the homogeneous problem concerning $\mathbf{\Pi^{11}}$ by determining the quantities $c_1(s)$, $c_2(s)$ from the boundary conditions. We have with $\sqrt{\delta + q_{12}s} =: \delta_s$,

$$\begin{aligned} &\omega_{1,hom}(x,s) \\ &\quad= \frac{1}{\sinh(\delta_s \cdot l)}\big[e^{dx}\sinh(\delta_s(l-x)) \cdot \omega_1^0(s) + e^{-d(l-x)}\sinh(\delta_s x)\cdot\omega_1^l(s)\big], \\ &\omega_{2,hom}(x,s) \\ &\quad= \frac{1}{q_{12}\sinh(\delta_s\cdot l)}\big[e^{dx}[-\mathfrak{s}\cdot\sinh(\delta_s(l-x)) - \delta_s\cosh(\delta_s(l-x))]\omega_1^0(s) \\ &\qquad\qquad + e^{-d(l-x)}[-\mathfrak{s}\cdot\sinh(\delta_s x) + \delta_s\cosh(\delta_s x)]w_1^l(s)\big]. \end{aligned} \tag{24}$$

Now we give a particular solution of the inhomogeneous problem (23), see [14], [26]. Let $\mathcal{W}$ be the fundamental matrix belonging to the homogeneous problem to (23), thus

$$\begin{aligned} \mathcal{W}(x,s) &= \begin{bmatrix} q_{11}e^{\lambda_1 x} & -q_{12}e^{\lambda_2 x} \\ (\lambda_1 - q_{11})e^{\lambda_1 x} & (-\lambda_2 + q_{11})e^{\lambda_2 x} \end{bmatrix} \\ &= \begin{bmatrix} q_{12}e^{dx+\delta_s x} & -q_{12}e^{dx-\delta_s x} \\ (-\mathfrak{s}+\delta_s)e^{dx+\delta_s x} & (\mathfrak{s}+\delta_s)e^{dx-\delta_s x} \end{bmatrix}. \end{aligned}$$

With

$$M_{11} := B_0^{11}\mathcal{W}(0,s) + B_l^{11}\mathcal{W}(l,s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathcal{W}(0,s) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\mathcal{W}(l,s)$$

we obtain by means of the method "variation of constants", see [14], as a particular solution of the inhomogeneous system (23),

$$\omega_{inh}(x,s) = \int_0^l \mathcal{G}^{11}(x,\xi,s)r(\xi,s)\,d\xi = \int_0^x \mathcal{G}^{11}_{\xi\le x} r(\xi,s)\,d\xi + \int_x^l \mathcal{G}^{11}_{x<\xi} r(\xi,s)\,d\xi,$$

with

$$\mathcal{G}^{11}(x,\xi,s) \;=\; \begin{cases} \mathcal{W}(x,s)M_{11}^{-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathcal{W}(0,s)\mathcal{W}^{-1}(\xi,s), & \xi \le x,\\ -\mathcal{W}(x,s)M_{11}^{-1}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\mathcal{W}(l,s)\mathcal{W}^{-1}(\xi,s), & x<\xi,\end{cases} \tag{25}$$

$$\mathcal{G}^{11}_{\xi\le x}(x,\xi,s) := \mathcal{W}(x,s)M_{11}^{-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathcal{W}(0,s)\mathcal{W}^{-1}(\xi,s), \quad \xi\le x,$$

$$\mathcal{G}^{11}_{x<\xi}(x,\xi,s) := -\mathcal{W}(x,s)M_{11}^{-1}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\mathcal{W}(l,s)\mathcal{W}^{-1}(\xi,s), \quad x<\xi.$$

Since the function $\mathcal{G}^{11}$ describes the influence of the left-hand and the right-hand boundary of S on the solution of the problem $\mathbf{\Pi^{11}}$ at the position x, $x \in (0,l)$, it is named influence function or Green's function. Because all $\mathcal{W}$-quantities and M_{11} in (25) are well known, we can specify the matrix $\mathcal{G}^{11}$ with its four elements. We abbreviate

$$\mathcal{S}(x) \underset{Df}{=} \sinh(\delta_s x), \qquad \mathcal{C}(x) \underset{Df}{=} \cosh(\delta_s x),$$

and we have

$$\mathcal{G}^{11}_{\xi\le x} = \frac{e^{d(x-\xi)}}{2\delta_s\mathcal{S}(l)}\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix},$$

where

$$G_{11} = \mathfrak{s}[\mathcal{C}(x+\xi-l)-\mathcal{C}(x-\xi-l)] - \delta_s[\mathcal{S}(x+\xi-l)+\mathcal{S}(x-\xi-l)],$$
$$G_{12} = q_{12}[\mathcal{C}(x+\xi-l)-\mathcal{C}(x-\xi-l)],$$
$$G_{21} = -q_{12}(\mathfrak{s}^2+\delta_s^2)\mathcal{C}(x+\xi-l) + q_{12}(\mathfrak{s}^2-\delta_s^2)\mathcal{C}(x-\xi-l) + 2q_{12}\mathfrak{s}\delta_s\mathcal{S}(x+\xi-l),$$
$$G_{22} = \mathfrak{s}[-\mathcal{C}(x+\xi-l)+\mathcal{C}(x-\xi-l)] + \delta_s[\mathcal{S}(x+\xi-l)-\mathcal{S}(x-\xi-l)]$$

and

$$\mathcal{G}^{11}_{x<\xi} = \frac{e^{d(x-\xi)}}{2\delta_s\mathcal{S}(l)}\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix},$$

where

$$H_{11} = \mathfrak{s}[\mathcal{C}(x+\xi-l)-\mathcal{C}(x-\xi+l)] - \delta_s[\mathcal{S}(x+\xi-l)+\mathcal{S}(x-\xi-l)],$$
$$H_{12} = q_{12}[\mathcal{C}(x+\xi-l)-\mathcal{C}(x-\xi+l)],$$
$$H_{21} = -q_{12}(\mathfrak{s}^2+\delta_s^2)\mathcal{C}(x+\xi-l) + q_{12}(\mathfrak{s}^2-\delta_s^2)\mathcal{C}(x-\xi+l) + 2q_{12}\mathfrak{s}\delta_s\mathcal{S}(x+\xi-l),$$
$$H_{22} = \mathfrak{s}[-\mathcal{C}(x+\xi-l)+\mathcal{C}(x-\xi-l)] + \delta_s[\mathcal{S}(x+\xi-l)-\mathcal{S}(x-\xi+l)].$$

With respect to (24) and (25) and the specialization $d=0$, $\mathfrak{s}=0$, $q_{21}=0$, $q_{12}=1$, we are falling back on the prototype, which was treated in [26] (classical heat conduction problem as a first-order system, but complete inhomogeneity).

