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Characterization of a b-metric space completeness via the existence of a fixed point of Ciric-Suzuki type quasi-contractive multivalued operators and applications

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Abstract

The aim of this paper is to introduce Ciric-Suzuki type quasi-contractive multivalued operators and to obtain the existence of fixed points of such mappings in the framework of b-metric spaces. Some examples are presented to support the results proved herein. We establish a characterization of strong b-metric and b-metric spaces completeness. An asymptotic estimate of a Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators is obtained. As an application of our results, existence and uniqueness of multivalued fractals in the framework of b-metric spaces is proved.

1 Introduction and preliminaries

Let (X, d) be a metric space. Let $CB(X)$ $(P(X))$ be the family of nonempty closed and bounded (nonempty subsets of X). For $A, B \in CB(X)$, let

$$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}$$

where $d(x,B) = \inf_{w\in B} d(x,w)$ and $\delta(A,B) = \sup\limits_{x\in A} d(x,B)$. The mapping H is said to be a Hausdorff metric on $CB(X)$ induced by d. The metric space

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$(CB(X), H)$ is complete if (X, d) is complete. For $f : X \to X$ and $T : X \to P(X)$, the pair (f, T) is called a hybrid pair of mappings. The fixed point problem of T is to find an $x \in X$ such that $x \in Tx$ (fixed point inclusion). The solution of a fixed point inclusion problem of T is called a fixed point of T. The set $F(T)$ denotes the set of fixed points of T. A point $x \in X$ is a coincidence point (common fixed point) of (f, T) if $fx \in Tx$ ($x = fx \in Tx$). Denote $C(f, T)$ and $F(f, T)$ by the set of coincidence and common fixed point of (f, T), respectively. The hybrid pair (f, T) is w-compatible ([1]) if $f(Tx) \subseteq T(fx)$ for all $x \in C(f, T)$. A mapping f is T-weakly commuting at $x \in X$ if $f^2(x) \in T(fx)$. The letters $\mathbb{R}^+$ and $\mathbb{N}^*$ will denote the set of nonnegative real numbers and the set of nonnegative integers, respectively.

A mapping $T : X \to CB(X)$ is called a *multivalued weakly Picard* operator (A MWP operator) ([34]), if for all $x \in X$ and for some $y \in Tx$, there exists a sequence $\{x_n\}$ satisfying (a_1) $x_0 = x$, $x_1 = y$, (a_2) $x_{n+1} \in Tx_n$, $n \in \mathbb{N}^*$ (a_3) $\{x_n\}$ converges to some $z \in F(T)$.

The sequence $\{x_n\}$ satisfying (a_1) and (a_2) is called a sequence of successive approximations (ssa at (x, y)) of T starting from (x, y).

If a single valued mapping T satisfies (a_1) to (a_3), then it is a Picard operator.

Let $T : X \longrightarrow P(X)$ be a MWP operator. Define the mapping $T^\infty : G(T) \to P(F(T))$ by

$$T^\infty(x, y) = \{z : \text{ there is an ssa at } (x, y) \text{ of } T \text{ that converging to } z\}$$

where $G(T) = \{(x, y) : x \in X, y \in Tx\}$ is called graph of T. A mapping $f : X \to X$ is called a selection of $T : X \longrightarrow P(X)$ if $C(f, T) = X$.

Definition 1.1. *([34]) Let (X, d) be a metric space and $c > 0$. A MWP operator $T : X \longrightarrow P(X)$ is called $c-$multivalued weakly Picard ($c-$MWP) operator if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \leq cd(x, y)$ for all $(x, y) \in G(T)$.*

One of the main result dealing with $c-$MWP operators is the following.

Theorem 1.2. *([34]) Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$. If T_i is a c_i-MWP operator for each $i \in \{1, 2\}$ and there exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$ for all $x \in X$. Then*

$$H(F(T_1), F(T_2)) \leq \lambda \max\{c_1, c_2\}.$$

Banach contraction principle (BCP) [7] states that if (X, d) is a complete metric space and $f : X \to X$ satisfies

$$d(fx, fy) \leq rd(x, y) \tag{1.1}$$

for all $x, y \in X$ with $r \in (0,1)$, then f has a unique fixed point.

Due to its applications in mathematics and other related disciplines, BCP has been generalized in many directions. Suzuki [39] proposed a contraction condition that does not imply the continuity of a mapping f. Suzuki type fixed point theorems are remarkable in the sense that these results characterize the completeness of underlying metric spaces ([39, Theorem 3]) whereas BCP does not ([15]).

A mapping $f : X \to X$ is called quasi-contraction [12, Theorem 1] if

$$d(fx, fy) \leq r \max \{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} \tag{1.2}$$

for all $x, y \in X$ with $r \in [0,1)$.

Nadler [31] proved a multivalued version of BCP as follows.

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$. If for all $x, y \in X$,*

$$H(Tx, Ty) \leq rd(x,y)$$

holds for some $r \in [0,1)$, then $F(T)$ is nonempty.

Amini-Harandi [2] generalized Theorem 1.3 as follows.

Theorem 1.4. *[2] Let (X, d) be a complete metric space and $T : X \to CB(X)$. If for all $x, y \in X$,*

$$H(Tx, Ty) \leq r \max \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.3}$$

holds for some $r \in \left[0, \frac{1}{2}\right)$. Then $F(T)$ is nonempty.

Define the mapping ξ_1: $[0,1) \to \left(\frac{1}{2}, 1\right]$ by $\xi_1(r) = \dfrac{1}{1+r}$.

Kikkawa and Suzuki [28] obtained an interesting generalization of Theorem 1.3 as follows.

Theorem 1.5. *[28] Let (X, d) be a complete metric space and $T : X \to CB(X)$. If there exists an $r \in [0,1)$ such that*

$$\xi_1(r)d(x,Tx) \leq d(x,y) \text{ implies that } H(Tx,Ty) \leq rd(x,y). \tag{1.4}$$

for all $x, y \in X$. Then $F(T)$ is nonempty.

The mapping satisfying (1.4) is called $r - KS$ multivalued operator.

Using axioms of choice, Haghi et al. [21] proved the following lemma.

Lemma 1.6. *[21] For a nonempty set X and $f : X \to X$, there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one.*

Euclidean distance is an important measure of "nearness" between two real or complex numbers. This notion has been generalized further in one to many directions (see [3]). Among which one of the most important generalization is the concept of a b-metric initiated by Czerwik [17]. The reader interested in fixed point results in setup of b-metric spaces is referred to ([3, 9, 14, 13, 16, 17, 18, 22, 29, 35]).

Definition 1.7. *[16] Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if there exists some real constant $b \geq 1$ such that for any $x, y, z \in X$, the following condition hold:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, y) \leq bd(x, z) + bd(z, y)$.

The pair (X, d) is termed as b-metric space with b-metric constant b. If (b_3) is replaced by

(b_4) $d(x, y) \leq d(x, z) + bd(z, y)$

then (X, d) is called a strong b-metric space (Kirk and Shahzad [26]) with strong b-metric constant $b \geq 1$.

If $b = 1$, then strong b-metric space is a metric space. Every metric is a strong b-metric and every strong b-metric is b-metric but converse does not hold in general ([4, 5, 13, 16, 35]).

Consistent with [16, 17, 18, 35], the following (definitions and lemmas) will be needed in the sequel.

Lemma 1.8. *[16, 17, 18, 35] Let (X, d) be a b-metric space, $x, y \in X$ and $A, B \in CB(X)$. The following statements hold:*

c_1) $(CB(X), H)$ *is a b-metric space.*

c_2) $d(x, B) \leq H(A, B)$ *for all* $x \in A$.

c_3) $d(x, A) \leq bd(x, y) + bd(y, A)$.

c_4) *For $h > 1$ and $z \in A$, there is a $w \in B$ such that $d(z, w) \leq hH(A, B)$.*

c_5) *For every $h > 0$ and $z \in A$, there is a $w \in B$ such that $d(z, w) \leq H(A, B) + h$.*

c_6) $d(w, A) = 0$ *if and only if* $w \in \bar{A} = A$.

c_7) *For* $\{x_n\} \subseteq X$, $d(x_0, x_n) \leq bd(x_0, x_1) + ... + b^{n-1}d(x_{n-2}, x_{n-1}) + b^{n-1}d(x_{n-1}, x_n)$.

Definition 1.9. *Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called:*

c_8) a Cauchy sequence if for any $\epsilon > 0$, there exists $n(\epsilon) \in N$ such that for each $n, m \geq n(\epsilon)$, we have $d(x_n, x_m) < \epsilon$,

c_9) a convergent sequence if there exists $x \in X$ such that for any $\epsilon > 0$, there exists $n(\epsilon) \in N$ with $d(x_n, x) < \epsilon$ for all $n \geq n(\epsilon)$. In this case, we write $\lim_{n\to\infty} x_n = x$.

