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Existence and uniqueness of positive solution for nonlinear difference equations involving $p(k)$-Laplacian operator

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Abstract

In this paper, we deal with the existence of at least one and of at least two positive solutions as well the uniqueness of a positive solution for an anisotropic discrete non-linear problem involving $p(k)$-Laplacian with Dirichlet boundary value conditions. The technical approach for the existence part is based on a local minimum theorem and on a two critical points theorem for differentiable functionals, and for uniqueness part is based on a Lipschitzian continuous condition on the nonlinearity term.

1 Introduction

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with $p(k)$-Laplacian operator, because of their applications in many fields. Results on this topic are usually achieved by using various fixed point theorems in cone; see [4, 30] and references therein for details. This kind of problems play a fundamental role in different fields of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. Important tools in

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the study of nonlinear difference equations are fixed point methods and upper and lower solution techniques; see, for instance, [16, 24] and references therein. It is well known that critical point theory is an important tool to deal with the problems for differential equations. More, recently, for example in [8, 14, 15, 18, 20, 22, 23, 25, 26, 27, 28] the existence and multiplicity of solutions for nonlinear discrete boundary value problems have been investigated by adopting variational methods.

The aim of this paper is to establish the existence of positive solutions for the following discrete boundary-value problem

$$\begin{cases} -\Delta(w(k-1)\phi_{p(k-1)}(\Delta u(k-1))) + q(k)\phi_{p(k)}(u(k)) = \lambda f(k, u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{1}$$

for every $k \in [1, T]$, where $T \geq 2$ is a fixed positive integer, λ is a positive real parameter, $[1, T]$ is the discrete interval $\{1, ..., T\}$, $p : [0, T+1] \to [2, \infty)$, $w : [0, T] \to [1, \infty)$, $q : [1, T+1] \to [1, \infty)$ are given functions, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\phi_{p(\cdot)}(s) = |s|^{p(\cdot)-2}s$ is the one-dimensional discrete $p(\cdot)$-Laplacian operator and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is the continuous function, i.e. for every $k \in [1, T]$ the function $f(k, \cdot)$ is contiunuous. Other assumptions will be provided throughout the paper in accordance with a method which we will apply.

Throughout the text we will use the following notations:

$$p^+ := \max_{k\in[0,T+1]} p(k), \quad p^- := \min_{k\in[0,T+1]} p(k),$$

$$w^+ := \max_{k\in[0,T]} w(k), \quad \overline{w} := \sum_{k=1}^{T+1} w(k-1),$$

$$q^+ := \max_{k\in[1,T+1]} q(k), \quad \overline{q} := \sum_{k=1}^{T+1} q(k).$$

The problem under consideration, problem (1), is the discrete variant of a kind of the variable exponent of an anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(w_i(x)|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) + q(x)|u|^{p_i(x)-2}u = \lambda f(k, u), \quad x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is given function satisfying certain properties and $q(x) \geq 1$, $w_i(x) \geq 1$ and $p_i(x) \geq 2$ are continuous functions on $\overline{\Omega}$ for every $x \in \Omega$ and every $i \in \{1, 2, \cdots, N\}$, λ is a positive real number.

Based on a local minimum theorem, Theorem 2.1, we ensure an exact interval of the parameter λ, in which problem (1) admits at least one positive solution.

We refer to the paper [1, 19] in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for different one-dimensional problems with two-point boundary value condition.

As an example, here, we point out the following special case of our main result.

Theorem 1.1. *Let $T \geq 2$ be a fixed positive integer and let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, such that $\lim_{t\to 0^+} \frac{\int_0^t f(s)ds}{t^2} > 0$. Then for any*

$$\lambda \in \left]0, \frac{1}{T(T+4)\int_0^{(2T+2)^{\frac{2T+6}{T+4}}} f(\xi)d\xi}\right[$$

the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^k \Delta u(k-1)) + |u(k)|^{k+1}u(k) = \lambda f(u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

admits at least one positive solution in the space $\{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\}$.