We still have to provide the inverse Laplace transform of the solution for $\mathbf{\Pi^{11}}$. Then we have the (formal) solution in the (x,t)-original domain. It can be noted that the functions that occur in (24) and $\mathcal{G}^{11}(x,\xi,s)$ are not inherently more difficult than the functions occurring in [26] (essentially inverse Laplace transforms for quotients of hyperbolic functions). We obtain formal solutions for the problem (22), i.e. for $\mathbf{P^{11}}$. In a similar way you can treat the problems $\mathbf{P^{ij}}$, see [26].

3.1.2. Treatment with the Fourier transform

We treat again as an example the problem $\mathbf{P^{11}}$, i.e. problem (22), but initially we not impose conditions on the coefficients $q_{ij} \in \mathbb{R}$ ("heat type" is no longer precondition). We use the Fourier cosine transform $\mathcal{F}_c$ with respect to t and the Fourier sine transform $\mathcal{F}_s$ with respect to t of a function $w(x,t)$ with the following designations:

$$\begin{aligned}
\mathcal{F}_c[w(x,t)] &= \hat{w}_c(x,\omega) = \int_0^\infty w(x,t)\cos\omega t\,dt, \\
&\qquad\qquad (\text{abbrev. } w(x,t) \circ\!\xrightarrow{\mathcal{F}_c}\!\bullet\, \hat{w}_c(x,\omega)), \\
\mathcal{F}_c^{-1}\mathcal{F}_c[w(x,t)] &= \mathcal{F}_c^{-1}[\hat{w}_c(x,\omega)] = w(x,t) = \frac{2}{\pi}\int_0^\infty \hat{w}_c(x,\omega)\cos\omega t\,d\omega, \\
\mathcal{F}_s[w(x,t)] &= \hat{w}_s(x,\omega) = \int_0^\infty w(x,t)\sin\omega t\,dt, \\
&\qquad\qquad (\text{abbrev. } w(x,t) \circ\!\xrightarrow{\mathcal{F}_s}\!\bullet\, \hat{w}_s(x,\omega)), \\
\mathcal{F}_s^{-1}\mathcal{F}_s[w(x,t)] &= \mathcal{F}_s^{-1}[\hat{w}_s(x,\omega)] = w(x,t) = \frac{2}{\pi}\int_0^\infty \hat{w}_s(x,\omega)\sin\omega t\,d\omega.
\end{aligned} \tag{26}$$

If we apply line by line to system (22) the transformation $\mathcal{F}_c$, so we have

$$\begin{aligned}
&\int_0^\infty [u_x - q_{11}u - q_{12}v - f_1(x,t)]\cos\omega t\,dt \\
&\qquad = \hat{u}_{c,x} - q_{11}\hat{u}_c - q_{12}\hat{v}_c - \hat{f}_{1,c} = 0, \\
&\int_0^\infty [u_t - v_x - q_{21}u - q_{22}v - f_2(x,t)]\cos\omega t\,dt \\
&\qquad = \omega\hat{u}_s - f(x) - \hat{v}_{c,x} - q_{21}\hat{u}_c - q_{22}\hat{v}_c - \hat{f}_{2,c} = 0.
\end{aligned}$$

If we apply line by line to system (22) the transformation $\mathcal{F}_s$, so we have

$$\begin{aligned}
&\int_0^\infty [u_x - q_{11}u - q_{12}v - f_1(x,t)]\sin\omega t\,dt \\
&\qquad = \hat{u}_{s,x} - q_{11}\hat{u}_s - q_{12}\hat{v}_s - \hat{f}_{1,s} = 0, \\
&\int_0^\infty [u_t - v_x - q_{21}u - q_{22}v - f_2(x,t)]\sin\omega t\,dt \\
&\qquad = -\omega\hat{u}_c - \hat{v}_{s,x} - q_{21}\hat{u}_s - q_{22}\hat{v}_s - \hat{f}_{2,s} = 0.
\end{aligned}$$

The rules on the use for the transform of derivatives $w_t(x,t)$ have been complied with, see e.g. [24], [21]. So we have in the Fourier (x,s)-image range the following boundary value problem $\mathbf{\Pi^{11}}$ for a system of linear ordinary differential equations

$$\begin{gathered}
\begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_x (x,\omega) = \begin{bmatrix} q_{11} & q_{12} & 0 & 0 \\ -q_{21} & -q_{22} & \omega & 0 \\ 0 & 0 & q_{11} & q_{22} \\ -\omega & 0 & -q_{21} & -q_{22} \end{bmatrix} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix} + \begin{bmatrix} \hat{f}_{1,c} \\ -\hat{f}_{2,c} - f(x) \\ \hat{f}_{1,s} \\ -\hat{f}_{2,s} \end{bmatrix}, \\
\hat{u}_c(0,\omega) = \hat{u}_c^0(\omega), \quad \hat{u}_c(l,\omega) = \hat{u}_c^l(\omega), \quad \hat{u}_s(0,\omega) = \hat{u}_s^0(\omega), \quad \hat{u}_s(l,\omega) = \hat{u}_s^l(\omega), \\
\text{(a parameter-dependent boundary value problem).}
\end{gathered} \tag{27}$$