Lemma 1.10. *[36] If a sequence $\{u_n\}$ in a b-metric space (X, d) satisfies $d(u_{n+1}, u_{n+2}) \leq hd(u_n, u_{n+1})$ for all $n \in \mathbb{N}$ and for some $0 \leq h < 1$, then it is a Cauchy sequence in X provided that $hb < 1$.*

Equivalently, a sequence $\{x_n\}$ in b-metric space X is Cauchy if and only if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p \in N$. A sequence $\{x_n\}$ is convergent to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Lemma 1.11. *Let (X, d) be a b-metric space, $A, B \in P(X)$. If there exists a $\lambda > 0$ such that (i) for each $\tilde{a} \in A$, there exists a $\tilde{b} \in B$ such that $d(\tilde{a}, \tilde{b}) \leq \lambda$, (ii) for each $\tilde{b} \in B$, there exists an $\tilde{a} \in A$ such that $d(\tilde{a}, \tilde{b}) \leq \lambda$, then $H(A, B) \leq \lambda$.*

A subset $Y \subset X$ is closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we must have $x \in Y$. A subset $Y \subset X$ is bounded if $\mathrm{diam}(Y)$ is finite, where $\mathrm{diam}(Y) = \sup\{d(a, b), a, b \in Y\}$. A b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

An et al. [4] studied the topological properties of b-metric spaces. In a b-metric space (X, d), d is not necessarily continuous in each variable. In a b-metric space (X, d), If d is continuous in one variable, then d is continuous in other variable. A ball $B_\varepsilon(x_0) = \{x : d(x, x_0) < \varepsilon\}$ in b-metric space (X, d) is not necessarily an open set. A ball in a b-metric space (X, d) is open if d is continuous in one variable (see [4]).

In what follows we assume that a b-metric d is continuous in one variable.

Aydi et al. [6] proved the following result as a generalization of Theorem 1.4 ([2, Theorem 1.4]).

Theorem 1.12. *[6] Let (X, d) be a complete b-metric space and $T : X \to CB(X)$. If there exists some $r \in [0, 1)$ with $r < \dfrac{1}{b^2 + b}$ such that*

$$H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

holds for all $x, y \in X$, then $F(T)$ is nonempty.

Define the mapping $\xi_2 : [0, 1) \to \left(\dfrac{1}{2}, 1\right]$ by $\xi_2(r) = \dfrac{1}{1 + br}$.

Kutbi et al. [29] obtained the following Suzuki type fixed point theorem result in the setup of b-metric spaces.

Theorem 1.13. *[29] Let (X, d) be a complete b-metric space and $T : X \to CB(X)$. If there exists some $r \in [0, 1)$ with $r < \frac{1}{b^2 + b}$ such that*

$$\xi_2(r)d(x, Tx) \leq bd(x, y) \tag{1.5}$$

implies that

$$H(Tx, Ty) \leq rd(x, y) \tag{1.6}$$

for $x, y \in X$, then $F(T)$ is nonempty.

Let (X, d) be a b-metric space, $f : X \to X$, $T : X \to CB(X)$ and $x, y \in X$. We use the notations

$$\begin{aligned} M_f(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}, \\ M_T(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \\ M_T^f(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx).\} \end{aligned}$$

Define

$$\Lambda = \left\{\xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} : \xi(s,t) \leq \frac{s}{b} - t\right\}$$

where b is the b-metric constant. Note that $\xi(bt, t) \leq 0$ and $\xi\left(s, \frac{s}{b}\right) \leq 0$ for all $s \in \mathbb{R}^+$.

Example 1.14. *For $i \in \{3, 4\}$, define $\xi_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by*

(1) $\xi_3(s, t) = \psi(s) - \varphi(t)$ *, where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are functions satisfying $\psi(t) \leq \frac{t}{b}$, $t \leq \varphi(t)$, and $b \geq 1$.*

(2) $\xi_4(s, t) = \frac{s}{b} - \frac{\psi(s,t)}{\varphi(s,t)}t$ *, where $\psi, \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are functions satisfying $\varphi(s, t) \leq \psi(s, t)$ for all $s, t \geq 0$.*

Definition 1.15. *Let (X, d) be a b-metric space. A mapping $T : X \to CB(X)$ is called a Ciric-Suzuki type quasi-contractive multivalued operator if there exists an $r \in [0, 1)$ satisfying $r < \frac{1}{b^2 + b}$ such that*

$$\xi(d(x, Tx), d(x, y)) \leq 0 \tag{1.7}$$

implies that

$$H(Tx, Ty) \leq rM_T(x, y) \tag{1.8}$$

for all $x, y \in X$, where $\xi \in \Lambda$.

If $CB(X) = \{\{x\} : x \in X\}$, then $T : X \to CB(X)$ is called a Ciric-Suzuki type quasi-contractive operator.

Definition 1.16. *Let (X, d) be a b-metric space, $f : X \to X$ and $T : X \to CB(X)$. A hybrid pair (f, T) is said to be Ciric-Suzuki type quasi-contractive hybrid pair if there exists an $r \in [0, 1)$ satisfying $r < \dfrac{1}{b^2 + b}$ such that*

$$\xi(d(fx, Tx), d(fx, fy)) \leq 0 \tag{1.9}$$

implies that

$$H(Tx, Ty) \leq rM_T^f(x, y) \tag{1.10}$$

for all $x, y \in X$ and for some $\xi \in \Lambda$.

In this paper, we obtain fixed point results for Ciric-Suzuki type quasi-contractive multivalued operators in b-metric space. Further, completeness characterization of strong b-metric and b-metric spaces via the existence of fixed point of Ciric-Suzuki type quasi-contractive operators is obtained. Our results extend, unify and generalize the comparable results in [2, 6, 12, 27, 29, 31, 33, 39]. As applications of our results:

1 We prove the existence of coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive single valued and multivalued operators.

2 We give an estimate of Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators.

3 We show that for a uniformly convergent sequence of Ciric-Suzuki type quasi-contractive multivalued operators, the corresponding sequence of fixed points set is uniformly convergent.

4 We obtain a unique multivalued fractal with respect to iterated multifunction system of Ciric-Suzuki type quasi-contractive multivalued operators.

2 Fixed points of Ciric-Suzuki type quasi-contractive multivalued operators

In this section, we obtain some fixed point results of Ciric-Suzuki type quasi-contractive multivalued operators in the framework of complete b-metric spaces.

We start with the following result.

Theorem 2.1. *Let (X, d) be a complete b-metric space and $T : X \longrightarrow CB(X)$ a Ciric-Suzuki type quasi-contractive multivalued operator. Then T is a MWP operator.*

Proof. Let u and v be given points in X. If $M_T(u,v)=0$, then $u=v\in Tu$. Define a sequence $\{u_n\}$ by $u_n=u=v$, for all $n\in\mathbb{N}^*$. Clearly, $u_n\in Tu_n$ and $\{u_n\}$ converges to $u=v\in F(T)$. Hence T is a MWP operator.

Suppose that $M_T(u,v)>0$ for all $u,v\in X$. As $r<\dfrac{1}{b^2+b}$, there exist $\alpha\in\mathbb{R}^+$ such that $\dfrac{r}{2}+\alpha=\dfrac{1}{2}\left(\dfrac{1}{b^2+b}\right)$. Clearly,

$$0<r+\alpha=\frac{1}{2}\left(\frac{1}{b^2+b}+r\right)=\beta\ (\text{ say})<1\ .$$

Let u_0 be any point in X and $u_1\in Tu_0$. Note that

$$\begin{aligned}\xi\left(d\left(u_0,Tu_0\right),d\left(u_0,u_1\right)\right) &\leq \frac{1}{b}d\left(u_0,Tu_0\right)-d\left(u_0,u_1\right)\\ &\leq d\left(u_0,Tu_0\right)-d\left(u_0,u_1\right)\\ &\leq d\left(u_0,u_1\right)-d\left(u_0,u_1\right)=0.\end{aligned}$$

