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Some decompositions of filters in residuated lattices

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Abstract

In this paper we introduce a new class of residuated lattice: residuated lattice with $(C_{\wedge \,\&\, \rightarrow})$ property and we prove that $(C_{\wedge \,\&\, \rightarrow}) \Leftrightarrow (C_{\rightarrow}) + (C_{\wedge})$. Also, we introduce and characterize $C_{\rightarrow}$, $C_{\vee}, C_{\wedge}$ and $C_{\wedge \,\&\, \rightarrow}$ filters in residuated lattices (i.e., we characterize the filters for which the quotient algebra that is constructed via these filters is a residuated lattice with $C_{\rightarrow}$ ($C_{\vee}$ or $C_{\wedge}$ or $C_{\wedge \,\&\, \rightarrow}$ property). We state and prove some results which establish the relationships between these filters and other filters of residuated lattices: BL filters, MTL filters, divisible filters and, by some examples, we show that these filters are different. Starting from the results of algebras, we present for MTL filters, BL filters and $C_{\wedge \,\&\, \rightarrow}$ filters the decomposition conditions.

1. Introduction

Residuated lattices, the algebraic counterpart of logics without contraction, were introduced in 1924 by Krull ([11]). They have been investigated by Balbes and Dwinger ([1]), Dilworth ([7]), and Ward and Dilworth ([14]). An important concept in the theory of residuated lattices and other algebraic structures used for formal fuzzy logic, is that of a *filter*. Filters can be used to define congruence relations and play an important role in studying logical

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systems and the related algebraic structures. There are a rich rage of classes of filters so, in [4], we proposed a new approach for the study of filters in residuated lattices.

In [10], starting from the well known condition $(C_\vee)$ satisfied in MTL algebras and BL algebras: $(\mathbf{C}_\vee) : x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$, for every x, y it were obtained interesting decompositions of prelinearity and divisibility in a residuated lattice with important consequences: $(prel) \Leftrightarrow (C_\to)+(C_\vee)$, $(prel) \Rightarrow (C_\wedge)$, $(div) \Rightarrow (C_\to)$ and $(C_\wedge)$, where $(C_\to) : (x \to y) \to (y \to x) = y \to x$, for every x, y and $(C_\wedge) : x \wedge y = [x \odot (x \to y)] \vee [y \odot (y \to x)]$, for every x, y.

In this paper, we introduce a new class of residuated lattices: residuated lattice with $(C_{\wedge\ \&\ \to})$ property: $x \wedge y = [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]$, for every x, y and we show the connections between these lattices and residuated lattices with $(C_\wedge\ )$ and $(C_{\ \to})$ property: $(C_{\wedge\ \&\ \to}) \Leftrightarrow (C_\to) + (C_\wedge)$.

Also, in this paper we introduce $C_\to$, $C_\vee$, $C_\wedge$ and $C_{\wedge\ \&\ \to}$ filters in residuated lattices. So, we characterize the filters for which the quotient algebra that is constructed via these filters is a residuated lattice with $C_\to$ ($C_\vee$, $C_\wedge$ or $C_{\wedge\ \&\ \to}$) property.

We state and prove some results which establish the relationships between these filters and other filters of a residuated lattice: BL filters, MTL filters, divisible filters and, by some examples, we show that these filters are different. Starting from the results of algebras, we present for MTL filters, BL filters and $C_{\wedge\ \&\ \to}$ filters the decomposition conditions. So, we make the following decompositions: $\mathbf{MTLF(L)} = \mathbf{C_\to F(L)} \cap \mathbf{C_\vee F(L)}$, $\mathbf{BLF(L)} = \mathbf{DivF(L)} \cap \mathbf{C_\vee F(L)}$ and $\mathbf{C_{\wedge\ \&\ \to} F(L)} = \mathbf{C_\to F(L)} \cap \mathbf{C_\wedge F(L)}$.

We prove that a residuated lattice L has $C_\to$ $(C_\vee, C_\wedge, C_{\wedge\ \&\ \to})$ property if and only if every filter of L is a $C_\to$ $(C_\vee, C_\wedge, C_{\wedge\ \&\ \to})$ filter if and only if $\{1\}$ is a $(C_\vee, C_\wedge, C_{\wedge\ \&\ \to})$ filter of L.

The conditions $C_\to, C_\vee, C_\wedge$ and $C_{\wedge\ \&\ \to}$ are topics of interest because their study provides new types of residuated lattices (and consequently new classes of filters) which refines the relations between different classes of residuated lattices (and consequently, between different classes of filters). A refinement of old hierarchies is thus obtained.

1 Preliminaries of residuated lattices

In this section we recall some definitions, properties and results for residuated lattices.

Definition 1. *([2], [9], [13], [14]) A* residuated lattice *is an algebra*

$(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *equipped with an order* $\leq$ *satisfying the following:*

(LR_1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice relative to the order* $\leq$*;*

(LR_2) $(L, \odot, 1)$ *is a commutative monoid;*

(LR_3) *The operations* $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.* $a \odot x \leq y$ *iff* $a \leq x \rightarrow y$ *for every* $a, x, y \in L$.

We denote by L a residuated lattice.

For $x \in L$ and a natural number n we define $x^* = x \rightarrow 0, x^{**} = (x^*)^*, x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

The following main rules of calculus in a residuated lattice L can be found, for example, in [3], [8], [9], [12], [14]:

(c_1) $1 \rightarrow x = x, x \rightarrow x = 1, x \rightarrow 1 = 1;$

(c_2) $x \leq y$ iff $x \rightarrow y = 1;$

(c_3) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;

(c_4) $x \leq y \rightarrow x, ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y;$

(c_5) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z);$

(c_6) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z), x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y);$

(c_7) $x \odot (x \rightarrow y) \leq y, x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z);$

(c_8) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z), x \rightarrow (y \vee z) \geq (x \rightarrow y) \vee (x \rightarrow z);$

(c_9) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z), (y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x);$

(c_{10}) $x \leq x^{**}, x \odot x^* = 0, x \rightarrow y^* = y \rightarrow x^*$, for $x, y, z \in L$.

Lemma 1. *Let* L *be a residuated lattice and* $x, y, z \in L$*. Then*

(c_{11}) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z.$

Proof. $x \odot (x \rightarrow y) \leq y$ and $y \odot (y \rightarrow z) \leq z$ so $x \odot (x \rightarrow y) \odot (y \rightarrow z) \leq y \odot (y \rightarrow z) \leq z$. We deduce that $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$, for every $x, y, z \in L$. □

In a residuated lattice L we consider the following identities:

(div) $x \odot (x \to y) = x \wedge y$ (*divisibility*);

$(prel)$ $(x \to y) \vee (y \to x) = 1$ (*prelinearity*).

Definition 2. *([9], [13]) The residuated lattice L is called*

(i) divisible *if L verifies* (div);

(ii) *an* MTL algebra *if L verifies* $(prel)$;

(iii) *a* BL algebra *if L verifies* (div) *and* $(prel)$.

Lemma 2. *A residuated lattice L is a divisible residuated lattice if and only if $x \odot (x \to y) \to y \odot (y \to x) = y \odot (y \to x) \to x \odot (x \to y)$, for every $x, y \in L$.*

Proof. Obviously, if L is divisible, then $x \odot (x \to y) = y \odot (y \to x)$, for every $x, y \in L$, so $x \odot (x \to y) \to y \odot (y \to x) = y \odot (y \to x) \to x \odot (x \to y) = 1$.