Gradually we will take back the difficulty level of the problem (27), so we can keep our approach clearly and obtain simpler results. In this sense, we consider now in the original (x,t)-domain the simple parabolic system

$$\begin{aligned} u_x &= \frac{1}{\kappa} v, \qquad \kappa \in \mathbb{R}\backslash\{0\}, \\ u_t - v_x &= f_2, \end{aligned}$$

which corresponds to the following equation

$$u_t = \kappa u_{xx} + f_2(x,t). \tag{28}$$

The system (27) will appear in the following form (the stripped-down problem $\mathbf{\Pi^{11}}$ in the (x,s)-image range)

$$\begin{gathered} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_x (x,\omega) = \begin{bmatrix} 0 & \frac{1}{\kappa} & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} \\ -\omega & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix} + \begin{bmatrix} 0 \\ -\hat{f}_{2,c} - f(x) \\ 0 \\ -\hat{f}_{2,s} \end{bmatrix}, \\ \hat{u}_c(0,\omega) = \hat{u}_c^0(\omega), \quad \hat{u}_c(l,\omega) = \hat{u}_c^l(\omega), \quad \hat{u}_s(0,\omega) = \hat{u}_s^0(\omega), \quad \hat{u}_s(l,\omega) = \hat{u}_s^l(\omega). \end{gathered} \tag{29}$$

Let $\mathbf{D}$ be the matrix of the coefficients of (29). The equation $|\mathbf{D} - \lambda \cdot \mathbf{I}| = 0$ gives with $\kappa = \frac{k}{2}$, $k = \pm 1$, the eigenvalues

$$\begin{aligned} k = 1: \quad & \lambda_{1,2} = (1 \pm i)\sqrt{\omega}, \quad \lambda_{3,4} = (-1 \pm i)\sqrt{\omega}, \\ k = -1: \quad & \lambda_{1,2} = (1 \pm i)\sqrt{\omega}, \quad \lambda_{3,4} = (-1 \pm i)\sqrt{\omega}. \end{aligned}$$

Both sets of eigenvalues are identical, they have been numbered for $k = 1$ and $k = -1$ in the same way. However, the eigenvectors are still a function of the coefficients of the matrix $\mathbf{D}$. These coefficients are different for $k = \pm 1$.

Let $\mathbf{X}_{k=+1}$ be the matrix of the eigenvectors x^i, $i = 1, \dots, 4$ of $\mathbf{D}$ for $k = +1$ and $\mathbf{X}_{k=-1}$ the matrix of the eigenvectors $\underline{\mathbf{x}}^i$, $i = 1, \dots, 4$ of $\mathbf{D}$ for $k = -1$, so we have

$$\mathbf{X}_{k=+1} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4] = \begin{bmatrix} 2 & 2 & 2 & 2 \\ (1+i)\sqrt{\omega} & (1-i)\sqrt{\omega} & (-1+i)\sqrt{\omega} & (-1-i)\sqrt{\omega} \\ 2i & -2i & -2i & 2i \\ (-1+i)\sqrt{\omega} & (-1-i)\sqrt{\omega} & (1+i)\sqrt{\omega} & (1-i)\sqrt{\omega} \end{bmatrix}$$

and

$$\mathbf{X}_{k=-1} = [\underline{\mathbf{x}}^1, \underline{\mathbf{x}}^2, \underline{\mathbf{x}}^3, \underline{\mathbf{x}}^4] = \begin{bmatrix} 2 & 2 & 2 & 2 \\ (-1-i)\sqrt{\omega} & (-1+i)\sqrt{\omega} & (1-i)\sqrt{\omega} & (1+i)\sqrt{\omega} \\ -2i & 2i & 2i & -2i \\ (-1+i)\sqrt{\omega} & (-1-i)\sqrt{\omega} & (1+i)\sqrt{\omega} & (1-i)\sqrt{\omega} \end{bmatrix}.$$

The general solutions of the two problems (29) for $k = \pm 1$ appear, with $c_i, d_i \in \mathbb{R}$,

$i = 1, \ldots, 4$, therefore as follows

$$
\begin{aligned}
\begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_{k=+1} (x,\omega) &= c_1 \mathbf{x}^1 e^{\lambda_1 x} + c_2 \mathbf{x}^2 e^{\lambda_2 x} + c_3 \mathbf{x}^3 e^{\lambda_3 x} + c_4 \mathbf{x}^4 e^{\lambda_4 x} =: \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_{+}, \\
\begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_{k=-1} (x,\omega) &= d_1 \underline{\mathbf{x}}^1 e^{\lambda_1 x} + d_2 \underline{\mathbf{x}}^2 e^{\lambda_2 x} + d_3 \underline{\mathbf{x}}^3 e^{\lambda_3 x} + d_4 \underline{\mathbf{x}}^4 e^{\lambda_4 x} =: \begin{bmatrix} \underline{\hat{u}}_c \\ \underline{\hat{v}}_c \\ \underline{\hat{u}}_s \\ \underline{\hat{v}}_s \end{bmatrix}_{-}.
\end{aligned}
\tag{30}
$$