As T is a Ciric-Suzuki type quasi-contractive multivalued operator, we obtain that

$$H(Tu_0,Tu_1)\leq rM_T(u_0,u_1). \tag{2.1}$$

By Lemma 1.8, there exists an element $u_2\in Tu_1$ such that

$$d(u_1,u_2)\leq H(Tu_0,Tu_1)+\alpha M_T(u_0,u_1). \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned}d(u_1,u_2) &\leq H(Tu_0,Tu_1)+\alpha M_T(u_0,u_1)\\ &\leq rM_T(u_0,u_1)+\alpha M_T(u_0,u_1)\\ &= \beta M_T(u_0,u_1)\\ &= \beta\max\left\{d(u_0,u_1),d(u_0,Tu_0),(u_1,Tu_1),d(u_0,Tu_1),d(u_1,Tu_0)\right\}\\ &\leq \beta\max\left\{d(u_0,u_1),d(u_0,u_1),(u_1,u_2),d(u_0,u_2),d(u_1,u_1)\right\}\\ &\leq \beta\max\left\{d(u_0,u_1),(u_1,u_2),b\left(d(u_0,u_1)+d(u_1,u_2)\right)\right\}\\ &= b\beta\left(d(u_0,u_1)+d(u_1,u_2)\right).\end{aligned}$$

That is

$$d(u_1,u_2)\leq b\beta\left(d(u_0,u_1)+d(u_1,u_2)\right). \tag{2.3}$$

As

$$\begin{aligned}\xi\left(d\left(u_1,Tu_1\right),d\left(u_1,u_2\right)\right) &\leq \frac{1}{b}d\left(u_1,Tu_1\right)-d\left(u_1,u_2\right)\\ &\leq d\left(u_1,Tu_1\right)-d\left(u_1,u_2\right)\\ &\leq d\left(u_1,u_2\right)-d\left(u_1,u_2\right)=0.\end{aligned}$$

We have

$$H(Tu_1, Tu_2) \leq rM_T(u_1, u_2). \tag{2.4}$$

Again by Lemma 1.8, there exists an element $u_3 \in Tu_2$ such that

$$d(u_2, u_3) \leq H(Tu_1, Tu_2) + \alpha M_T(u_1, u_2). \tag{2.5}$$

By (2.4) and (2.5), we obtain that

$$\begin{aligned}
d(u_2, u_3) &\leq H(Tu_1, Tu_2) + \alpha M_T(u_1, u_2) \\
&\leq rM_T(u_1, u_2) + \alpha M_T(u_1, u_2) \\
&= \beta M_T(u_1, u_2) \\
&= \beta \max\{d(u_1, u_2), d(u_1, Tu_1), (u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\} \\
&\leq \beta \max\{d(u_1, u_2), d(u_1, u_2), (u_2, u_3), d(u_1, u_3), d(u_2, u_2)\} \\
&\leq \beta \max\{d(u_1, u_2), (u_2, u_3), b\,(d(u_1, u_2) + d(u_2, u_3))\} \\
&= b\beta\,(d(u_1, u_2) + d(u_2, u_3))\,.
\end{aligned}$$

That is

$$d(u_2, u_3) \leq b\beta\,(d(u_1, u_2) + d(u_2, u_3))\,. \tag{2.6}$$

Continuing this way, we can obtain a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and it satisfies:

$$d(u_n, u_{n+1}) \leq b\beta\,(d(u_{n-1}, u_n) + d(u_n, u_{n+1})) \tag{2.7}$$

$n \in \mathbb{N}^*$. If $\delta_n = d(u_n, u_{n+1})$, then from (2.7), we have $\delta_n \leq \gamma\delta_{n-1}$, where $\gamma = \dfrac{b\beta}{1 - b\beta}$. Now by $b \geq 1$ and $r < \dfrac{1}{b^2 + b}$, we have

$$b\beta = \frac{b}{2}\left(\frac{1}{b^2 + b} + r\right) < \frac{1}{1 + b} \text{ and } \gamma = \frac{b\beta}{1 - b\beta} < \frac{1}{b}.$$

That is $b\gamma < 1$. By Lemma 1.10 , $\{u_n\}$ is a Cauchy sequence and hence

$$\lim_{n\to\infty} d(u_n, z) = 0 \tag{2.8}$$

for some $z \in X$. Now we claim that

$$d(z, Tx) \leq r \max\{d(z, x), d(x, Tx)\} \tag{2.9}$$

for all $x \neq z$. As $\lim_{n\to\infty} d(u_n, z) = 0$, there exists $n_0 \in \mathbb{N}$ such that $d(u_n, z) <$

$\frac{1}{3b}d(z,x)$ for all $n \geq n_0$ and $x \neq z$. Note that

$$\begin{aligned}
\xi\left(d(u_n, Tu_n), d(u_n, x)\right) &\leq \frac{1}{b}d(u_n, Tu_n) - d(u_n, x) \\
&\leq \frac{1}{b}d(u_n, u_{n+1}) - d(u_n, x) \\
&\leq \frac{1}{b}\left(bd(u_n, z) + bd(z, u_{n+1})\right) - d(u_n, x) \\
&\leq \frac{2}{3b}d(z, x) - d(u_n, x) \\
&= \frac{1}{b}\left(d(z, x) - \frac{1}{3}d(z, x)\right) - d(u_n, x) \\
&\leq \frac{1}{b}\left(d(z, x) - bd(u_n, z)\right) - d(u_n, x) \\
&\leq \frac{1}{b}\left(bd(u_n, x)\right) - d(u_n, x) = 0
\end{aligned}$$

for all $n \geq n_0$. That is

$$\xi\left(d(u_n, Tu_n), d(u_n, x)\right) \leq 0 \tag{2.10}$$

for all $n \geq n_0$. Thus

$$\begin{aligned}
d(u_{n+1}, Tx) &\leq H(Tu_n, Tx) \\
&\leq rM_T(u_n, x) \\
&= r\max\left\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\right\} \\
&\leq r\max\left\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\right\}
\end{aligned}$$

for all $n \geq n_0$. Now, by taking limit as $n \to \infty$ on both sides of the above inequality, it follows that

$$d(z, Tx) \leq r\max\left\{d(z, x), d(x, Tx), d(z, Tx)\right\}.$$

If $\max\left\{d(z, x), d(x, Tx), d(z, Tx)\right\} = d(z, Tx)$, then we obtain that

$$d(z, Tx) \leq rd(z, Tx) < \beta d(z, Tx) < d(z, Tx),$$

a contradiction and hence (2.9) holds for all $x \neq z$. Now we show that $z \in Tz$. Assume on contrary that $z \notin Tz$. Clearly, $r < \frac{1}{b^2 + b}$ implies that $2rb < 1$. We now choose $a \in Tz$ such that $a \neq z$ and $d(z, a) < d(z, Tz) + \left(\frac{1}{2rb} - 1\right)d(z, Tz)$. That is

$$2brd(z, a) < d(z, Tz). \tag{2.11}$$

Note that

$$\begin{aligned}\xi\left(d(z,Tz),d(z,a)\right) &\leq \frac{1}{b}d(z,Tz)-d(z,a)\\ &\leq d(z,Tz)-d(z,a)\leq d(z,a)-d(z,a)=0.\end{aligned}$$

Hence

$$\begin{aligned}H(Tz,Ta) &\leq rM_T(z,a)\\ &\leq r\max\left\{d(z,a),d(z,Tz),d(a,Ta),d(z,Ta),d(a,Tz)\right\}\\ &\leq r\max\left\{d(z,a),d(z,a),d(a,Ta),d(z,Ta),d(a,a)\right\}\\ &= r\max\left\{d(z,a),d(a,Ta),d(z,Ta)\right\}.\end{aligned}$$

If $\max\left\{d(z,a),d(a,Ta),d(z,Ta)\right\}=d(a,Ta)$, then we have

$$d(a,Ta)\leq H(Tz,Ta)\leq rd(a,Ta)$$

which implies either $a\in Ta$ or $d(a,Ta)<d(a,Ta)$, a contradiction. Hence

$$H(Tz,Ta)\leq r\max\left\{d(z,a),d(z,Ta)\right\}.$$

If $\max\left\{d(z,a),d(a,Ta),d(z,Ta)\right\}=d(z,Ta)$, then (2.9) gives that

$$\begin{aligned}H(Tz,Ta) &\leq rd(z,Ta)\\ &\leq r^2\max\{d(z,a),d(a,Ta)\}\\ &\leq r\max\{d(z,a),d(a,Ta)\}.\end{aligned}$$

As $\max\{d(z,a),d(a,Ta)\}=d(a,Ta)$, is not possible, we have

$$H(Tz,Ta)\leq rd(z,a). \tag{2.12}$$

From (2.9) and (2.12), we obtain that

$$d(z,Ta)\leq r\max\{d(z,a),d(a,Ta)\}\leq r\max\{d(z,a),H(Tz,Ta)\}\leq rd(z,a). \tag{2.13}$$