Conversely, if we suppose that $x \odot (x \to y) \to y \odot (y \to x) = y \odot (y \to x) \to x \odot (x \to y)$, for every $x, y \in L$, then for $x = x \wedge y$ we obtain that $(x \wedge y) \odot ((x \wedge y) \to y) \to y \odot (y \to (x \wedge y)) = y \odot (y \to (x \wedge y)) \to (x \wedge y) \odot ((x \wedge y) \to y)$.

Since $y \to (x \wedge y) = (y \to x) \wedge (y \to y) = y \to x$, $y \odot (y \to x) \leq x \wedge y$ for every $x, y \in L$ we obtain that $(x \wedge y) \to y \odot (y \to x) = y \odot (y \to x) \to (x \wedge y)$ $\Leftrightarrow (x \wedge y) \to y \odot (y \to x) = 1 \Leftrightarrow x \wedge y \leq y \odot (y \to x) \Leftrightarrow x \wedge y = y \odot (y \to x)$, for every $x, y \in L$, so L is a divisible residuated lattice. $\square$

Lemma 3. *A residuated lattice L is an MTL algebra if and only if $[(x \odot (x \to y)) \to z] \wedge [(y \odot (y \to x)) \to z] = (x \to z) \vee (y \to z)$, for every $x, y, z \in L$.*

Proof. First we suppose that $[(x \odot (x \to y)) \to z] \wedge [(y \odot (y \to x)) \to z] = (x \to z) \vee (y \to z)$, for every $x, y, z \in L$.

Then for $z = x \wedge y$ we obtain $[(x \odot (x \to y)) \to (x \wedge y)] \wedge [(y \odot (y \to x)) \to (x \wedge y)] = (x \to (x \wedge y)) \vee (y \to (x \wedge y)) \Leftrightarrow 1 = (x \to y) \vee (y \to x)$, for every $x, y \in L$, so L is an MTL algebra.

Conversely, we suppose that L is an MTL algebra and let $x, y, z \in L$.

Since $x \to z \overset{(c_4)}{\leq} (x \to y) \to (x \to z) \overset{(c_8)}{=} x \odot (x \to y) \to z$ and $y \to z \overset{(c_5)}{\leq} (x \to y) \to (x \to z) \overset{(c_8)}{=} x \odot (x \to y) \to z$ we deduce that $(x \to z) \vee (y \to z) \leq x \odot (x \to y) \to z$ and similarly, $(y \to z) \vee (x \to z) \leq y \odot (y \to x) \to z$.

Let $t \in L$ such that $t \leq x \odot (x \to y) \to z$, $y \odot (y \to x) \to z$.

We deduce that $1 = t \to [x \odot (x \to y) \to z] = (x \to y) \to [t \to (x \to z)]$ and $1 = t \to [y \odot (y \to x) \to z] = (y \to x) \to [t \to (y \to z)]$. Hence $x \to y \leq t \to (x \to z)$ and $y \to x \leq t \to (y \to z)$. But $t \to [(x \to z) \vee (y \to z)]$

$\overset{(c_8)}{\geq}$ $[t \to (x \to z)] \vee [t \to (y \to z)] \geq (x \to y) \vee (y \to x) = 1$, since L is an MTL algebra.

Hence $t \leq (x \to z) \vee (y \to z)$ so $[(x \odot (x \to y)) \to z] \wedge [(y \odot (y \to x)) \to z] = (x \to z) \vee (y \to z)$, for every $x, y, z \in L$. $\square$

In [10], starting from the well known condition $(C_\vee)$ satisfied in MTL algebras and BL algebras:

$$(\mathbf{C}_\vee): \ x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x],$$

for every x, y it were obtained interesting decompositions of prelinearity and divisibility in a residuated lattice with important consequences:

$$(prel) \Leftrightarrow (C_\to) + (C_\vee) \text{ and } (prel) \Rightarrow (C_\wedge)$$

$$(div) \Rightarrow (C_\to) \text{ and } (div) \Rightarrow (C_\wedge),$$

where

$$(C_\to) : (x \to y) \to (y \to x) = y \to x,$$

$$(C_\wedge) : x \wedge y = [x \odot (x \to y)] \vee [y \odot (y \to x)],$$

for every x, y.

Lemma 4. *Let L be a residuated lattice with $(C_\to)$ property. Then*

$$(x \to y) \odot [(x \to y) \to (y \to x)] = (y \to x) \odot [(y \to x) \to (x \to y)] = (x \to y) \odot (y \to x),$$

for every $x, y \in L$.

Proof. Since $(x \to y) \to (y \to x) = y \to x$, for every $x, y \in L$, $(x \to y) \odot [(x \to y) \to (y \to x)] = (x \to y) \odot (y \to x)$ and $(y \to x) \odot$ $[(y \to x) \to (x \to y)] = (y \to x) \odot (x \to y)$. $\square$

2 Residuated lattices with $(C_{\wedge \& \to})$ property

Definition 3. *We say that a residuated lattice L has $(C_{\wedge\ \&\ \to})$ property if it satisfies the following condition:*

$$(C_{\wedge\ \&\ \to}): \ x \wedge y = [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))].$$

Example 1. *We consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from [12], Example 1.11, p. 7. It is immediate to prove that L satisfies the condition $(C_{\wedge\ \&\ \to})$.*

Theorem 1. *Let L be a residuated lattice. Then* $(C_\wedge) + (C_\rightarrow) \Leftrightarrow (C_{\wedge\ \&\ \rightarrow})$.

Proof. $(C_\wedge)+(C_\rightarrow) \Rightarrow (C_{\wedge\ \&\ \rightarrow})$: We have $x \wedge y \overset{(C_\wedge)}{=} [x \odot (x \to y)] \vee [y \odot (y \to x)] \overset{(C_\rightarrow)}{=} [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]$. It follows that $(C_{\wedge\ \&\ \rightarrow})$ holds.

$(C_{\wedge\ \&\ \rightarrow}) \Rightarrow (C_\rightarrow)$: From $(C_{\wedge\ \&\ \rightarrow})$ we deduce that $\{[x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]\} \to (x \wedge y) = 1 \overset{(c_9)}{\Leftrightarrow} \{[x \odot ((y \to x) \to (x \to y))] \to x \wedge y\} \wedge \{[y \odot ((x \to y) \to (y \to x))] \to x \wedge y\} = 1 \overset{(c_9)}{\Leftrightarrow} ([x \odot ((y \to x) \to (x \to y))] \to x) \wedge ([x \odot ((y \to x) \to (x \to y))] \to y) \wedge ([y \odot ((x \to y) \to (y \to x))] \to x) \wedge ([y \odot ((x \to y) \to (y \to x))] \to y) = 1 \Leftrightarrow 1 \wedge ([x \odot ((y \to x) \to (x \to y))] \to y) \wedge ([y \odot ((x \to y) \to (y \to x))] \to x) \wedge 1 = 1 \Leftrightarrow x \odot ((y \to x) \to (x \to y)) \to y = y \odot ((x \to y) \to (y \to x)) \to x = 1$. Thus, $((y \to x) \to (x \to y)) \to (x \to y) = 1$.

We deduce that $(y \to x) \to (x \to y) \leq x \to y$, so $(x \to y) \to (y \to x) = y \to x$, for every $x, y \in L$. It follows that $(C_\rightarrow)$ holds.