Now we give real fundamental systems for the solutions of (30), see [15],

$$
\begin{aligned}
\begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_{+} &= c_1 e^{\sqrt{\omega}x} \begin{bmatrix} 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \\ -2\sin\sqrt{\omega}x \\ -\sqrt{\omega}[\cos\sqrt{\omega}x + \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + c_2 e^{\sqrt{\omega}x} \begin{bmatrix} 2\sin\sqrt{\omega}x \\ \sqrt{\omega}[\sin\sqrt{\omega}x + \cos\sqrt{\omega}x] \\ 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + c_3 e^{-\sqrt{\omega}x} \begin{bmatrix} 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[-\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \\ 2\sin\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + c_4 e^{-\sqrt{\omega}x} \begin{bmatrix} 2\sin\sqrt{\omega}x \\ \sqrt{\omega}[-\sin\sqrt{\omega}x + \cos\sqrt{\omega}x] \\ -2\cos\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x + \sin\sqrt{\omega}x] \end{bmatrix},
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\begin{bmatrix} \underline{\hat{u}}_c \\ \underline{\hat{v}}_c \\ \underline{\hat{u}}_s \\ \underline{\hat{v}}_s \end{bmatrix}_{-} &= d_1 e^{\sqrt{\omega}x} \begin{bmatrix} 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[-\cos\sqrt{\omega}x + \sin\sqrt{\omega}x] \\ +2\sin\sqrt{\omega}x \\ \sqrt{\omega}[-\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + d_2 e^{\sqrt{\omega}x} \begin{bmatrix} 2\sin\sqrt{\omega}x \\ \sqrt{\omega}[-\sin\sqrt{\omega}x - \cos\sqrt{\omega}x] \\ -2\cos\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + d_3 e^{-\sqrt{\omega}x} \begin{bmatrix} 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[+\cos\sqrt{\omega}x + \sin\sqrt{\omega}x] \\ -2\sin\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x - \sin\sqrt{\omega}x] \end{bmatrix} \\
&\quad + d_4 e^{-\sqrt{\omega}x} \begin{bmatrix} 2\sin\sqrt{\omega}x \\ \sqrt{\omega}[+\sin\sqrt{\omega}x - \cos\sqrt{\omega}x] \\ 2\cos\sqrt{\omega}x \\ \sqrt{\omega}[\cos\sqrt{\omega}x + \sin\sqrt{\omega}x] \end{bmatrix},
\end{aligned}
\tag{32}
$$

with $c_i, d_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

We need to fulfil the boundary conditions in the case $\mathbf{P^{11}}$ for the functions $\hat{u}_c$, $\underline{\hat{u}}_c$, $\hat{u}_s$, $\underline{\hat{u}}_s$ for our problems (29), $k = \pm 1$. We have linear algebraic systems for c_i, d_i, $i = 1, \ldots, 4$, from the real fundamental systems (31) and (32) and from the specified boundary conditions (27).

$$\begin{aligned}
\hat{u}_c(0,\omega) &= 2c_1 + 2c_3 = \hat{u}_c^0(\omega),\\
\hat{u}_c(l,\omega) &= 2[\cos\sqrt{\omega}l e^{\sqrt{\omega}l}c_1 + \sin\sqrt{\omega}l e^{\sqrt{\omega}l}c_2 + \cos\sqrt{\omega}l e^{-\sqrt{\omega}l}c_3 + \sin\sqrt{\omega}l e^{-\sqrt{\omega}l}c_4]\\
&= \hat{u}_c^l(\omega),\\
\hat{u}_s(0,\omega) &= 2c_2 - 2c_4 = \hat{u}_s^0(\omega),\\
\hat{u}_s(l,\omega) &= 2[-\sin\sqrt{\omega}l e^{\sqrt{\omega}l}c_1 + \cos\sqrt{\omega}l e^{\sqrt{\omega}l}c_2\\
&\quad + \sin\sqrt{\omega}l e^{-\sqrt{\omega}l}c_3 - \cos\sqrt{\omega}l e^{-\sqrt{\omega}l}c_4]\\
&= \hat{u}_s^l(\omega),\\
\underline{\hat{u}}_c(0,\omega) &= 2d_1 + 2d_3 = \hat{u}_c^0(\omega),\\
\underline{\hat{u}}_c(l,\omega) &= 2[\cos\sqrt{\omega}l e^{\sqrt{\omega}l}d_1 + \sin\sqrt{\omega}l e^{\sqrt{\omega}l}d_2 + \cos\sqrt{\omega}l e^{-\sqrt{\omega}l}d_3 + \sin\sqrt{\omega}l e^{-\sqrt{\omega}l}d_4]\\
&= \hat{u}_c^l(\omega),\\
\underline{\hat{u}}_s(0,\omega) &= -2d_2 + 2d_4 = \hat{u}_s^0(\omega),\\
\underline{\hat{u}}_s(l,\omega) &= 2[\sin\sqrt{\omega}l e^{\sqrt{\omega}l}d_1 - \cos\sqrt{\omega}l e^{\sqrt{\omega}l}d_2 - \sin\sqrt{\omega}l e^{-\sqrt{\omega}l}d_3 + \cos\sqrt{\omega}l e^{-\sqrt{\omega}l}d_4]\\
&= \hat{u}_s^l(\omega).
\end{aligned}$$

We finally get (the other components are not taken into account at this point) with $\Delta(l,\omega) := \cosh 2\sqrt{\omega}l - \cos 2\sqrt{\omega}l$,

$$\begin{aligned}
&\Delta(l,\omega)\cdot\hat{u}_c(x,\omega)\\
&\quad = \hat{u}_c^0(\omega)[\cosh\sqrt{\omega}(2l-x)\cdot\cos\sqrt{\omega}x - \cosh\sqrt{\omega}x\cdot\cos\sqrt{\omega}(2l-x)]\\
&\qquad + \hat{u}_c^l(\omega)[\cosh\sqrt{\omega}(l+x)\cdot\cos\sqrt{\omega}(l-x)\\
&\qquad\qquad - \cosh\sqrt{\omega}(l-x)\cdot\cos\sqrt{\omega}(l+x)]\\
&\qquad + \hat{u}_s^0(\omega)[\sinh\sqrt{\omega}x\cdot\sin\sqrt{\omega}(2l-x) - \sinh\sqrt{\omega}(2l-x)\cdot\sin\sqrt{\omega}x]\\
&\qquad + \hat{u}_s^l(\omega)[\sinh\sqrt{\omega}(l-x)\cdot\sin\sqrt{\omega}(l+x)\\
&\qquad\qquad - \sinh\sqrt{\omega}(l+x)\cdot\sin\sqrt{\omega}(l-x)],
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
&\Delta(l,\omega)\cdot\underline{\hat{u}}_c(x,\omega)\\
&\quad = \hat{u}_c^0(\omega)[\cosh\sqrt{\omega}(2l-x)\cdot\cos\sqrt{\omega}x - \cosh\sqrt{\omega}x\cdot\cos\sqrt{\omega}(2l-x)]\\
&\qquad + \hat{u}_c^l(\omega)[\cosh\sqrt{\omega}(l+x)\cdot\cos\sqrt{\omega}(l-x)\\
&\qquad\qquad - \cosh\sqrt{\omega}(l-x)\cdot\cos\sqrt{\omega}(l+x)]\\
&\qquad + \hat{u}_s^0(\omega)[\sinh\sqrt{\omega}(2l-x)\cdot\sin\sqrt{\omega}x - \sinh\sqrt{\omega}x\cdot\sin\sqrt{\omega}(2l-x)]\\
&\qquad + \hat{u}_s^l(\omega)[-\sinh\sqrt{\omega}(l-x)\cdot\sin\sqrt{\omega}(l+x)\\
&\qquad\qquad + \sinh\sqrt{\omega}(l+x)\cdot\sin\sqrt{\omega}(l-x)].
\end{aligned}\tag{34}$$