Now, by (2.11), (2.12), and (2.13), we have

$$\begin{aligned}d(z,Tz) &\leq bd(z,Ta)+bH(Tz,Ta)\\ &\leq brd(z,a)+brd(z,a)\\ &= 2brd(z,a)<d(z,Tz),\end{aligned}$$

a contradiction. Hence $z\in Tz$. □

Remark 2.2. *We obtain Theorem 1.12 as a special case of Theorem 2.1.*

Remark 2.3. *Theorem 1.13 follows from 2.1. Indeed, define the mapping* ξ *by* $\xi(s,t) = \dfrac{\xi_2(r)}{b}s - t$, *where* $\xi_2(r) = \dfrac{1}{1+br}$. *Clearly,* $\xi(s,t) \leq \dfrac{s}{b} - t$ *as* $\xi_2(r) \leq 1$. *Take* $s = d(x,Tx)$, $t = d(x,y)$ *and*

$$\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} = d(x,y).$$

Corollary 2.4. *Let* (X,d) *be a complete b-metric space and* $T : X \longrightarrow CB(X)$. *If for any* $x, y \in X$, $d(x,Tx) \leq bd(x,y)$ *implies that*

$$H(Tx,Ty) \leq r\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for some $r \in \left[0, \dfrac{1}{b^2+b}\right)$. *Then* T *is a MWP operator.*

Example 2.5. *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *and* $d : X \times X \to \mathbb{R}^+$ *be defined as* $d(x_1,x_2) = d(x_1,x_3) = 3$, $d(x_1,x_4) = d(x_1,x_5) = 12$, $d(x_2,x_5) = d(x_3,x_4) = d(x_3,x_5) = 9$, $d(x_2,x_4) = 8$, $d(x_2,x_3) = 6$, $d(x_4,x_5) = 2$, $d(x,x) = 0$ *and* $d(x,y) = d(y,x)$ *for all* $x,y \in X$. *As* $12 = d(x_1,x_4) \nleq d(x_1,x_2) + d(x_2,x_4) = 11$, d *is not a metric on* X. *On the other hand,* (X,d) *is a complete b-metric space with parameter* $b \geq \dfrac{12}{11} > 1$. *Suppose that* $\xi(s,t) = \dfrac{s}{b} - t \in \Lambda$, $r = \dfrac{2}{5}$. *Then* $r < \dfrac{121}{276} = \dfrac{1}{b^2+b}$. *Define the mapping* $T : X \longrightarrow CB(X)$ *by*

$$Tx = \begin{cases} \{x_1\} \text{ if } x = x_1, x_2, x_3, \\ \{x_2\} \text{ if } x = x_4, \\ \{x_3\} \text{ if } x = x_5. \end{cases}$$

Note that $H(Tx,Ty) = 0 \leq rM_T(x,y)$ *for all* $x,y \in \{x_1,x_2,x_3\}$. *If* $x = x_1$ *and* $y \in \{x_4,x_5\}$, *then* $H(Tx,Ty) = d(x,y) = 3 \leq 4.8 = rd(x,y) \leq rM_T(x,y)$. *If* $x = x_2$ *and* $y = x_4$, *then we have* $H(Tx_2,Tx_4) = d(x_1,x_2) = 3 \leq 3.2 = rd(x_2,x_4) \leq rM_T(x_2,x_4)$. *For,* $x \in \{x_2,x_3\}$ *and* $y \in \{x_4,x_5\}$, *we have* $H(Tx,Ty) = 3 \leq 3.6 = rd(x,y) \leq rM_T(x,y)$. *Note that*

$$\begin{aligned} \xi\left(d(x_4,Tx_4), d(x_4,x_5)\right) &= \frac{11d(x_4,x_2)}{12} - d(x_4,x_5) = \frac{16}{3} > 0, \text{ and} \\ \xi\left(d(x_5,Tx_5), d(x_5,x_4)\right) &= \frac{11d(x_5,x_3)}{12} - d(x_5,x_4) = \frac{25}{4} > 0. \end{aligned}$$

Hence, for all $x,y \in X$, *we have* $\xi\left(d(x,Tx), d(x,y)\right) \leq 0$ *implies that* $H(Tx,Ty) \leq rM_T(x,y)$. *Thus all the conditions of Theorem 2.1 are satisfied. On the other hand, if we take* $x = x_4$, $y = x_5$, *then we have*

$$\begin{aligned} H(Tx_4,Tx_5) &= d(x_2,x_3) = 6 \text{ and} \\ M_T(x_4,x_5) &= \max\{d(x_4,x_5), d(x_4,Tx_4), d(x_5,Tx_5), d(x_4,Tx_5), d(x_5,Tx_4)\} \\ &= \max\{d(x_4,x_5), d(x_4,x_2), d(x_5,x_3), d(x_4,x_3), d(x_5,x_2)\} = 9. \end{aligned}$$

Hence $H(Tx_4, Tx_5) = 6 \nleq 3.6 = 9r = rM_T(x_4, x_5)$ *for any* $r < \dfrac{121}{276} = \dfrac{1}{b^2+b}$. *Thus, Theorem 1.12 is not applicable in this case. Hence Theorem 2.1 is a proper generalization of Theorem 1.12 which in turn generalize Theorems 1.3, 1.4 and [12, Theorem 1].*

Example 2.6. *Let* $X = \{x_1, x_2, x_3\}$ *and* $d : X \times X \to \mathbb{R}^+$ *be defined as* $d(x_1, x_2) = 4$, $d(x_1, x_3) = 1$, $d(x_2, x_3) = 2$, $d(x, x) = 0$ *and* $d(x, y) = d(y, x)$ *for all* $x, y \in X$. *As* $4 = d(x_1, x_2) \nleq d(x_1, x_3) + d(x_3, x_2) = 3$, d *is not a metric on* X. *Indeed* (X, d) *is a b-metric space with* $b \geq \dfrac{4}{3} > 1$. *Define the mapping* $T : X \longrightarrow CB(X)$ *by*

$$Tx = \begin{cases} \{x_1, x_3\} \text{ if } x = x_1, x_3, \\ \{x_1\} \text{ if } x = x_2. \end{cases}$$

Let $\xi(s, t) = \dfrac{s}{b} - t \in \Lambda$ *and* $r = \dfrac{3}{10}$. *Clearly,* $r < \dfrac{9}{28} = \dfrac{1}{b^2+b}$. *If* $x, y \in \{x_1, x_3\}$, *then* $H(Tx, Ty) = 0 \leq rM_T(x, y)$. *If* $x \in \{x_1, x_3\}$ *and* $y = x_2$, *then* $H(Tx, Ty) = 1 \leq 1.2 \leq rM_T(x, y)$. *Hence for any* $x, y \in X$, $\xi(d(x, Tx), d(x, y)) \leq 0$ *implies that* $H(Tx, Ty) \leq rM_T(x, y)$. *Thus, all the conditions of Theorem 2.1 are satisfied. On the other hand, if* $x = x_2$, $y = x_3$, *then* $\xi_2(r)d(x_3, Tx_3) = 0 \leq bd(x_3, x_2) = 2$, *and* $H(Tx_3, Tx_2) = d(x_1, x_3) = 1$. *So,* $H(Tx_3, Tx_2) = 1 \nleq 0.6 = 2r = rd(x_3, x_2)$ *for any* $r < \dfrac{9}{28} = \dfrac{1}{b^2+b}$. *Hence Theorem 1.13 is not applicable in this case. This implies that Theorem 2.1 is a proper generalization of Theorem 1.13 which itself is a generalization of Theorem 1.5, and Theorem 1.3.*

Corollary 2.7. *Let* (X, d) *be a complete b-metric space and* $f : X \longrightarrow X$ *a Ciric-Suzuki type quasi-contractive operator. Then* $F(f) = \{u\}$, *and the sequence* $\{f^n x\}$ *converges to* u *for any choice of an element* $x \in X$.