$(C_{\wedge\ \&\ \rightarrow}) \Rightarrow (C_\wedge)$: Since $(C_{\wedge\ \&\ \rightarrow})$ implies $(C_\rightarrow)$ it follows that $x \wedge y = [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))] \overset{(C_\rightarrow)}{=} [x \odot (x \to y)] \vee [y \odot (y \to x)]$, so $(C_\wedge)$ holds. □

Theorem 2. *Let L be a residuated lattice. L has $(C_{\wedge\ \&\ \rightarrow})$ property if and only if*

$$(x \wedge y) \to z = [((y \to x) \to (x \to y)) \to (x \to z)] \wedge [((x \to y) \to (y \to x)) \to (y \to z)],$$

for every $x, y, z \in L$.

Proof. First, we suppose that L has $(C_{\wedge\ \&\ \rightarrow})$ property.

We deduce that

$$([x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]) \to z = (x \wedge y) \to z \overset{(c_9)}{\Leftrightarrow}$$

$$[x \odot ((y \to x) \to (x \to y)) \to z] \wedge [y \odot ((x \to y) \to (y \to x)) \to z] = (x \wedge y) \to z \Leftrightarrow$$

$$(x \wedge y) \to z = [((y \to x) \to (x \to y)) \to (x \to z)] \wedge [((x \to y) \to (y \to x)) \to (y \to z)].$$

Conversely, for $z = x$ we obtain $1 = ((x \to y) \to (y \to x)) \to (y \to x)$, so $(x \to y) \to (y \to x) = y \to x$ for every $x, y \in L$ and $(C_\rightarrow)$ holds.

For $z = x \wedge y$ we obtain $[x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))] \leq x \wedge y$ and for $z = [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]$

we obtain $x \wedge y \leq [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))]$. We deduce that $x \wedge y = [x \odot ((y \to x) \to (x \to y))] \vee [y \odot ((x \to y) \to (y \to x))] \overset{(C_{\to})}{=} [x \odot (x \to y)] \vee [y \odot (y \to x)]$, so $(C_{\wedge})$ holds. $\square$

Remark 1. *MTL algebras and divisible residuated lattices have $(C_{\wedge\ \&\ \to})$ property. Indeed, since $(prel) \Rightarrow (C_{\wedge}), (C_{\to})$ and $(div) \Rightarrow (C_{\wedge}), (C_{\to})$ we deduce that $(prel) \Rightarrow (C_{\wedge}) + (C_{\to}) \Leftrightarrow (C_{\wedge\ \&\ \to})$ and $(div) \Rightarrow (C_{\wedge}) + (C_{\to}) \Leftrightarrow (C_{\wedge\ \&\ \to})$.*

Theorem 3. *Let L be a residuated lattice with $(C_{\wedge\ \&\ \to})$ property. Then $(x \to y) \odot (y \to x) = (x \to y) \wedge (y \to x)$, for every $x, y \in L$.*

Proof. Since from Theorem 1, $(C_{\wedge\ \&\ \to}) \Leftrightarrow (C_{\wedge}) + (C_{\to})$ we deduce that

$$(x \to y) \odot (y \to x) = [(x \to y) \odot (y \to x)] \vee [(y \to x) \odot (x \to y)] \overset{(C_{\to})}{=}$$

$$= [(x \to y) \odot ((x \to y) \to (y \to x))] \vee [(y \to x) \odot ((y \to x) \to (x \to y))] \overset{(C_{\wedge})}{=}$$

$$= (x \to y) \wedge (y \to x).$$ $\square$

Remark 2. *We consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from [6], Example 5.11, p.152. Then L is not an MTL algebra and neither a divisible residuated lattice. Moreover, $(b \to c) \odot (c \to b) = d \odot d = 0 \neq d = d \wedge d = (b \to c) \wedge (c \to b)$.*

Open problem. Find an example of residuated lattice with $(x \to y) \odot (y \to x) = (x \to y) \wedge (y \to x)$, for every x, y, not verifying $(C_{\wedge\ \&\ \to})$ property.

Remark 3. *Every MTL algebras satisfies the equality $(x \to y) \odot (y \to x) = (x \to y) \wedge (y \to x)$, for every $x, y \in L$. Indeed since $(x \to y) \vee (y \to x) = 1 \Rightarrow (x \to y) \odot (y \to x) = (x \to y) \wedge (y \to x)$.*

Remark 4. *Every divisible residuated lattice satisfies the equality $(x \to y) \odot (y \to x) = (x \to y) \wedge (y \to x)$, for every $x, y \in L$. Indeed, $(x \to y) \wedge (y \to x) = (x \to y) \odot [(x \to y) \to (y \to x)] = (x \to y) \odot (y \to x)$.*

Remark 5. *There are residuated lattices which have $(C_{\to})$ property but do not have $(C_{\wedge\ \&\ \to})$ property. For example the residuated lattice $L = \{0, n, a, b, c, d, m, 1\}$ from Example 1, pp. 243, [10], is a residuated lattice with $(C_{\to})$ property. We have $[a \odot ((d \to a) \to (a \to d))] \vee [d \odot ((a \to d) \to (d \to a))] = [a \odot (a \to d)] \vee [d \odot (d \to a)] = (a \odot d) \vee (d \odot a) = 0 \vee 0 = 0$, but $a \wedge d = n \neq 0$.*

Remark 6. *There are residuated lattices which have* $(C_\wedge)$ *property but do not have* $(C_{\wedge\ \&\ \rightarrow})$ *property. For example the residuated lattice* $L = \{0, a, b, c, d, 1\}$ *from Example 1, pp. 249, [10], is a residuated lattice with* $(C_\wedge)$ *property. We have* $[b \odot ((c \rightarrow b) \rightarrow (b \rightarrow c))] \vee [c \odot ((b \rightarrow c) \rightarrow (c \rightarrow b))] = [b \odot (d \rightarrow d)] \vee [c \odot (d \rightarrow d)] = (b \odot 1) \vee (c \odot 1) = b \vee c = d$*, but* $b \wedge c = a \neq d$*.*

3 Filters in residuated lattices

Definition 4. *([13]) A nonempty subset* F *of a residuated lattice* L *is called* deductive system *if :*

(F_1) $1 \in F$;

(F_2) *If* $x, x \rightarrow y \in F$*, then* $y \in F$*.*

An equivalent definition for a deductive system is ([13]):

(F'_1) If $x \leq y$ and $x \in F$, then $y \in F$;

(F'_2) If x, $y \in F$, then $x \odot y \in F$.

Following this equivalence, a deductive system of L is also called an *implicative filter* (*i-filter* or *filter*, for short). We denote by $\mathbf{F(L)}$ the set of all filters of L. If $F, G \in \mathbf{F(L)}$ then $F \cap G \in \mathbf{F(L)}$, see [13].

Lemma 5. *Let* $F \in \mathbf{F(L)}$*,* $x, y, z \in L$ *such that* $x \rightarrow y, y \rightarrow z \in F$*. Then* $x \rightarrow z \in F$*.*

Proof. Since $x \rightarrow y, y \rightarrow z \in F$ we deduce that $(x \rightarrow y) \odot (y \rightarrow z) \in F$. From (c_{11}), $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$, so $x \rightarrow z \in F$. ☐

If $F \in \mathbf{F(L)}$, then the relation $\sim_F$ defined on L by $(x, y) \in \sim_F$ iff $x \rightarrow y, y \rightarrow x \in F$ iff $(x \rightarrow y) \odot (y \rightarrow x) \in F$ is a congruence relation on L (see [13]).