Thus, the problems $\mathbf{\Pi^{11}}$, $k = \pm 1$, in (29) are solved in the Fourier image range, with the restriction to the homogeneous case and to the component $\hat{u}_c$ resp. $\underline{\hat{u}}_c$.

There remains the inverse transformation according (26), i.e.

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \hat{u}_c(x,\omega)\cos\omega t\,d\omega, \qquad \underline{u}(x,t) = \frac{2}{\pi}\int_0^\infty \hat{\underline{u}}_c(x,\omega)\cos\omega t\,d\omega.$$

Unfortunately, the tables for the inverse Fourier transform of the quotients of hyperbolic and trigonometric functions are not so well developed that one could write down the solutions in the Fourier (x,t)-domain immediately. In [8] methods are shown in a broader context (Cauchy's residue theorem) in order to accomplish this back-transformations in special cases.

A further simplification of the homogeneous problem that belongs to (29):

We consider the limiting case $l \to \infty$, that is, we go over from the strip with width l into the quarter plane. With (33), (34) we have

$$\hat{u}_c^\infty(x,\omega) := \lim_{l\to\infty}\hat{u}_c(x,\omega,l) = \hat{u}_c^0(\omega)e^{-\sqrt{\omega}x}\cos\sqrt{\omega}x - \hat{u}_s^0(\omega)e^{-\sqrt{\omega}x}\sin\sqrt{\omega}x, \quad (35)$$

$$\hat{\underline{u}}_c^\infty(x,\omega) := \lim_{l\to\infty}\hat{\underline{u}}_c(x,\omega,l) = \hat{u}_c^0(\omega)e^{-\sqrt{\omega}x}\cos\sqrt{\omega}x + \hat{u}_s^0(\omega)e^{-\sqrt{\omega}x}\sin\sqrt{\omega}x. \quad (36)$$

We will now try to specify the Fourier originals to $\hat{u}_c^\infty(x,\omega)$ and $\hat{\underline{u}}_c^\infty(x,\omega)$. For this purpose, the transformation $\mathcal{F}_c^{-1}$ on both sides of (35) and (36) is applied. It is known (see [19]) that

$$g(x,t) := \frac{x}{\sqrt{2\pi}}\cdot\frac{e^{-\frac{x^2}{2t}}}{t^{\frac{3}{2}}} \circ\!\xleftrightarrow{\mathcal{F}_c}\!\bullet\, e^{-\sqrt{\omega}x}\cos\sqrt{\omega}x =: g_c(x,\omega),$$

$$g(x,t) \;=\; \frac{x}{\sqrt{2\pi}}\cdot\frac{e^{-\frac{x^2}{2t}}}{t^{\frac{3}{2}}} \circ\!\xleftrightarrow{\mathcal{F}_s}\!\bullet\, e^{-\sqrt{\omega}x}\sin\sqrt{\omega}x =: g_s(x,\omega).$$

For the solution in the quarter plane ($k = 1$), we then obtaining by (35),

$$\mathcal{F}_c^{-1}[\hat{u}_c^\infty(x,\omega)] = u^\infty(x,t) = \mathcal{F}_c^{-1}[\hat{u}_c^0(\omega)g_c(x,\omega)] - \mathcal{F}_c^{-1}[\hat{u}_s^0(\omega)g_s(x,\omega)].$$

We apply on the right-hand side of this representation the relevant competent convolution theorem for the inverse transformation of a product [4], [5], [9], [21], [24] and obtain

$$u^\infty(x,t) = 1\cdot\int_0^t u_0(\tau)\cdot g(t-\tau)\,d\tau = \frac{x}{\sqrt{2\pi}}\int_0^t u_0(\tau)\frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\,d\tau,$$

and this is the known solution of (28) for the quarter plan in the homogeneous case with $u(x,0) = f(x) = 0$, $\kappa = \frac{1}{2}$, see [20].

One would expect with (36), that one can find analogously the solution in the (x,t)-domain for the quarter plane (now $k = -1$, see also [10] as a singular hint):

$$\mathcal{F}_c^{-1}[\hat{\underline{u}}_c^\infty(x,\omega)] = \underline{u}^\infty(x,t) = \mathcal{F}_c^{-1}[\hat{u}_c^0(\omega)g_c(x,\omega)] + \mathcal{F}_c^{-1}[\hat{u}_s^0(\omega)g_s(x,\omega)].$$

This is so far not succeeded, but the following term occurs

$$w(x,t) = \frac{x}{\sqrt{2\pi}}\int_t^\infty u_0(\tau)\frac{e^{-\frac{x^2}{2(\tau-t)}}}{(\tau-t)^{\frac{3}{2}}}\,d\tau,$$

which satisfies the homogeneous diffusion equation (28) with $\kappa = -\frac{1}{2}$.