Based on a two-non-zero critical points theorem, Theorem 2.2, we obtain the existance of at least two positive solutions for some values of the parameter λ.

The inspiration to study the multipicity of positive solutions lies in the paper [17], where the Authors have been examined the Dirichlet boundary value problem for difference equations involving the discrete p-Laplacian operator. We also refer to the paper [7] in which Theorem 2.2 has been successfully employed to show the existence of non-zero solutions for the second-order discrete boundary value problem.

The paper is arranged as follows. In Section 2 we recall some basic definitions and the main tools which we use to show our results. We also provide a few inequalities which play very important role in our investigations. In Section 3 we provide our main result that contains the existence theorem. In section 4 we focus on the multiplicity result and in Section 5 we consider the uniqueness of solution under suitable condition on the nonlinearity term. Examples are also provided.

2 Preliminaries

Our first tool and approach is based on the variational principle of Ricceri established in [31, Theorem 2.1] which we recall here, for reader's convenience, in the following form given in [12, Theorem 2.1].

Theorem 2.1. *(Bonanno and Molica Bisci [12, Theorem 2.1(a)]) Let X be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda = \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$ and for any $r > \inf_X \Phi$ let ϕ be the function defined as*

$$\phi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

Then, for any $r > \inf_X \Phi$ and for any $\lambda \in (0, \frac{1}{\phi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

Our second main tool and approach is based on a two non-zero critical points theorem established in [9], which is a suitable combination of the classical Ambrosetti—Rabinowitz theorem (see: [5]) and a local minimum theorem established in [6].

Before providing the mentioned result we recall that a continuously differentiable functional I defined on a real Banach space X satisfies the Palais-Smale condition, the (PS) - condition for short, if every sequence $\{u_n\}$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \longrightarrow 0$ in X^* as $n \longrightarrow \infty$ has a convergent subsequence.

The existence result of multiple critical points reads

Theorem 2.2. *(Bonanno and D'Aguì [9, Theorem 2.1]) Let X be a real finite dimensional Banach space and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{\Phi^{-1}(]-\infty,r])} \Psi}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2}$$

and, for each

$$\lambda \in \Lambda := \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{\Phi^{-1}(]-\infty,r])} \Psi} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the function I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

In order to give the variational formulation of problem (1) we introduce the T-dimensional Banach space

$$W := \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| := \left(\sum_{k=1}^{T+1} \left(w(k-1)|\Delta u(k-1)|^2 + q(k)|u(k)|^2\right)\right)^{\frac{1}{2}}.$$

In the space W we will also consider the following norms

$$\|u\|_{p^+} := \left(\sum_{k=1}^{T+1} \left(w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^+}\right)\right)^{\frac{1}{p^+}},$$

$$\|u\|_{\max} := \max_{k\in[1,T]} |u(k)|$$

and (like in [13]) the norm

$$\|u\|_{p(\cdot)} := \inf\left\{\mu > 0 : \sum_{k=1}^{T+1} \left(w(k-1)\left|\frac{\Delta u(k-1)}{\mu}\right|^{p(k-1)} + q(k)\left|\frac{u(k)}{\mu}\right|^{p(k)}\right) \le 1\right\}.$$

The latest norm is called the Luxemburg norm.

Put

$$K := (2\max\{\overline{w}, \overline{q}\})^{\frac{2-p^+}{2p^+}}.$$

Note that $K \le 1$.

To prove our results we will use several inequalities which connect the above norms. The important role in our investigations plays the following inequality

$$K\|u\| \le \|u\|_{p^+} \le 2^{\frac{p^+-2}{2p^+}} K\|u\|, \tag{3}$$

which are obtained by using twice Weighted Hölder's inequality.