The quotient algebra $L/\sim_F$ denoted by L/F becomes a residuated lattice in a natural way, with the operations induced from those of L.

For $x \in L$ we denote by x/F the congruence class of x modulo $\sim_F$. The order relation on L/F is given by $x/F \leq y/F$ iff $x \rightarrow y \in F$. Obviously, $x/F = 1/F$ iff $x \in F$.

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{RL}$ of residuated lattices.

Definition 5. *([4]) A filter* $F \in \mathbf{F(L)}$ *will be called a* $\mathcal{V}$ filter *if* $L/F \in \mathcal{V}$*.*

We denote by $\mathcal{V}\mathbf{F(L)}$ the set of all $\mathcal{V}$ filters of L.

In what follows we will consider in place of $\mathcal{V}$ different subvarieties of residuated lattices.

3.1 MTL filters, divisible filters and BL filters in a residuated lattice

Definition 6. *([15]) A filter F of a residuated lattice L is called* MTL filter *if L/F is an MTL algebra.*

We denote by $\mathbf{MTLF(L)}$ the set of all MTL filters of L.

Theorem 4. *([15]) For $F \in \mathbf{F(L)}$ the following conditions are equivalent:*

(i) $F \in \mathbf{MTLF(L)}$;

(ii) $(x \to y) \vee (y \to x) \in F$ *for every* $x, y \in L$.

Lemma 6. *Let F be a filter of a residuated lattice L. $F \in \mathbf{MTLF(L)}$ if and only if $[(x \vee y) \to x] \vee [(x \vee y) \to y] \in F$ for every $x, y \in L$.*

Proof. Obviously, from Theorem 4 and (c_9) we have $[(x \vee y) \to x] \vee [(x \vee y) \to y] = [(x \to x) \wedge (y \to x)] \vee [(x \to y) \wedge (y \to y)] = (y \to x) \vee (x \to y)$. □

Definition 7. *([5]) A filter F of a residuated lattice L is called* divisible filter *if L/F is a divisible residuated lattice.*

We denote by $\mathbf{DivF(L)}$ the set of all divisible filters of L.

Theorem 5. *([5]) For $F \in \mathbf{F(L)}$ the following conditions are equivalent:*

(i) $F \in \mathbf{DivF(L)}$;

(ii) $(x \wedge y) \to [x \odot (x \to y)] \in F$ *for every* $x, y \in L$.

Lemma 7. *A filter F of a residuated lattice L is divisible if and only if $[x \odot (x \to y) \to y \odot (y \to x)] \to [y \odot (y \to x) \to x \odot (x \to y)] \in F$, for every $x, y \in L$.*

Proof. If $F \in \mathbf{DivF(L)}$, then L/F is divisible, so, using Lemma 2, $[x \odot (x \to y) \to y \odot (y \to x)] \to [y \odot (y \to x) \to x \odot (x \to y)] \in F$.

Conversely, for $y = x \wedge y$, we obtain $(x \wedge y) \to [x \odot (x \to y)] \in F$. Using Theorem 5, we deduce that $F \in \mathbf{DivF(L)}$. □

Definition 8. *([5]) A filter F of a residuated lattice L is called* BL filter *if L/F is a BL algebra.*

We denote by $\mathbf{BLF(L)}$ the set of all BL filters of L.

In [5] it is proved the following result : $\mathbf{BLF(L)} = \mathbf{MTLF(L)} \cap \mathbf{DivF(L)}$.

3.2 $C_{\to}$ filters in a residuated lattice

Definition 9. *A filter F of a residuated lattice L is called* $C_{\to}$ filter *if L/F is a residuated lattice with $C_{\to}$ property.*

We denote by $\mathbf{C_{\to}F(L)}$ the set of all $C_{\to}$ filters of L.

Example 2. *We consider the residuated lattices $L = \{0, a, b, c, d, 1\}$ from Remark 7. L has $C_{\to}$ property and $F = \{1\}$ is a $C_{\to}$ filter of L since $L/F \approx L$.*

Theorem 6. *For a filter F of a residuated lattice L the following conditions are equivalent:*

(*i*) $F \in \mathbf{C_{\to}F(L)}$;

(*ii*) $[(x \to y) \to (y \to x)] \to (y \to x) \in F$, *for every* $x, y \in L$;

(*iii*) *If for* $x, y, z \in L$, $(x \to y) \to (y \to (z \to x)) \in F$ *then* $y \to (z \to x) \in F$;

(*iv*) $[(x \to y) \to (y^n \to x)] \to (y^n \to x) \in F$, *for every natural number* $n \geq 1$ *and every* $x, y \in L$;

(*v*) *If for* $x, y, z \in L$ *with* $z \leq y$, $(x \to y) \to (z \to x) \in F$ *then* $z \to x \in F$;

(*vi*) $[(x \to y) \to (y \to x)] \to [((y \to x) \to z) \to z] \in F$, *for every* $x, y, z \in L$;

(*vii*) $[(x \to y) \to (y \to (z \to x))] \to (y \to (z \to x)) \in F$, *for every* $x, y, z \in L$;

(*viii*) $[(x \to y) \to (y \to x)] \to [(x \to z) \to (y \to z)] \in F$, *for every* $x, y, z \in L$.

Proof. $(i) \Rightarrow (ii)$. We suppose that $F \in \mathbf{C_{\to}F(L)}$. Then L/F is a residuated lattice with $C_{\to}$ property so $(x/F \to y/F) \to (y/F \to x/F) = y/F \to x/F$, for every $x, y \in L$. Thus $[(x/F \to y/F) \to (y/F \to x/F)] \to (y/F \to x/F) = 1/F$ and $[(x \to y) \to (y \to x)] \to (y \to x) \in F$.

$(ii) \Rightarrow (i)$. We suppose that $[(x \to y) \to (y \to x)] \to (y \to x) \in F$, for every $x, y \in L$. Since by (c_4), $y \to x \leq (x \to y) \to (y \to x)$ we deduce that $(y \to x) \to [(x \to y) \to (y \to x)] = 1 \in F$, so $(x/F \to y/F) \to (y/F \to x/F) = y/F \to x/F$, for every $x, y \in L$, thus L/F is a residuated lattice with $C_{\to}$ property and $F \in \mathbf{C_{\to}F(L)}$.

$(ii) \Rightarrow (iii)$. Suppose that $(x \to y) \to (y \to (z \to x)) \in F$, so $z \to [(x \to y) \to (y \to x)] \in F$. Since by hypothesis, $[(x \to y) \to (y \to x)] \to (y \to x) \in F$, from Lemma 5 we deduce that $z \to (y \to x) \in F$, so $y \to (z \to x) \in F$.

$(iii) \Rightarrow (ii)$. Since $1 = [(x \to y) \to (y \to x)] \to [(x \to y) \to (y \to x)] \overset{(c_8)}{=} ((x \to y) \odot [(x \to y) \to (y \to x)]) \to (y \to x) = (x \to y) \to [y \to (((x \to y) \to (y \to x)) \to x)]$

$\in F$, by hypothesis, $y \to [((x \to y) \to (y \to x)) \to x] \in F$, i.e., $[(x \to y) \to (y \to x)] \to (y \to x) \in F$.