Proof. It follows from Theorem 2.1 that $F(f)$ is nonempty and for all $x \in X$, the sequence $f^n x \to u$ as $n \to \infty$. To prove the uniqueness of fixed point of f; let $u, v \in F(f)$ with $u \neq v$. Note that $\xi(d(u, fu), d(u, v)) \leq \frac{1}{b}d(u, fu) - d(u, v) = -d(u, v) \leq 0$. Thus, we have

$$\begin{aligned} d(u, v) &= d(fu, fv) \leq rM_f(u, v) \\ &= r\max\{d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, fu)\} \\ &= rd(u, v) < d(u, v), \end{aligned}$$

a contradiction and hence $F(f)$ is singleton. $\square$

Corollary 2.8. *Let (X,d) be a complete b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$, $d(x, fx) \leq bd(x, y)$ implies that $d(fx, fy) \leq rd(x, y)$ for some $r \in \left[0, \dfrac{1}{b^2+b}\right)$. Then $F(f) = \{u\}$ and the sequence $\{f^n x\}$ converges to u for any choice of an element $x \in X$.*

Corollary 2.9. *Let (X,d) be a complete b-metric space and $f : X \longrightarrow X$ a mapping. If there exists a $\xi \in \Lambda$ and an $r \in [0,1)$ with $r < \frac{1}{b^2+b}$ such that $\xi(d(x, fx), d(x, y)) \leq 0$ implies that $d(fx, fy) \leq rd(x, y)$ for all $x, y \in X$,. Then $F(f) = \{u\}$, and the sequence $\{f^n x\}$ converges to u for any choice of an element $x \in X$.*

Proof. It follows from Corollary 2.7. □

Corollary 2.10. *Let (X,d) be a complete b-metric space and $f : X \longrightarrow X$ a mapping. If there exists a $r \in [0,1)$ with $r < \frac{1}{b^2+b}$ such that $\eta(r)d(x, fx) \leq bd(x, y)$ implies that $d(fx, fy) \leq rd(x, y)$ for all $x, y \in X$, where $\eta : [0,1) \to (0,1]$. Then $F(f) = \{u\}$, and the sequence $\{f^n x\}$ converges to u for any choice of an element $x \in X$.*

Proof. Consider $\xi(s,t) = \frac{\eta(r)}{b}s - t \leq \frac{s}{b} - t$. Hence $\xi \in \Lambda$. If $s = d(x, fx)$ and $t = d(x, y)$ then $\xi(d(x, fx), d(x, y)) = \frac{\eta(r)}{b}s - t \leq 0$. Hence result follows from Corollary 2.9. □

Corollary 2.11. *Let (X,d) be a complete strong b-metric space and $f : X \longrightarrow X$ a mapping. If there exists a $r \in [0,1)$ with $r < \frac{1}{b^2+b}$ such that $\eta(r)d(x, fx) \leq bd(x, y)$ implies that $d(fx, fy) \leq rd(x, y)$ for all $x, y \in X$, where $\eta : [0,1) \to (0,1]$. Then $F(f) = \{u\}$, and the sequence $\{f^n x\}$ converges to u for any choice of an element $x \in X$.*

Proof. It follows from Corollary 2.10 as every strong b-metric is b-metric. □

3 Characterization of a b-metric space completeness

Connel studied properties of fixed point sets and presented an example [15, Example 3] of a separable and locally contractible incomplete metric space that has a fixed point property (FPP) for contraction mappings. This shows that BCP does not characterize metric completeness (see also [20]). Kannan [24, 25] proved a fixed point theorem which is independent of BCP. Subrahmanyam [38] proved that if underlying metric space X has FPP for Kannan type contractions, then X is complete. Suzuki [39] presented a fixed point theorem that also characterize metric completeness of X. For more details on FPP and completeness properties of metric spaces, see [11].

In this section, we present some results about the strong b-metric and b-metric completeness characterizations via fixed point results obtained in section 2.

Jovanovic et al. [23] proved the following version of BCP in b-metric spaces.

Theorem 3.1. *Let (X, d) be a complete b-metric space and $T : X \to X$ a map such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$ and some $r \in \left[0, \frac{1}{b}\right)$. Then $F(T)$ is singleton.*

Dung et al. [19] replaced the condition $0 \leq r < \frac{1}{b}$ with $0 \leq r < 1$ and proved that BCP can be transported in b-metric spaces without imposing any additional condition on a contraction constant r.

They proved the following result.

Theorem 3.2. *Let (X, d) be a complete b-metric space and $T : X \to X$ a map such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$ and some $r \in [0, 1)$. Then $F(T)$ is singleton.*

Park and Rhoads [32] commented on characterization of metric completeness.

We present analogous comments in b-metric spaces.

Let (X, d) be a b-metric space and B a class of mappings of a b-metric space X such that if any map in B has a fixed point then X is complete. Let A be a class of mappings of a b-metric space X containing B such that completeness of X implies the existence of fixed point of any map in A.

Theorem 3.3. *(compare [32]) If (X, d) is a b-metric space, then*

X is complete if and only if any map in A has a fixed point.

Proof. If X is complete then, any map in A has a fixed point. Conversely, let any map in A has a fixed point, then any map in B has a fixed point. Then by assumption on B, X is complete. □

We present the following lemma that is needed to prove the main result in this section.

Lemma 3.4. *Let (X, d) be a strong b-metric space and $\{x_n\}$ a Cauchy sequence in X. Then $d(x, x_n)$ is a Cauchy sequence in $\mathbb{R}$ for all x in X.*

Proof. Note that

$$d(x, x_n) \leq d(x, x_m) + bd(x_m, x_n)$$

for each $n, m \in \mathbb{N}$. Thus, we have

$$|d(x, x_n) - d(x, x_m)| \leq bd(x_m, x_n)$$

for each $n, m \in \mathbb{N}$. The result follows as $\{x_n\}$ is a Cauchy sequence in X. $\square$

The following result gives the characterization of completeness of a strong b-metric space.

Theorem 3.5. *Let (X, d) be a strong b-metric space. For $r \in [0, 1)$ with $r < \frac{1}{b^2+b}$, let $A_{r,\eta}$ be a class of mappings T on X which satisfies the following :*

(a) *For any $x, y \in X$*

$$\eta(r)d(x, Tx) \leq bd(x, y) \text{ implies that } d(Tx, Ty) \leq rd(x, y) \tag{3.1}$$

where $\eta : [0, 1) \to (0, 1]$.

Let $B_{r,\eta}$ be the class of mappings T on X satisfying (a) and the following:

(b) *$T(X)$ is countably infinite.*

(c) *Every subset of $T(X)$ is closed.*

Then the following are equivalent:

(i) *(X, d) is complete,*

(ii) *Every mapping $T \in A_{r,\eta}$ has a fixed point for all $r \in [0, 1)$ with $r < \frac{1}{b^2+b}$.*

(iii) *There exists an $r \in (0, 1)$ with $r < \frac{1}{b^2+b}$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. It follows from Corollary 2.11 that (i) implies (ii). As $B_{r,\eta} \subseteq A_{r,\eta}$, so (ii) implies (iii). We now show that (iii) implies (i). Suppose that (X, d) is not complete. That is, there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function $f : X \to [0, \infty)$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$ for $x \in X$. By Lemma 3.4, $\{d(x, u_n)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$. Hence f is well defined. Note that $f(x) > 0$ for every $x \in X$ and $\lim_{n\to\infty} f(u_n) = 0$. Consequently, for every $x \in X$ there exists a $\upsilon \in \mathbb{N}$ such that

$$f(u_\upsilon) \leq \left(\frac{r\eta(r)}{3b^3 + r\eta(r)} \right) f(x). \tag{3.2}$$

Define $T(x) = u_v$. Then

$$f(Tx) \leq \left(\frac{r\eta(r)}{3b^3 + r\eta(r)} \right) f(x) \text{ and } Tx \in \{u_n : n \in \mathbb{N}\} \tag{3.3}$$

for all $x \in X$. From (3.3), we have $f(Tx) < f(x)$, and hence $Tx \neq x$ for all $x \in X$. That is, T has no fixed point. As $T(X) \subset \{u_n : n \in \mathbb{N}\}$, so (b) holds. It is easy to show that (c) holds. Note that, for all $x, y \in X$

$$\begin{aligned}
&f(x) - f(y) \leq bd(x, y) \\
&f(y) - f(x) \leq bd(x, y) \\
&f(x) - f(Tx) \leq bd(x, Tx) \text{ and} \\
&d(Tx, Ty) \leq f(Tx) + bf(Ty).
\end{aligned}$$

Fix $x, y \in X$ such that $\eta(r)d(x, Tx) \leq bd(x, y)$. We now show that (3.1) holds. Observe that

$$\begin{cases} d(x, y) \geq \dfrac{\eta(r)}{b} d(x, Tx) \geq \dfrac{\eta(r)}{b^2}(f(x) - f(Tx)) \\ \geq \dfrac{\eta(r)}{b^2} \left(1 - \dfrac{r\eta(r)}{3b^3 + r\eta(r)}\right) f(x) = \dfrac{3b\eta(r)}{3b^3 + r\eta(r)} f(x). \end{cases} \tag{3.4}$$

We now divide the proof in two cases.