3.2. The canonic-elliptic system: Treatment with the Fourier transform

We start from (CES) in a local coordinate system (x, y) and with our standard designations, note Remark 2.1, and investigate analogous to problem (22) the following initial value problem $\mathbf{P^{11}}$ for the elliptic case in the strip $S = (0, l) \times (0, \infty)$:

$$\begin{aligned} v_y + u_x &= q_{11}u + q_{12}v + f_1(x, y), \qquad u, v \in \mathbf{C^1}[(0, \infty) \times (0, \infty)], \\ u_y - v_x &= q_{21}\, u + q_{22}v + f_2(x, y), \qquad f_i \in \mathbf{C}[(0, \infty) \times (0, \infty)], \\ \lim_{x\to+0} u(x, y) &= u^0(y), \quad \lim_{x\to l-0} u(x, y) = u^l(y), \qquad u^0, u^l \in \mathbf{C}[(0, \infty)], \\ \lim_{y\to+0} u(x, y) &= f_0(x), \quad \lim_{y\to+0} v(x, y) = g_0(x), \qquad f_0(x), g_0(x) \in \mathbf{C}[(0, l)], \\ & \qquad f_0, g_0, u^0, u^l \text{ given}, \ q_{ij} \in \mathbb{R}, \ i, j \in \{1, 2\}. \end{aligned} \tag{37}$$

We apply the transformation $\mathcal{F}_c$ and the transformation $\mathcal{F}_s$ line by line and completely to the system (37) and obtain, similarly to the procedure in the parabolic case in the Fourier image range, the system of ordinary differential equations

$$\begin{gathered} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_x (x, \omega) = \begin{bmatrix} q_{11} & q_{12} & 0 & -\omega \\ -q_{21} & -q_{22} & \omega & 0 \\ 0 & \omega & q_{11} & q_{12} \\ -\omega & 0 & -q_{21} & -q_{22} \end{bmatrix} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix} + \begin{bmatrix} \hat{f}_{1,c} + g_0(x) \\ -\hat{f}_{2,c} - f_0(x) \\ \hat{f}_{1,s} \\ -\hat{f}_{2,s} \end{bmatrix}, \\ \hat{u}_c(0, \omega) = \hat{u}_c^0(\omega), \quad \hat{u}_c(l, \omega) = \hat{u}_c^l(\omega), \quad \hat{u}_s(0, \omega) = \hat{u}_s^0(\omega), \quad \hat{u}_s(l, \omega) = \hat{u}_s^l(\omega), \\ \text{(a parameter-dependent boundary value problem).} \end{gathered} \tag{38}$$

Equations (38) are the problem $\mathbf{\Pi^{11}}$ in the Fourier image range, which is assigned to problem $\mathbf{P^{11}}$.

Gradually we will take back the difficulty level of the problem (37), so we can keep our approach clearly and obtain simpler results.

We put in (37) $q_{12} = q_{21} = 0$, $q_{11} = q_{22} = q$, $f_1(x, y) = f(x, y)$, $f_2(x, y) \equiv 0$, and so we consider the initial value problem $\mathbf{P^{11}}$ for the following system

$$\begin{aligned} v_y + u_x &= q \cdot u + f(x, y), \\ u_y - v_x &= q \cdot v. \end{aligned} \tag{39}$$

We transfer the system (39) into the Helmholtz equation, what is easily achieved by additional differentiations in (39),

$$u_{xx} + u_{yy} = q^2 u + qf + f_x.$$

Conversely, is given a differential equation

$$u_{xx} + u_{yy} = q^2 u + h(x, y),$$

we can determine f from $f_x = -qf + h(x, y)$ and work with (39). We proceed as already described in the treatment of (29). From the simplified system (38) we get

$$\begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix}_x = \begin{bmatrix} q & 0 & 0 & -\omega \\ 0 & -q & \omega & 0 \\ 0 & \omega & q & 0 \\ -\omega & 0 & 0 & -q \end{bmatrix} \begin{bmatrix} \hat{u}_c \\ \hat{v}_c \\ \hat{u}_s \\ \hat{v}_s \end{bmatrix} + \begin{bmatrix} \hat{f}_c + g_0(x) \\ -f_0(x) \\ \hat{f}_s \\ 0 \end{bmatrix}, \tag{61}$$
$$\hat{u}_c(0,\omega) = \hat{u}_c^0(\omega), \quad \hat{u}_c(l,\omega) = \hat{u}_c^l(\omega), \quad \hat{u}_s(0,\omega) = \hat{u}_s^0(\omega), \quad \hat{u}_s(l,\omega) = \hat{u}_s^l(\omega).$$

From the system (61) we only solve the homogeneous problem, that is $f_0 = g_0 = 0$, and we only give the components $\hat{u}_s(x,\omega)$ and $\hat{v}_c(x,\omega)$. With $\delta := \sqrt{q^2+\omega^2}$ and $\Delta_q := \cosh 2\sqrt{q^2+\omega^2}l - 1$ we have

$$\begin{aligned} \hat{u}_s(x,\omega) &= \frac{\hat{u}_s^0}{\Delta_q}\{-\cosh\delta x + \cosh\delta(2l-x)\} + \frac{\hat{u}_s^l}{\Delta_q}\{\cosh\delta(l+x) - \cosh\delta(l-x)\}, \\ \hat{v}_c(x,\omega) &= \frac{\hat{u}_s^0}{\Delta_q}\Big\{\frac{q}{\omega}\cosh\delta x - \frac{\delta}{\omega}\sinh\delta x - \frac{q}{\omega}\cosh\delta(2l-x) - \frac{\delta}{\omega}\sinh\delta(2l-x)\Big\} \\ &\quad + \frac{\hat{u}_s^l}{\Delta_q}\Big\{\frac{\delta}{\omega}\sinh\delta(l+x) + \frac{\delta}{\omega}\sinh\delta(l-x) \\ &\qquad\qquad - \frac{q}{\omega}\cosh\delta(l+x) + \frac{q}{\omega}\cosh\delta(l-x)\Big\}. \end{aligned}$$

And again we are missing powerful tables to the Fourier transform.