$(ii) \Rightarrow (iv)$. In (ii) for a natural number $n \geq 1$ replacing y by y^n we obtain $[(x \to y^n) \to (y^n \to x)] \to (y^n \to x) \in F$, for every $x, y \in L$. Since $y^n \leq y \overset{(c_3)}{\Rightarrow} x \to y^n \leq x \to y \Rightarrow (x \to y) \to (y^n \to x) \overset{(c_3)}{\leq} (x \to y^n) \to (y^n \to x) \overset{(c_3)}{\Rightarrow} [(x \to y^n) \to (y^n \to x)] \to (y^n \to x) \leq [(x \to y) \to (y^n \to x)] \to (y^n \to x)$. Since F is a filter it follows that $[(x \to y) \to (y^n \to x)] \to (y^n \to x) \in F$.

$(iv) \Rightarrow (ii)$. For $n = 1$.

$(v) \Rightarrow (iii)$. Let $x, y, z \in L$ such that $(x \to y) \to (y \to (z \to x)) \in F$, i.e., $(x \to y) \to (y \odot z \to x) \in F$. Since $y \odot z \leq y$, from (v) we deduce that $(y \odot z) \to x = y \to (z \to x) \in F$.

$(ii) \Rightarrow (v)$. Let $x, y, z \in L$ with $z \leq y$ such that $(x \to y) \to (z \to x) \in F$. From (ii), $[(x \to z) \to (z \to x)] \to (z \to x) \in F$.

Since $z \leq y$ from (c_3), $x \to z \leq x \to y$ so $(x \to y) \to (z \to x) \leq (x \to z) \to (z \to x)$. But F is a filter, so $(x \to z) \to (z \to x) \in F$. We deduce that $z \to x \in F$.

$(ii) \Rightarrow (vi)$. Since $[(x \to y) \to (y \to x)] \to (y \to x) \in F$ and $[(x \to y) \to (y \to x)] \to (y \to x) \overset{(c_6)}{\leq} [(y \to x) \to z] \to [((x \to y) \to (y \to x)) \to z]$ $\overset{(c_8)}{=} [(x \to y) \to (y \to x)] \to [((y \to x) \to z) \to z]$ for every $x, y, z \in L$ it results that $[(x \to y) \to (y \to x)] \to [((y \to x) \to z) \to z] \in F$.

$(vi) \Rightarrow (ii)$. For $z = x$ we have $[(x \to y) \to (y \to x)] \to [((y \to x) \to x) \to x] \overset{(c_4)}{=} [(x \to y) \to (y \to x)] \to (y \to x) \in F$.

$(ii) \Rightarrow (vii)$. We have $[(x \to y) \to (y \to (z \to x))] \to (y \to (z \to x))$ $\overset{(c_8)}{=} [(x \to y) \to (z \to (y \to x))] \to [z \to (y \to x)] = [z \to ((x \to y) \to (y \to x))] \to [z \to (y \to x)]$. Using (c_6) we deduce that $[(x \to y) \to (y \to x)] \to (y \to x) \leq [z \to ((x \to y) \to (y \to x))] \to [z \to (y \to x)]$. From (ii), it results that $[(x \to y) \to (y \to (z \to x))] \to (y \to (z \to x)) \in F$, for every $x, y, z \in L$.

$(vii) \Rightarrow (ii)$. For $z = 1$ we obtain (ii).

$(ii) \Rightarrow (viii)$. Since $[(x \to y) \to (y \to x)] \to (y \to x) \in F$ and $1 = (y \to x) \to [(x \to z) \to (y \to z)] \in F$, applying Lemma 5 we deduce that $[(x \to y) \to (y \to x)] \to [(x \to z) \to (y \to z)] \in F$, for every $x, y, z \in L$.

$(viii) \Rightarrow (ii)$. We take $z = x$. $\square$

Theorem 7. *Let L be a residuated lattice. L has $C_{\to}$ property if and only if any filter of L is a $C_{\to}$ filter.*

Proof. We suppose that L has $C_{\to}$ property and let F be a filter of L. Since

$1 \in F$ we deduce that $[(x \to y) \to (y \to x)] \to (y \to x) = 1 \in F$, for every $x, y \in L$, so $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$.

Conversely, if any filter of L is a $C_{\to}$ filter, then $\{1\}$ is a $C_{\to}$ filter, so $(x \to y) \to (y \to x) \to (y \to x) = 1$, thus $(x \to y) \to (y \to x) \leq (y \to x)$ for every $x, y \in L$.

Since by (c_4), $y \to x \leq (x \to y) \to (y \to x)$ we deduce that $(x \to y) \to (y \to x) = (y \to x)$ for every $x, y \in L$, thus L has $C_{\to}$ property. □

Corollary 1. *A residuated lattice L has $C_{\to}$ property if and only if $\{1\}$ is a $C_{\to}$ filter of L.*

Proposition 1. *Let L be a residuated lattice and $F, G \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$. Then $F \cap G \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F, G \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$, then by Theorem 6, (ii), $[(x \to y) \to (y \to x)] \to (y \to x)$ is in F and also in G, so $[(x \to y) \to (y \to x)] \to (y \to x) \in F \cap G$, that is, $F \cap G \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$. □

Proposition 2. *Suppose that F and G are two filters of a residuated lattice L and $F \subseteq G$. If $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$, then $G \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$, from Theorem 6, (ii), we deduce that $[(x \to y) \to (y \to x)] \to (y \to x) \in F$. By hypothesis, $F \subseteq G$, so $[(x \to y) \to (y \to x)] \to (y \to x) \in G$, for every $x, y \in L$. □

Lemma 8. *Let $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$. Then*

(i) *$[(x \to z) \to (y \to x)] \to (y \to x) \in F$, for every $x, y, z \in L$ with $y \leq z$;*

(ii) *$[(y^* \to x^*) \to (y \to x)] \to (y \to x) \in F$, for every $x, y \in L$;*

(iii) *$(x \to y) \odot [(x \to y) \to (y \to x)] \to (x \to y) \odot (y \to x) \in F$, for every $x, y \in L$.*

Proof. (i). Let $x, y, z \in L$ with $y \leq z$. Then $x \to y \leq x \to z$, so by (c_3), $(x \to z) \to (y \to x) \leq (x \to y) \to (y \to x)$ and $[(x \to y) \to (y \to x)] \to (y \to x)$ $\leq [(x \to z) \to (y \to x)] \to (y \to x)$. Since $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$ from Theorem 6 (ii), $[(x \to y) \to (y \to x)] \to (y \to x) \in F$ so we obtain $[(x \to z) \to (y \to x)] \to (y \to x) \in F$.

(ii). From (i) for $z = y^{**}$, since $y \leq y^{**}$, we deduce that $[(x \to y^{**}) \to (y \to x)] \to (y \to x) \overset{(c_{10})}{=} [(y^* \to x^*) \to (y \to x)] \to (y \to x) \in F$.

(iii). Since $F \in \mathbf{C}_{\to}\mathbf{F}(\mathbf{L})$ from Theorem 6 (ii), $[(x \to y) \to (y \to x)] \to (y \to x) \in F$, for every $x, y \in L$. But $[(x \to y) \to (y \to x)] \to (y \to x) \overset{(c_7)}{\leq} (x \to y) \odot [(x \to y) \to (y \to x)] \to (x \to y) \odot (y \to x)$. We deduce that $(x \to y) \odot [(x \to y) \to (y \to x)] \to (x \to y) \odot (y \to x) \in F$, for every $x, y \in L$. □

3.3 $C_\vee$ filters in a residuated lattice

Definition 10. *A filter F of a residuated lattice L is called* $\mathrm{C}_\vee$ filter *if L/F is a residuated lattice with $C_\vee$ property.*

We denote by $\mathbf{C}_\vee\mathbf{F}(\mathbf{L})$ the set of all $C_\vee$ filters of L.