Case (1) Suppose that $f(y) \geq 2bf(x)$. Then

$$\begin{aligned}
d(Tx, Ty) \quad &\leq \quad f(Tx) + bf(Ty) \\
&\leq \frac{r\eta(r)}{3b^3 + r\eta(r)} fx + \frac{br\eta(r)}{3b^3 + r\eta(r)} fy \\
&\leq \frac{r}{3b}(fx + fy) + \frac{2r}{3b}(fy - 2bfx) = \frac{r}{3}\left(\frac{1}{b} fx + \frac{1}{b} fy + \frac{2}{b} fy - \frac{4}{b} fx\right) \\
&\leq \frac{r}{3}\left(\frac{3}{b} fy - \frac{3}{b} fx\right) \leq r\left(\frac{1}{b} fy - \frac{1}{b} fx\right) \leq rd(x, y).
\end{aligned}$$

Case (2) If $f(y) < 2bf(x)$, then by (3.4) we have

$$\begin{aligned}
d(Tx, Ty) &\leq bf(Tx) + f(Ty) \\
&\leq \frac{br\eta(r)}{3b^3 + r\eta(r)} fx + \frac{r\eta(r)}{3b^3 + r\eta(r)} fy \\
&\leq \frac{br\eta(r)}{3b^3 + r\eta(r)} fx + \frac{2br\eta(r)}{3b^3 + r\eta(r)} fx \\
&= \frac{3br\eta(r)}{3b^3 + r\eta(r)} fx = r\frac{3b\eta(r)}{3b^3 + r\eta(r)} fx \leq rd(x, y).
\end{aligned}$$

Hence $\eta(r)d(x,Tx) \leq bd(x,y)$ implies that

$$d(Tx,Ty) \leq rd(x,y)$$

for all $x, y \in X$. From (iii), a mapping T has a fixed point which gives a contradiction. Hence X is complete and consequently (iii) implies (i). □

Remark 3.6. *Let $\{x_n\}$ be a Cauchy sequence in a $b-$ metric space X. If $\{x_n\}$ is convergent to some $u \in X$, then for any $x \in X$, $\{d(x,x_n)\}$ is convergent in $\mathbb{R}$ and hence Cauchy in $\mathbb{R}$. If $\{x_n\}$ is not convergent, then from triangular inequality of b-metric, it does not follow necessarily the Cauchyness of $d(x,x_n)$ in $\mathbb{R}$. Assume that $\digamma$ is the class of b-metrics d and for any Cauchy sequence $\{x_n\}$ in X and for any x in X, $\{d(x,x_n)\}$ is Cauchy in $\mathbb{R}$. Consider a metric space (X,ρ) with $d(x,y) = (\rho(x,y))^p$ for $p > 1$. Then d is a b-metric on X (see [26]). Hence $\digamma$ is nonempty.*

Now we present the following result which deals with characterization of a completeness of b-metric space.

Theorem 3.7. *Let (X,d) be a b-metric space such that $d \in \digamma$. For $r \in [0,1)$ with $r < \frac{1}{b^2+b}$, let $A_{r,\eta}$ be a class mappings T on X which satisfies the following:*

(a) *For $x, y \in X$*

$$\eta(r)d(x,Tx) \leq bd(x,y) \text{ implies that } d(Tx,Ty) \leq rd(x,y) \tag{3.5}$$

where $\eta : [0,1) \to (0,1]$.

Let $B_{r,\eta}$ be the class of mappings T on X satisfying (a) and the following conditions:

(b) *$T(X)$ is countably infinite.*

(c) *Every subset of $T(X)$ is closed.*

Then the following are equivalent:

(i) *(X,d) is complete,*

(ii) *Every mapping $T \in A_{r,\eta}$ has a fixed point for all $r \in [0,1)$ with $r < \frac{1}{b^2+b}$.*

(iii) *There exists an $r \in (0,1)$ with $r < \frac{1}{b^2+b}$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. By Corollary 2.10 (i) implies (ii). As $B_{r,\eta} \subseteq A_{r,\eta}$, so we have (ii) implies (iii). Now we prove that (iii) implies (i). Assume that (iii) holds. Suppose that (X, d) is not complete. Define the function $f : X \to [0, \infty)$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$ for $x \in X$. By given assumption, $\{d(x, u_n)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$. Hence f is well defined. Note that $f(x) > 0$ for every $x \in X$ and $\lim_{n\to\infty} f(u_n) = 0$. Consequently, for every $x \in X$, there exists a $\upsilon \in \mathbb{N}$ such that

$$f(u_\upsilon) \leq \left(\frac{r\eta(r)}{3b^4 + rb\eta(r)}\right) f(x). \tag{3.6}$$

Define $T(x) = u_\upsilon$, then we have

$$f(Tx) \leq \left(\frac{r\eta(r)}{3b^4 + rb\eta(r)}\right) f(x) \text{ and } Tx \in \{u_n : n \in \mathbb{N}\} \tag{3.7}$$

for all $x \in X$. The rest of the proof is obtained following similar arguments to those arguments similar to those in the proof of Theorem 3.7. □

4 Coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive operators

In this section, we apply Theorem 2.1 to obtain the existence of coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive multivalued operators and single-valued self mappings in the setup of b-metric spaces.

Theorem 4.1. *Let (X, d) be a b-metric space and (f, T) a Ciric-Suzuki type quasi-contractive hybrid pair with $T(X) \subseteq f(X)$ and $f(X)$ a complete subspace of X. Then $C(f, T)$ is nonempty. Furthermore, $F(f, T)$ is nonempty if any of the following conditions hold:*

C_1- *The hybrid pair (f, T) is $w-$compatible, $\lim_{n\to\infty} f^n(x) = u$ for some $u \in X$ and $x \in C(f, T)$ and f is continuous at u.*

C_2- *The mapping f is $T-$weakly commuting at some $x \in C(f, T)$ and $f^2x = fx$.*

C_3- *The mapping f is continuous at at some $x \in C(f, T)$ and $\lim_{n\to\infty} f^n(u) = x$ for some $u \in X$.*

Proof. By Lemma 1.6, there is a set $E \subseteq X$ such that $f : E \to X$ is one-to-one and $f(E) = f(X)$. Define the mapping $\mathcal{T} : f(E) \to CB(X)$ by $\mathcal{T}fx = Tx$ for all $f(x) \in f(E)$. The mapping $\mathfrak{T}$ is well defined because f is one–to-one. As (f,T) is Ciric-Suzuki type quasi-contractive hybrid pair, for any $x,y \in X$

$$\begin{array}{l} \xi\left(d(fx,Tx),d(fx,fy)\right) \leq 0 \\ \text{implies that} \\ H(Tx,Ty) \leq r\max\left\{d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\right\} \end{array} \tag{4.1}$$

for some $r \in \left[0, \dfrac{1}{b^2+b}\right)$ and $\xi \in \Lambda$. Thus for all $fx, fy \in f(E)$,

$$\left\{\begin{array}{l} \xi\left(d(fx,\mathfrak{T}fx),d(fx,fy)\right) \leq 0 \\ \text{implies the} \\ H(\mathfrak{T}fx,\mathfrak{T}fy) \leq r\max\left\{d(fx,fy),d(fx,\mathfrak{T}fx),d(fy,\mathfrak{T}fy),d(fx,\mathfrak{T}fy),d(fy,\mathfrak{T}fx)\right\} \end{array}\right.$$

for some $r \in \left[0, \dfrac{1}{b^2+b}\right)$ and $\xi \in \Lambda$. As $f(X)$ is complete so is $f(E)$. It follows from Theorem 2.1 that the mapping $\mathfrak{T}$ on $f(E)$ is MWP operator. Thus we may choose a point $u \in f(E)$ such that $u \in \mathfrak{T}u$. Since $u \in f(E) = f(X)$, there exists $x \in X$ such that $fx = u$. Hence $fx \in \mathfrak{T}fx = Tx$, that is, $x \in C(f,T)$. To prove $F(f,T) \neq \emptyset$: Suppose that (C_1) holds. Now, $\lim\limits_{n\to\infty} f^n(x) = u$ for some $u \in X$ and the continuity of f at u imply that $fu = u$ and hence $\lim\limits_{n\to\infty} f^n(x) = fu$. From $w-$compatibility of a pair (f,T), we have $f^n(x) \in T\left(f^n(x)\right)$, that is $f^n(x) \in C(f,T)$ for all $n \in \mathbb{N}$. Suppose that $f^n(x) \neq f(u)$ for all n. Indeed, if $f^n(x) = f(u)$ for some n, then we have $u = fu = f^n(x) \in T(f^{n-1}(x)) = T(u)$ and hence the result. Note that

$$\begin{aligned} & \xi\left(d(f^n(x),T\left(f^{n-1}(x)\right)),d(ff^{n-1}(x),fu)\right) \\ \leq\ & \frac{1}{b}d(f^n(x),T\left(f^{n-1}(x)\right)) - d(ff^{n-1}(x),fu) = 0 - d(ff^{n-1}(x),fu) < 0. \end{aligned}$$