In the case $q = 0$ (Laplace equation) we obtain

$$\begin{aligned} \hat{u}_s(x,\omega) &= \frac{\hat{u}_s^0}{\Delta_0}\{-\cosh\omega x + \cosh\omega(2l-x)\} \\ &\quad + \frac{\hat{u}_s^l}{\Delta_0}\{\cosh\omega(l+x) - \cosh\omega(l-x)\}, \\ \hat{v}_c(x,\omega) &= \frac{\hat{u}_s^0}{\Delta_0}\{-\sinh\omega x - \sinh\omega(2l-x)\} \\ &\quad + \frac{\hat{u}_s^l}{\Delta_0}\{\sinh\omega(l+x) + \sinh\omega(l-x)\}. \end{aligned} \tag{40}$$

For the Laplace equation, we restrict the problem even more by $u^l(y) = 0$ in (61), so we have from (40) by means of $\mathcal{F}_s^{-1}$ as a representation of the solution, see [8],

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\hat{u}(\omega)_s^0}{\Delta_0} \cdot [\cosh\omega(2l-x) - \cosh\omega x]\sin\omega y\, d\omega. \tag{41}$$

From the representation (41) is easily shown that $u(x,y)$ satisfies the Laplace equation and the required initial conditions. If we go over from the strip with width l into the quarter plane, $(l \to \infty)$, we obtain

$$u^\infty(x,y) = \frac{2}{\pi}\int_0^\infty \hat{u}_s^0(\omega) \cdot e^{-\omega x} \cdot \sin\omega y\, d\omega.$$

If we note (26) and

$$\hat{u}_s^0(\omega) = \int_0^\infty u^0(\eta)\sin\omega\eta\,d\eta \qquad \text{and} \qquad \frac{2}{\pi}\frac{x}{y^2+x^2} \circ \xleftrightarrow{\mathcal{F}_c} \bullet\; e^{-\omega x},$$

so comes as the solution for the quarter plane, in accordance with [5],

$$u^\infty(x,y) = \frac{1}{\pi}\int_0^\infty u_0(\eta)\Big[\frac{x}{(y-\eta)^2+x^2} - \frac{x}{(y+\eta)^2+x^2}\Big]\,d\eta.$$

3.3. The canonic-hyperbolic system: Treatment with the Laplace transform

We start from (CHS) and use now because of the close relationship to the wave equation, the time coordinate t instead of y, we again use the matrix Q for our linear problem and formulate the following initial value problem in a rectangle $R = (0, x_e) \times (0, t_e)$,

$$\begin{aligned} &u_x = q_{11}u(x,t) + q_{12}v(x,t) + f_1(x,t), && \\ &v_t = q_{21}u(x,t) + q_{22}v(x,t) + f_2(x,t), && (x,t)\in R,\ f_1, f_2 \in C[\overline{\mathbb{R}_2^+}], \\ &u(0,t) = u_0(t), && t\in[0,t_e],\ t_e>0,\ u_0\in C[\overline{\mathbb{R}_1^+}], \\ &v(x,0) = v_0(x), && x\in[0,x_e],\ x_e>0,\ v_0\in C[\overline{\mathbb{R}_1^+}]. \end{aligned} \tag{42}$$

This problem we will treat with the one-dimensional Laplace transform. Let

$$L[u(x,t);x](s) = \int_0^\infty e^{-s\xi}u(\xi,t)\,d\xi = \omega_1(s,t), \quad (\text{abrrev. } u(x,t)\circ\!\rightarrow\!\bullet\,\omega_1(s,t)),$$

be the Laplace transform with respect to x. We have then for the other quantities in (42),

$$\begin{aligned} &v(x,t)\circ\!\rightarrow\!\bullet\,\omega_2(s,t), \qquad v_t(x,t)\circ\!\rightarrow\!\bullet\,\omega_{2,t}(s,t), \\ &u_x(x,t)\circ\!\rightarrow\!\bullet\, s\cdot\omega_1(s,t) - u_0(t), \qquad f_i(x,t)\circ\!\rightarrow\!\bullet\,\varphi_i(s,t), \quad i=1,2. \end{aligned}$$

So we get in the image range

$$\begin{aligned} s\cdot\omega_1(s,t) - u_0(t) &= q_{11}\omega_1 + q_{12}\omega_2 + \varphi_1(s,t), \\ \omega_{2,t}(s,t) &= q_{21}\omega_1 + q_{22}\omega_2 + \varphi_2(s,t), \\ \omega_2(s,0) &= \omega_{2,0}(s) := L[v_0(x);x](s). \end{aligned} \tag{43}$$

We eliminate in (43) $\omega_1(s,t)$ and obtain a parameter-dependent first order differential equation in the variable t for $\omega_2(s,t)$ together with the initial value

$$\begin{aligned} &\omega_{2,t}(s,t) = \frac{q_{22}s - |Q|}{s-q_{11}}\omega_2(s,t) + \frac{q_{21}}{s-q_{11}}[\varphi_1(s,t) + u_0(t)] + \varphi_2(s,t), \\ &\omega_2(s,0) = \omega_{2,0}(s). \end{aligned}$$

We solve this initial value problem for an inhomogeneous linear ordinary differential equation of first order with respect to the identity

$$\frac{q_{22}s-|Q|}{s-q_{11}}=\frac{q_{12}q_{21}}{s-q_{11}}+q_{22},\qquad q_{12}q_{21}=:q\neq 0,$$

and so we have

$$\begin{aligned}\omega_2(s,t)\\ &=e^{q_{22}t}\cdot\big[e^{(\frac{qt}{s-q_{11}})}-1\big]\omega_{2,0}(s)+e^{q_{22}t}\omega_{2,0}(s)\\ &\quad+q_{21}\int_0^t e^{q_{22}(t-\tau)}[\varphi_1(s,\tau)+u_0(\tau)+(s-q_{11})\varphi_2(s,\tau)]\frac{e^{(\frac{q(t-\tau)}{s-q_{11}})}}{s-q_{11}}\,d\tau.\end{aligned}\tag{44}$$