Theorem 8. *For a filter F of a residuated lattice L the following conditions are equivalent:*

(*i*) $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$;

(*ii*) $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$, *for every* $x, y \in L$;

(*iii*) *If* $((x \to y) \to (z \to y)) \wedge ((y \to x) \to (z \to x)) \in F$ *then* $z \to (x \vee y) \in F$, *for* $x, y, z \in L$.

Proof. $(i) \Rightarrow (ii)$. If $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$, L/F is a residuated lattice with $C_\vee$ property so $[(x/F \to y/F) \to y/F] \wedge [(y/F \to x/F) \to x/F] = x/F \vee y/F$, for every $x, y \in L$. Thus $([(x/F \to y/F) \to y/F] \wedge [(y/F \to x/F) \to x/F]) \to (x/F \vee y/F) = 1/F$ and $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$, for every $x, y \in L$.

$(ii) \Rightarrow (i)$. We suppose that $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$, for every $x, y \in L$. Since $x, y \leq (x \to y) \to y$ and $x, y \leq (y \to x) \to x$ it follows that $x \vee y \leq (x \to y) \to y$ and $x \vee y \leq (y \to x) \to x$. We deduce that $(x \vee y) \to [((x \to y) \to y) \wedge ((y \to x) \to x)] = 1 \in F$, so $(x/F \vee y/F) = [((x/F \to y/F) \to y/F) \wedge ((y/F \to x/F) \to x/F)]$ for every $x, y \in L$, thus L/F is a residuated lattice with $C_\vee$ property and $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$.

$(ii) \Rightarrow (iii)$. Suppose that $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$, for every $x, y \in L$. We have $[(x \to y) \to (z \to y)] \wedge [(y \to x) \to (z \to x)] \overset{(c_8)}{=}$ $[z \to ((x \to y) \to y)] \wedge [z \to ((y \to x) \to x)] \overset{(c_9)}{=} z \to [(x \to y) \to y) \wedge ((y \to x) \to x)]$. By hypothesis and Lemma 5 we deduce that $z \to (x \vee y) \in F$.

$(iii) \Rightarrow (ii)$. Since $((x \to y) \to (z \to y)) \wedge ((y \to x) \to (z \to x)) = z \to [((x \to y) \to y) \wedge ((y \to x) \to x)]$ and $1 = [((x \to y) \to y) \wedge ((y \to x) \to x)] \to [((x \to y) \to y) \wedge ((y \to x) \to x)] \in F$, by hypothesis, $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$. $\square$

Theorem 9. *Let L be a residuated lattice. L has $C_\vee$ property if and only if any filter of L is a $C_\vee$ filter.*

Proof. We suppose that L has $C_\vee$ property and let F be a filter of L. Since $1 \in F$ we deduce that $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) = 1 \in F$, for every $x, y \in L$, so $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$. Conversely, if any filter of L is a $C_\vee$ filter, then $\{1\}$ is a $C_\vee$ filter, so $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) = 1$, so $((x \to y) \to y) \wedge ((y \to x) \to x) \leq x \vee y$, for every $x, y \in L$. Since $x \vee y \leq ((x \to y) \to y) \wedge ((y \to x) \to x)$ we conclude that $((x \to y) \to y) \wedge ((y \to x) \to x) = x \vee y$ for every $x, y \in L$, thus L has $C_\vee$ property. $\square$

Corollary 2. *A residuated lattice L has $C_\vee$ property if and only if $\{1\}$ is a $C_\vee$ filter of L.*

Example 3. *We consider the following residuated lattice ([10], Example 1, pp. 249): Let $L = \{0, a, b, c, d, 1\}$, with $0 < a < b, c < d < 1$, but b, c are incomparable, with the operations:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	d	1	d	1	1
c	b	d	d	1	1	1
d	a	d	d	d	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

.

The lattice L has $(C_\wedge)$ and $(C_\vee)$ properties, so $F = \{1\}$ is a $C_\vee$ filter of L.

Proposition 3. *Let L be a residuated lattice and $F, G \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$. Then $F \cap G \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F, G \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$, then by Theorem 8, (ii), $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y)$ is in F and also in G, so $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F \cap G$, that is, $F \cap G \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$. $\square$

Proposition 4. *Suppose that F and G are two filters of a residuated lattice L and $F \subseteq G$. If $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$, then $G \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F \in \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$, from Theorem 8, (ii), we deduce that $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in F$. By hypothesis, $F \subseteq G$, so $[((x \to y) \to y) \wedge ((y \to x) \to x)] \to (x \vee y) \in G$, for every $x, y \in L$. $\square$

3.4 $C_\wedge$ filters in a residuated lattice

Definition 11. *A filter F of a residuated lattice L is called* $\mathrm{C}_\wedge$ filter *if L/F is a residuated lattice with $C_\wedge$ property.*

We denote by $\mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$ the set of all $C_\wedge$ filters of L.

Theorem 10. *For a filter F of a residuated lattice L the following conditions are equivalent:*

(*i*) $F \in \mathbf{C}_\wedge \mathbf{F}(\mathbf{L})$;

(*ii*) $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F$, *for every* $x, y \in L$;

(*iii*) *If* $[(x \to y) \to (x \to z)] \wedge [(y \to x) \to (y \to z)] \in F$ *then* $(x \wedge y) \to z \in F$, *for* $x, y, z \in L$.

Proof. $(i) \Rightarrow (ii)$. If $F \in \mathbf{C}_\wedge \mathbf{F}(\mathbf{L})$, L/F is a residuated lattice with $C_\wedge$ property so $x/F \wedge y/F = [x/F \odot (x/F \to y/F)] \vee [y/F \odot (y/F \to x/F)]$, for every $x, y \in L$. Thus $(x/F \wedge y/F) \to [x/F \odot (x/F \to y/F) \vee y/F \odot (y/F \to x/F)] = 1/F$ so $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F$, for every $x, y \in L$.

$(ii) \Rightarrow (i)$. We suppose that $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F$ for every $x, y \in L$. Since $x \odot (x \to y) \leq x, y$ and $y \odot (y \to x) \leq x, y$ we deduce that $x \odot (x \to y) \leq x \wedge y$ and $y \odot (y \to x) \leq x \wedge y$, so $[x \odot (x \to y)] \vee [y \odot (y \to x)] \leq x \wedge y$.

Thus, $([x \odot (x \to y)] \vee [y \odot (y \to x)]) \to (x \wedge y) = 1 \in F$, so $x/F \wedge y/F = [x/F \odot (x/F \to y/F)] \vee [y/F \odot (y/F \to x/F)]$ for every $x, y \in L$, i.e., L/F is a residuated lattice with $C_\wedge$ property and $F \in \mathbf{C}_\wedge \mathbf{F}(\mathbf{L})$.