Hence

$$\begin{aligned} d(f^nx,Tu) \quad &\leq \quad H(Tf^{n-1}x,Tu) \\ &\leq r\max\left\{d(f^nx,fu),d(f^nx,Tf^{n-1}x),d(fu,Tu),d(f^nx,Tu),d(fu,Tf^{n-1}x)\right\} \\ &\leq r\max\left\{d(f^nx,fu),d(f^nx,f^nx),d(fu,Tu),d(f^nx,Tu),d(fu,f^nx)\right\} \\ &\leq r\max\left\{d(f^nx,fu),d(f^nx,f^nx),d(fu,Tu),d(f^nx,Tu),d(fu,f^nx)\right\}. \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that $d(fu,Tu) \leq rd(fu,Tu)$. Hence $d(fu,Tu) = 0$ implies that $u = fu \in Tu$. That is, $F(f,T)$ is nonempty. If (C_2) holds, then $f^2x = fx$ for some $x \in C(f,T)$. Also, f is $T-$weakly commuting, $fx = f^2x \in Tfx$. Hence $fx \in F(f,T)$. If (C_3) holds , then we have $\lim\limits_{n\to\infty} f^n(u) = x$ for some $u \in X$ and $x \in C(f,T)$.

By continuity of f, $x = fx \in Tx$. Hence in all the three cases, we have $F(f,T) \neq \emptyset$. $\square$

Corollary 4.2. *Let (X,d) be a b-metric space, $f : X \to X$, $T : X \to CB(X)$ with $T(X) \subseteq f(X)$ and $f(X)$ a complete subspace of X. If for any $x, y \in X$*

$$\xi\left(d(fx,Tx), d(fx,fy)\right) \leq 0 \text{ implies that } H(Tx,Ty) \leq rd(fx,fy)$$

where $r < \dfrac{1}{b^2+b}$ and $\xi \in \Lambda$. Then $C(f,T)$ is nonempty. Furthermore, $F(f,T)$ is nonempty if any of the following conditions hold:

C_4- *The hybrid pair (f,T) is $w-$compatible, $\lim_{n\to\infty} f^n(x) = u$ for some $u \in X$ and $x \in C(f,T)$ and f is continuous at u.*

C_5- *The mapping f is $T-$weakly commuting at some $x \in C(f,T)$ and $f^2x = fx$.*

C_6- *The mapping f is continuous at at some $x \in C(f,T)$ and $\lim_{n\to\infty} f^n(u) = x$ for some $u \in X$.*

5 Stability and uniform convergence results

In this section, we find an upper bound of Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators and then study the uniform convergence of such sets in the setup of b-metric spaces.

Theorem 5.1. *Let (X,d) be a complete b-metric space and $T_1, T_2 : X \to P(X)$. Suppose that T_i is Ciric-Suzuki type quasi-contractive multivalued operator for each $i \in \{1,2\}$. If there exists $\lambda > 0$ such that*

$$H(T_1x, T_2x) \leq \lambda \tag{5.1}$$

for all $x \in X$. Then $F(T_i)$ is closed subset of X and T_i is a MWP operator for each $i \in \{1,2\}$. Also, the following holds:

$$H(F(T_1), F(T_2)) \leq \frac{\lambda}{1 - b \max\limits_{i\in\{1,2\}} \gamma_i} \tag{5.2}$$

where

$$\gamma_i = \frac{b\beta_i}{1-b\beta_i},\ \beta_i = r_i + \alpha_i,\ \text{and } \alpha_i = \frac{1}{2}\left(\frac{1}{b^2+b} - r_i\right) \text{ for } i \in \{1,2\}.$$

Proof. By Theorem 2.1, $F(T_i)$ is nonempty for each $i \in \{1,2\}$. Let $\{x_n\}$ be a sequence in $F(T_1)$ such that $x_n \to z$ as $n \to \infty$. Note that

$$\begin{aligned} \xi\left(d(x_n, T_1x_n), d(z, x_n)\right) &\leq \frac{1}{b}d(x_n, T_1x_n) - d(z, x_n) \\ &\leq d(x_n, T_1x_n) - d(z, x_n) \\ &\leq d(x_n, x_n) - d(z, x_n) = -d(z, x_n) \leq 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} d(z, T_1z) &\leq bd(z, x_n) + bd(x_n, T_1z) \\ &\leq bd(z, x_n) + bH(T_1z, T_1x_n) \\ &\leq bd(z, x_n) + br_1 \max\{d(z, x_n), d(z, T_1z), d(T_1x_n, x_n), d(x_n, T_1z), d(z, T_1x_n)\} \\ &\leq bd(z, x_n) + br_1 \max\{d(z, x_n), d(z, T_1z), d(x_n, T_1z)\}. \end{aligned}$$

On taking the limit as $n \to \infty$ we obtain that

$$d(z, T_1z) \leq br_1 d(z, T_1z) \leq \frac{1}{b+1} d(z, T_1z).$$

As $b \geq 1$, so $d(z, T_1z) = 0$, that is, $z \in T_1z$. Hence $F(T_1)$ is closed. Similarly, $F(T_2)$ is a closed subset of X. Following arguments similar to those in the proof of Theorem 2.1, we conclude that T_i is MWP operator for each $i \in \{1,2\}$. We now show that (5.2) holds for all x in X. As $r_i < \dfrac{1}{b^2+b} < 1$, there exist $\alpha_i \in \mathbb{R}^+$ such that $\dfrac{r_i}{2} + \alpha_i = \dfrac{1}{2}\left(\dfrac{1}{b^2+b}\right)$ which gives that

$$r_i + \alpha = \frac{1}{2}\left(\frac{1}{b^2+b} + r_i\right).$$

We set $\beta_i = r_i + \alpha_i$. Note that $0 < \beta_i < 1$ and $\alpha_i > 0$. Following arguments similar to those in the proof of Theorem 2.1 with $x_0 \in F(T_1)$ and $x_1 \in T_2x_0$, we obtain a Cauchy sequence $\{x_n\}$ in X such that $x_{n+1} \in T_2x_n$ for all $n \geq 1$ and it satisfies:

$$d(x_n, x_{n+1}) \leq \gamma_2 d(x_{n-1}, x_n)$$

and

$$d(x_n, x_{n+1}) \leq \gamma_2 d(x_{n-1}, x_n) \leq (\gamma_2)^2 d(x_{n-2}, x_{n-1}) \leq ... \leq (\gamma_2)^n d(x_0, x_1). \tag{5.3}$$

where $\gamma_2 = \dfrac{b\beta_2}{1-b\beta_2}$. We choose an element u in X such that $x_n \to u$ as $n \to \infty$ and $u \in T_2 u$. From (5.3), we obtain that

$$\begin{aligned} d(x_n, x_{n+p}) &\leq bd(x_n, x_{n+1}) + ... + b^{p-1} d(x_{n+p-2}, x_{n+p-1}) + b^{p-1} d(x_{n+p-1}, x_{n+p}) \\ &\leq b\gamma_2^n d(x_0, x_1) + ... + b^{p-1}\gamma_2^{n+p-2} d(x_0, x_1) + b^{p-1}\gamma_2^{n+p-1} d(x_0, x_1) \\ &\leq b\gamma_2^n d(x_0, x_1) \left(1 + b\gamma_2 + ... + (b\gamma_2)^{p-2} + \frac{1}{b}(b\gamma_2)^{p-1}\right) \\ &\leq b\gamma_2^n d(x_0, x_1) \left(1 + b\gamma_2 + ... + (b\gamma_2)^{p-2} + (b\gamma_2)^{p-1}\right) \\ &\leq \frac{(b\gamma_2)^n (1-(b\gamma_2)^p)}{1-b\gamma_2} d(x_0, x_1). \end{aligned}$$

Thus, we have

$$d(x_n, x_{n+p}) \leq \frac{(b\gamma_2)^n (1-(b\gamma_2)^p)}{1-b\gamma_2} d(x_0, x_1). \tag{5.4}$$

On taking limit as $p \to \infty$ on both sides of the above inequality, we have

$$d(x_n, u) \leq \frac{(b\gamma_2)^n}{1-b\gamma_2} d(x_0, x_1). \tag{5.5}$$

Also, from (5.1) and (5.5), we have

$$d(x_0, u) \leq \frac{1}{1-b\gamma_2} d(x_0, x_1) \leq \frac{\lambda}{1-b\gamma_2}. \tag{5.6}$$

Similarly, for each $z_0 \in T_2 z_0$, we get $v \in T_1 v$ such that

$$d(z_0, v) \leq \frac{1}{1-b\gamma_1} d(z_0, z_1) \leq \frac{\lambda}{1-b\gamma_1}. \tag{5.7}$$

It follows from (5.6), (5.7) and Lemma 1.11 that

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1-\max\{b\gamma_1, b\gamma_2\}} = \frac{\lambda}{1-b\max\limits_{i\in\{1,2\}}\gamma_i}.$$

□

The following theorem generalizes the results in [30, 37] for a sequence of Ciric-Suzuki type quasi-contractive multivalued operators in b-metric spaces.