This is the solution in the image range. She is now transformed back into the original Laplace domain. For this we need the Bessel functions $J_\nu(x)$, the modified Bessel functions $I_\nu(x)$, the entire transcendental function ${}_0F_1(c;x)$, see [17], we need them on this place only for real arguments, $x\in\mathbb{R}$, $c>0$,

$${}_0F_1(c;x)=1+\frac{x}{1!c}+\frac{x^2}{2!c(c+1)}+\ldots=\sum_{l=0}^{\infty}\frac{\Gamma(c)x^l}{l!\Gamma(c+l)},\qquad\Gamma(l+1)=l!.$$

We use the following correspondences [18], [22]

$$\begin{aligned}\frac{1}{s}e^{\frac{a}{s}}\bullet\!\to\circ\, I_0(2\sqrt{ax}),\qquad a>0,\\ \frac{1}{s}e^{\frac{-a}{s}}\bullet\!\to\circ\, J_0(2\sqrt{ax}),\qquad a>0,\\ e^{\frac{a}{s}}-1\bullet\!\to\circ\sqrt{\frac{a}{x}}I_1(2\sqrt{ax}),\qquad a>0,\\ e^{\frac{-a}{s}}-1\bullet\!\to\circ-\sqrt{\frac{a}{x}}J_1(2\sqrt{ax}),\qquad a>0\end{aligned}\tag{45}$$

and we will continue to work with the hypergeometric function ${}_0F_1$, it seems our problem to be more appropriate because of the simpler arguments

$$\begin{aligned}I_\nu(z)&=\frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu{}_0F_1\Big(\nu+1;\frac{z^2}{4}\Big),\\ J_\nu(z)&=\frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu{}_0F_1\Big(\nu+1;-\frac{z^2}{4}\Big),\\ I_0(2\sqrt{qtx})&={}_0F_1(1;qt,x),\\ \sqrt{\frac{|q|t}{x}}I_1(2\sqrt{qtx})&=qt\cdot{}_0F_1(2;qtx),\\ J_0(2\sqrt{|q|tx})&={}_0F_1(1;-|q|tx),\\ \sqrt{\frac{|q|t}{x}}J_1(2\sqrt{|q|tx})&=|q|t\cdot{}_0F_1(2;-|q|tx).\end{aligned}\tag{46}$$

In (44) there are five additively related items. These are now individually inverse transformed according to the rules of the Laplace transform (concerning damping, folding) in accordance with the formula groups (45), (46). This gives

$$\begin{aligned}
&v(x,t)\\
&= e^{q_{22}t}v_0(x) + \int_0^t e^{q_{22}(t-\tau)} f_2(x,\tau)\,d\tau\\
&\quad + q_{21}\int_0^t e_0^{q_{11}x+q_{22}(t-\tau)} F_1(1; q_{12}q_{21}x(t-\tau))u_0(\tau)\,d\tau\\
&\quad + q_{12}q_{21}\cdot t\cdot \int_0^x e_0^{q_{11}(x-\xi)+q_{22}t} F_1(2; q_{12}q_{21}(x-\xi)t)v_0(\xi)\,d\xi\\
&\quad + q_{21}\int_0^t\int_0^x e_0^{q_{11}(x-\xi)+q_{22}(t-\tau)} F_1(1; q_{12}q_{21}(x-\xi)(t-\tau))f_1(\xi,\tau)\,d\xi d\tau\\
&\quad + q_{12}q_{21}\int_0^t\int_0^x (t-\tau)e_0^{q_{11}(x-\xi)+q_{22}(t-\tau)} F_1(2; q_{12}q_{21}(x-\xi)(t-\tau))f_2(\xi,\tau)\,d\xi d\tau.
\end{aligned}$$

Analogously, we obtain $u(x,t)$, if we apply the Laplace transform with respect to the variable t and then eliminate the function $\omega_2(x,s)$ (it remains an ordinary differential equation for $\omega_1(x,s)$). After the back transformation, we obtain

$$\begin{aligned}
&u(x,t)\\
&= e^{q_{11}x}u_0(t) + \int_0^x e^{q_{11}(x-\xi)} f_1(\xi,t)\,d\xi\\
&\quad + q_{12}\int_0^x e_0^{q_{11}(x-\xi)+q_{22}t} F_1(1; q_{12}q_{21}(x-\xi)t)v_0(\xi)\,d\xi\\
&\quad + q_{12}q_{21}\cdot x\cdot \int_0^t e_0^{q_{11}x+q_{22}(t-\tau)} F_1(2; q_{12}q_{21}x(t-\tau))u_0(\tau)\,d\tau\\
&\quad + q_{12}\int_0^t\int_0^x e_0^{q_{11}(x-\xi)+q_{22}(t-\tau)} F_1(1; q_{12}q_{21}(x-\xi)(t-\tau))f_1(\xi,\tau)\,d\xi d\tau\\
&\quad + q_{12}q_{21}\int_0^t\int_0^x (x-\xi)e_0^{q_{11}(x-\xi)+q_{22}(t-\tau)} F_1(2; q_{12}q_{21}(x-\xi)(t-\tau))f_2(\xi,\tau)\,d\xi d\tau.
\end{aligned}$$

Hereby the known solution pair $(u,v)^T$ [25], [6], [2] is again derived in a different way.

By inserting the solution pair $(u,v)^T$ into the differential equation (42) one shows that $(u,v)^T$ solves the initial value problem (42) and $(u,v)^T$ is thus not only a formal solution. Numerical procedures of Runge-Kutta type for the treatment of problem (42) with general nonlinear right-hand sides and vector valued functions u and v can be found in [3].

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