$(ii) \Rightarrow (iii)$. Let $x, y, z \in L$ such that $((x \to y) \to (x \to z)) \wedge ((y \to x) \to (y \to z)) \in F$. We have $[(x \to y) \to (x \to z)] \wedge [(y \to x) \to (y \to z)] = [(x \odot (x \to y)) \to z] \wedge [(y \odot (y \to x)) \to z] \overset{(c_9)}{=} [(x \odot (x \to y)) \vee (y \odot (y \to x))] \to z$. Since by hypothesis, $(x \wedge y) \to [(x \odot (x \to y)) \vee (y \odot (y \to x))] \in F$, from Lemma 5 we deduce that $(x \wedge y) \to z \in F$.

$(iii) \Rightarrow (ii)$. Since $1 = [(x \odot (x \to y)) \vee (y \odot (y \to x))] \to [(x \odot (x \to y)) \vee (y \odot (y \to x))]$, then for $z = x \odot (x \to y) \vee y \odot (y \to x)$, we obtain $[(x \to y) \to (x \to z)] \wedge [(y \to x) \to (y \to z)] = 1 \in F$. Now, by hypothesis we obtain $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F$. □

Theorem 11. *Let L be a residuated lattice. L has $C_\wedge$ property if and only if any filter of L is a $C_\wedge$ filter.*

Proof. We suppose that L has $C_\wedge$ property and let F be a filter of L. Since $1 \in F$ we deduce that $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] = 1 \in F$, for every $x, y \in L$, so $F \in \mathbf{C}_\wedge \mathbf{F}(\mathbf{L})$.

Conversely, if any filter of L is a $C_\wedge$ filter, then $\{1\}$ is a $C_\wedge$ filter, so $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] = 1$, so $x \wedge y \leq x \odot (x \to y) \vee y \odot (y \to x)$, for every $x, y \in L$. Since $x \odot (x \to y) \vee y \odot (y \to x) \leq x \wedge y$ we conclude that $x \wedge y = x \odot (x \to y) \vee y \odot (y \to x)$ for every $x, y \in L$, thus L has $C_\wedge$ property. □

Corollary 3. *A residuated lattice L has $C_\wedge$ property if and only if $\{1\}$ is a $C_\wedge$ filter of L.*

Example 4. *If we consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example 3 then $F = \{1\}$ is a $C_\wedge$ filter of L.*

Proposition 5. *Let L be a residuated lattice and $F, G \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$. Then $F \cap G \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F, G \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$, then by Theorem 10, (ii), $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)]$ is in F and also in G, so $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F \cap G$, that is, $F \cap G \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$. □

Proposition 6. *Suppose that F and G are two filters of a residuated lattice L and $F \subseteq G$. If $F \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$, then $G \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$.*

Proof. Let $x, y \in L$. Since $F \in \mathbf{C}_\wedge\mathbf{F}(\mathbf{L})$, from Theorem 10, (ii), we deduce that $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in F$. By hypothesis, $F \subseteq G$, so $(x \wedge y) \to [x \odot (x \to y) \vee y \odot (y \to x)] \in G$, for every $x, y \in L$. □

4 Some decompositions of filters in residuated lattices

Theorem 12. *In any residuated lattice L,*

$$\mathbf{MTLF}(\mathbf{L}) = \mathbf{C}_\to\mathbf{F}(\mathbf{L}) \cap \mathbf{C}_\vee\mathbf{F}(\mathbf{L}).$$

Proof. $F \in \mathbf{C}_\to\mathbf{F}(\mathbf{L}) \cap \mathbf{C}_\vee\mathbf{F}(\mathbf{L})$ iff L/F is a residuated lattice with $C_\to$ and $C_\vee$ properties iff L/F is an MTL algebra (see [10], Theorem 3.2.1) iff $F \in \mathbf{MTLF}(\mathbf{L})$. □

Remark 7. *Every MTL filter of a residuated lattice L is a $C_\to$ filter of L, i.e.,*

$$\mathbf{MTLF}(\mathbf{L}) \subsetneq \mathbf{C}_\to\mathbf{F}(\mathbf{L}).$$

Indeed, from Theorem 12, $\mathbf{MTLF}(\mathbf{L}) \subseteq \mathbf{C}_\to\mathbf{F}(\mathbf{L})$. We show that $\mathbf{MTLF}(\mathbf{L}) \neq \mathbf{C}_\to\mathbf{F}(\mathbf{L})$. We consider the following example ([10], Example 1, pp. 240) of a residuated lattice which has $\mathbf{C}_\to$ property. Let $L = \{0, a, b, c, d, 1\}$, with $0 < a < b, c < d < 1$, but b, c are incomparable, and the following operations:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	c	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	c	c	c
d	0	0	b	c	d	d
1	0	a	b	c	d	1

.

From Corollary 1, $F = \{1\}$ is a $\mathbf{C}_{\rightarrow}$ filter but it is not an MTL filter since $(b \to c) \vee (c \to b) = c \vee b = d \notin F$.

Remark 8. *Every MTL filter of a residuated lattice L is a $C_{\vee}$ filter of L, i.e.,*

$$\mathbf{MTLF(L)} \subsetneqq \mathbf{C}_{\vee}\mathbf{F(L)}.$$

Indeed, from Theorem 12, $\mathbf{MTLF(L)} \subseteq \mathbf{C}_{\vee}\mathbf{F(L)}$. We show that $\mathbf{MTLF(L)} \neq \mathbf{C}_{\vee}\mathbf{F(L)}$. We consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example 3. L has $\mathbf{C}_{\vee}$ property. From Corollary 2, $F = \{1\}$ is a $\mathbf{C}_{\vee}$ filter but it is not an MTL filter since $(b \to c) \vee (c \to b) = d \vee d = d \notin F$.

Proposition 7. *In any residuated lattice L, every MTL filter of L is a $\mathbf{C}_{\wedge}$ filter of L, i.e.,*

$$\mathbf{MTLF(L)} \subsetneqq \mathbf{C}_{\wedge}\mathbf{F(L)}.$$

Proof. Suppose that $F \in \mathbf{MTLF(L)}$. Then L/F is an MTL algebra so L/F has $\mathbf{C}_{\wedge}$ property (see [10], Proposition 3.2.5) and finally $F \in \mathbf{C}_{\wedge}\mathbf{F(L)}$.

Obviously, $\mathbf{C}_{\wedge}\mathbf{F(L)} \neq \mathbf{MTLF(L)}$. Indeed, if we consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example 3 which has $\mathbf{C}_{\wedge}$ property, then $F = \{1\}$ is a $\mathbf{C}_{\wedge}$ but is not an MTL filter, see Remark 8. $\square$

Proposition 8. *In any residuated lattice L, every divisible filter of L is a $\mathbf{C}_{\rightarrow}$ filter of L, i.e.,*

$$\mathbf{DivF(L)} \subsetneqq \mathbf{C}_{\rightarrow}\mathbf{F(L)}.$$

Proof. Suppose that $F \in \mathbf{DivF(L)}$. Then L/F is a divisible residuated lattice so L/F has $\mathbf{C}_{\rightarrow}$ property (see [10], Proposition 3.2.3) and finally $F \in \mathbf{C}_{\rightarrow}\mathbf{F(L)}$.