Theorem 5.2. *Let (X, d) be a complete b-metric space and $T_n : X \to P(X)$, a sequence of Ciric-Suzuki type quasi-contractive multivalued operator for each $n \in \mathbb{N}$. If $\{T_n\}$ converges to T_0 uniformly on X, then* $\lim\limits_{n\to\infty} H(F(T_n), F(T_0)) = 0$.

Proof. Let γ_i for each $i \in \mathbb{N}^*$ be as given in the proof of Theorem 5.1. Then $\gamma_i > 0$ for $i \in \mathbb{N}^*$ and $b \max_{i \in \mathbb{N}^*} \gamma_i < 1$. As $\{T_n\}$ converges to T_0 uniformly on X, so for any $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$\sup_{x \in X} H(T_n(x), T_0(x)) < \left(1 - b \max_{i \in \mathbb{N}^*} \gamma_i\right) \varepsilon$$

for all $n \geq n_0$. If we set, $\lambda = \left(1 - b \max_{i \in \mathbb{N}^*} \gamma_i\right) \varepsilon$, then $H(T_n(x), T_0(x)) < \lambda$ for all $n \geq n_0$ and $x \in X$. By Theorem 5.1, we have

$$H(F(T_n), F(T_0)) \leq \frac{\lambda}{\left(1 - b \max_{i \in \mathbb{N}^*} \gamma_i\right)} = \varepsilon$$

for all $n \geq n_0$. □

6 Multivalued fractals in b-metric spaces

Let (X, d) be a b-metric space and $T_i : X \to K(X)$, where $K(X)$ a collection of nonempty compact subsets of X.

The system $T = (T_1, T_2, ..., T_k)$ is called an iterated multifunction system (briefly IMS). If T_i is upper semicontinuous for each $i = 1, 2, ..., k$, then the single valued operator $\mathfrak{T}_T : K(X) \to K(X)$ defined by $\mathfrak{T}_T(A) = \bigcup_{i=1}^{k} T_i(A)$ is called multi fractal generated by the IMS $T = (T_1, T_2, ..., T_k)$. Since the image of a compact set under an upper semicontinuous multivalued mapping is compact, therefore operator $\mathfrak{T}_T$ is well defined ([8, 10, 14]).

A set $\mathring{A} \in K(X)$ is called multivalued fractal with respect to IMS $T = (T_1, T_2, ..., T_k)$ if and only if $\mathring{A} \in F(\mathfrak{T}_T)$.

Theorem 6.1. *Let (X, d) be a b-metric space and $T_i : X \to K(X)$ upper semicontinuous multivalued operators for each $i \in \{1, 2, ..., k\}$. Suppose that for any $x, y \in X$,*

$$\begin{aligned} &\xi\left(d(x, T_i x), d(x, y)\right) \leq 0 \text{ implies that} \\ &H(T_i x, T_i y) \leq r_i \max\left\{d(x, y), d(x, T_i y), d(y, T_i x)\right\} \end{aligned}$$

where $r_i < \dfrac{1}{b^2 + b}$ for each $i \in \{1, 2, ..., k\}$ and $\xi \in \Lambda$. If $\dfrac{1}{b} d(x, T_i x) \leq d(x, y)$ for all $x \in A$, $y \in B$ and $i \in \{1, 2, ..., k\}$. Then $\mathfrak{T}_T : (K(X), H) \to (K(X), H)$ is a Ciric-Suzuki type quasi-contractive operator, that is

$$\begin{aligned} &\xi\left(H(A, \mathfrak{T}_T A), H(A, B)\right) \leq 0 \text{ implies that} \\ &H(\mathfrak{T}_T A, \mathfrak{T}_T B) \leq r \max\left\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\right\} \end{aligned} \tag{6.1}$$

for all $A, B \in K(X)$. *Also, there exists a unique multivalued fractal* $\mathring{A} \in K(X)$ *such that* $\lim_{n\to\infty} H(\mathfrak{T}_T^n A, \mathring{A}) = 0$ *for every* $A \in K(X)$.

Proof. For each $i \in \{1, 2, ..., k\}$, we have $\frac{1}{b} d(x, T_i x) \leq d(x, y)$ for all $x \in A, y \in B$. Thus $\xi(d(x, T_i x), d(x, y)) \leq 0$ for all $x \in A, y \in B$. Hence, for each $i \in \{1, 2, ..., k\}$

$$H(T_i x, T_i y) \leq r_i \max\{d(x, y), d(x, T_i x), d(y, T_i y), d(x, T_i y), d(y, T_i x)\} \quad (6.2)$$

for all $x \in A, y \in B$. By (6.2), we have

$$\begin{aligned}
\delta(T_i A, T_i B) &= \sup_{x\in A}\left(\inf_{y\in B} \delta(T_i x, T_i y)\right) \\
&= \sup_{x\in A}\inf_{y\in B} \delta(T_i x, T_i y) \leq \sup_{x\in A}\inf_{y\in B} H(T_i x, T_i y) \\
&\leq \sup_{x\in A}\inf_{y\in B} r_i \max\{d(x, y), d(x, T_i y), d(y, T_i x)\} \\
&\leq r_i \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in A}\inf_{y\in B} d(x, T_i y), \sup_{x\in A}\inf_{y\in B} d(y, T_i x)\right\} \\
&\leq r_i \max\{\delta(A, B), \delta(A, T_i B), \delta(B, T_i A)\} \\
&= r_i \max\{\delta(A, B), \delta(A, \mathfrak{T}_T B), \delta(B, \mathfrak{T}_T A)\} \\
&\leq r_i \max\{H(A, B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\} \\
&\leq r_i \max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\}
\end{aligned}$$

for all $A, B \in K(X)$, for each $i \in \{1, 2, ..., k\}$. That is,

$$\delta(T_i A, T_i B) \leq r_i \max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\} \quad (6.3)$$

for all $A, B \in K(X)$, for each $i \in \{1, 2, ..., k\}$. Similarly,

$$\delta(T_i B, T_i A) \leq r_i \max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\} \quad (6.4)$$

for all $A, B \in K(X)$, for each $i \in \{1, 2, ..., k\}$. Also, from (6.3) and (6.4) we obtain that

$$H(T_i A, T_i B) \leq r_i \max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\} \quad (6.5)$$

for all $A, B \in K(X)$, for each $i \in \{1, 2, ..., k\}$. Note that

$$\begin{aligned}
H\left(\bigcup_{i=1}^k T_i A, \bigcup_{i=1}^k T_i B\right) &\leq \max_{i=1,2,\ldots,k}\{H(T_i A, T_i B)\} \\
\leq \max_{i=1,2,\ldots,k} &(r_i \max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\}) \\
\leq \left(\max_{i=1,2,\ldots,k} r_i\right) &\max\{H(A, B), H(A, \mathfrak{T}_T A), H(B, \mathfrak{T}_T B), H(A, \mathfrak{T}_T B), H(B, \mathfrak{T}_T A)\}.
\end{aligned}$$

Hence

$$H(\mathfrak{I}_T A, \mathfrak{I}_T B) \leq r \max\{H(A,B), H(A,\mathfrak{I}_T A), H(B,\mathfrak{I}_T B), H(A,\mathfrak{I}_T B), H(B,\mathfrak{I}_T A)\},$$

where, $r = \max_{i\in\{1,2,\ldots,k\}} r_i$. Consequently, $\xi(H(A,\mathfrak{I}_T A), H(A,B)) \leq 0$ implies that

$$H(\mathfrak{I}_T A, \mathfrak{I}_T B) \leq r \max\{H(A,B), H(A,\mathfrak{I}_T A), H(B,\mathfrak{I}_T B), H(A,\mathfrak{I}_T B), H(B,\mathfrak{I}_T A)\}$$

for all $A, B \in K(X)$. It now follows from Corollary 2.7 that $F(\mathfrak{I}_T) = \{\mathring{A}\}$ and $\lim_{n\to\infty} H(\mathfrak{I}_T^n A, \mathring{A}) = 0$ for every $A \in K(X)$. $\square$

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