Obviously, $\mathbf{C}_{\rightarrow}\mathbf{F(L)} \neq \mathbf{DivF(L)}$. Indeed, if we consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example by Remark 7, then $F = \{1\}$ is a $\mathbf{C}_{\rightarrow}$ filter but it is not a divisible filter since $(b \wedge c) \to [b \odot (b \to c)] = a \to (b \odot c) = a \to 0 = d \notin F$. $\square$

Proposition 9. *In any residuated lattice L, every divisible filter of L is a $\mathbf{C}_{\wedge}$ filter of L, i.e.,*

$$\mathbf{DivF(L)} \subsetneqq \mathbf{C}_{\wedge}\mathbf{F(L)}.$$

Proof. Suppose that $F \in \mathbf{DivF(L)}$. Then L/F is a divisible residuated lattice so L/F has $\mathbf{C}_{\wedge}$ property (see [10], Proposition 3.2.4) and finally $F \in \mathbf{C}_{\wedge}\mathbf{F(L)}$. Obviously, $\mathbf{C}_{\wedge}\mathbf{F(L)} \neq \mathbf{DivF(L)}$. Indeed, if we consider the residuated lattice $L = \{0, a, b, c, d, 1\}$ from Example 3 which has $\mathbf{C}_{\wedge}$ property, then $F = \{1\}$ is a $\mathbf{C}_{\wedge}$ filter but it is not a divisible filter since $(b \wedge d) \to [d \odot (d \to b)] = b \to (d \odot d) = b \to a = d \notin F$. $\square$

Corollary 4. *In any residuated lattice L,*

$$\mathbf{BLF(L)} = \mathbf{DivF(L)} \cap \mathbf{C}_{\vee}\mathbf{F(L)}.$$

Proof. Since $\mathbf{BLF(L)} = \mathbf{MTLF(L)} \cap \mathbf{DivF(L)}$ (see [5]) and $\mathbf{MTLF(L)} \subseteq \mathbf{C}_{\vee}\mathbf{F(L)}$ we deduce that $\mathbf{BLF(L)} \subseteq \mathbf{C}_{\vee}\mathbf{F(L)} \cap \mathbf{DivF(L)}$. Conversely, let $F \in \mathbf{DivF(L)} \cap \mathbf{C}_{\vee}\mathbf{F(L)}$. Since $\mathbf{DivF(L)} \subseteq \mathbf{C}_{\rightarrow}\mathbf{F(L)}$ we deduce that $F \in \mathbf{C}_{\rightarrow}\mathbf{F(L)} \cap \mathbf{C}_{\vee}\mathbf{F(L)} = \mathbf{MTLF(L)}$. Thus,

$$F \in \mathbf{MTLF(L)} \cap \mathbf{DivF(L)} = \mathbf{BLF(L)}.$$

□

Definition 12. *A filter F of a residuated lattice L is called $C_{\wedge\&\rightarrow}$* filter *if L/F is a residuated lattice with $C_{\wedge\&\rightarrow}$ property.*

We denote by $\mathbf{C}_{\wedge\&\rightarrow}\mathbf{F(L)}$ the set of all $C_{\wedge\&\rightarrow}$ filters of L.

Theorem 13. *In any residuated lattice L,*

$$\mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)} = \mathbf{C}_{\rightarrow}\mathbf{F(L)} \cap \mathbf{C}_{\wedge}\mathbf{F(L)}.$$

Proof. $F \in \mathbf{C}_{\rightarrow}\mathbf{F(L)} \cap \mathbf{C}_{\wedge}\mathbf{F(L)}$ iff L/F is a residuated lattice with $C_{\rightarrow}$ and $C_{\wedge}$ properties iff L/F has $C_{\wedge\ \&\ \rightarrow}$ property (see Theorem 1) iff $F \in \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}$. □

From Theorems 7, 11, Corollary 1 and Corollary 3 we deduce that:

Corollary 5. *A residuated lattice L has $C_{\wedge\&\rightarrow}$ property if and only if any filter of L is a $C_{\wedge\&\rightarrow}$ filter if and only if $\{1\}$ is a $C_{\wedge\&\rightarrow}$ filter of L.*

Corollary 6. *In any residuated lattice L,*

$$\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} \subseteq \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}.$$

Proof. From Remark 7, Propositions 7, 8, 9 and Theorem 13. □

Corollary 7. *Let L be an MTL algebra. Then*

$$\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}.$$

Proof. Since L is an MTL algebra, every filter of L is an MTL filter and a $\mathbf{C}_{\wedge\ \&\ \rightarrow}$ filter (because, $(prel) \Rightarrow (C_{\rightarrow})$ and $(C_{\wedge})$), so, $\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{F(L)} \cup \mathbf{DivF(L)} = \mathbf{F(L)}$ and $\mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)} = \mathbf{F(L)}$.

We obtain that $\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}$. □

Corollary 8. *Let L be a divisible residuated lattice. Then*

$$\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}.$$

Proof. Since L is divisible, every filter of L is divisible and a $\mathbf{C}_{\wedge\ \&\ \rightarrow}$ filter (because, $(div) \Rightarrow (C_{\rightarrow})$ and $(C_{\wedge})$), so,

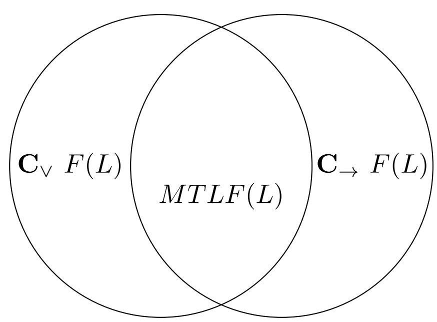
$$\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{MTLF(L)} \cup \mathbf{F(L)} = \mathbf{F(L)}$$

and $\mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)} = \mathbf{F(L)}$. We obtain that $\mathbf{MTLF(L)} \cup \mathbf{DivF(L)} = \mathbf{C}_{\wedge\ \&\ \rightarrow}\mathbf{F(L)}$. $\square$

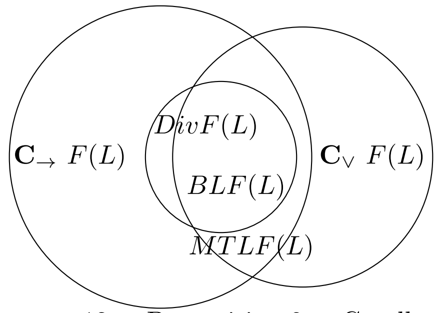
5 Conclusion and future work

In [10], A. Iorgulescu presents new properties, obtained by decomposing the old properties $(prel)$ and (div), and studies the new algebras obtained by adding these new properties to residuated lattices. Filters play an important role in studying algebraic structures. So, in this paper, we have added new classes of filters ($C_{\rightarrow}, C_{\vee}, C_{\wedge}$ and $C_{\wedge\ \&\ \rightarrow}$ filters) to the old classes of filters in residuated lattices (MTL filters and divisible filters). In this way, we refine the relations between different classes of filters and study the decompositions of MTL filters, BL filters and $C_{\wedge\&\rightarrow}$ filters in residuated lattices.

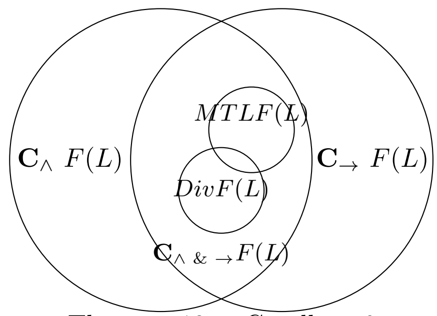
The connections between these filters are resumed in the following Figures:



Theorem 12



Theorem 12 + Proposition 8 + Corollary 4



Theorem 13 + Corollary 6

In our future work, we are going to consider other new decompositions of filters in residuated lattices.

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