Indeed, on the one hand

$$\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2$$

$$\leq \left(\sum_{k=1}^{T+1} w(k-1)\{1\}^{\frac{p^+}{p^+-2}}\right)^{\frac{p^+-2}{p^+}} \left(\sum_{k=1}^{T+1} w(k-1)\left(|\Delta u(k-1)|^2\right)^{\frac{p^+}{2}}\right)^{\frac{2}{p^+}}$$

$$\leq \overline{w}^{\frac{p^+-2}{p^+}} \left(\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+}\right)^{\frac{2}{p^+}}.$$

In the same manner we get

$$\sum_{k=1}^{T+1} q(k)|u(k)|^2 \leq \overline{q}^{\frac{p^+-2}{p^+}} \left(\sum_{k=1}^{T+1} q(k)|u(k)|^{p^+}\right)^{\frac{2}{p^+}}.$$

Taking the above inequalities and $\frac{2}{p^+} \leq 1$ into account, we get

$$\|u\|^2 \leq (\max\{\overline{w}, \overline{q}\})^{\frac{p^+-2}{p^+}}$$

$$\times \left(\left(\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+}\right)^{\frac{2}{p^+}} + \left(\sum_{k=1}^{T+1} q(k)|u(k)|^{p^+}\right)^{\frac{2}{p^+}}\right)$$

$$\leq 2^{1-\frac{2}{p^+}} (\max\{\overline{w}, \overline{q}\})^{\frac{p^+-2}{p^+}}$$

$$\times \left(\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} + \sum_{k=1}^{T+1} q(k)|u(k)|^{p^+}\right)^{\frac{2}{p^+}} = K^{-2}\|u\|_{p^+}^2.$$

Hence $K\|u\| \leq \|u\|_{p^+}$.

On the other hand taking $\frac{p^+}{2} \geq 1$ into account, we can conclude that

$$\|u\|_{p^+}^{p^+} \leq (\max\{\overline{w}, \overline{q}\})^{\frac{2-p^+}{2}}$$

$$\times \left(\left(\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2 \right)^{\frac{p^+}{2}} + \left(\sum_{k=1}^{T+1} q(k)|u(k)|^2 \right)^{\frac{p^+}{2}} \right)$$

$$\leq (\max\{\overline{w}, \overline{q}\})^{\frac{2-p^+}{2}}$$

$$\times \left(\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2 + \sum_{k=1}^{T+1} q(k)|u(k)|^2 \right)^{\frac{p^+}{2}} = 2^{\frac{p^+-2}{2}} K^{p^+} \|u\|^{p^+}.$$

Hence $\|u\|_{p^+} \leq 2^{\frac{p^+-2}{2p^+}} K\|u\|$. Thus inequalities (3) holds.

We also use the following inequality

$$\|u\|_{\max} \leq (2T+2)^{\frac{1}{2}} \|u\| \tag{4}$$

which can be simply verify by the discrete Hölder inequality.

Note that in the space W all norms are equivalent, since W is finite dimensional, therefore there exist two positive constants $L_1 < \frac{2}{K}$ and $L_2 > 1$ such that

$$L_1 \|u\|_{p(\cdot)} \leq \|u\| \leq L_2 \|u\|_{p(\cdot)}. \tag{5}$$

Let $\psi : W \to \mathbb{R}$ be a functional given by the formula

$$\psi(u) := \sum_{k=1}^{T+1} \Big[w(k-1)|\Delta u(k-1)|^{p(k-1)} + q(k)|u(k)|^{p(k)} \Big]. \tag{6}$$

It is easy to check that for any $u \in W$ the following properties hold

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \psi(u) \leq \|u\|_{p(\cdot)}^{p^-} \leq \|u\|_{p(\cdot)}^2, \tag{7}$$

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^2 \leq \|u\|_{p(\cdot)}^{p^-} \leq \psi(u) \leq \|u\|_{p(\cdot)}^{p^+}, \tag{8}$$

and

$$\psi(u) < 1 \Rightarrow \|u\|_{p^+}^{p^+} \leq \psi(u) \leq \|u\|^2. \tag{9}$$

Put

$$F(k,t) := \int_0^t f(k,\xi) d\xi$$

for every $(k,t) \in [1,T] \times \mathbb{R}$.

To study problem (1) we consider the functional $I_\lambda : W \to \mathbb{R}$ defined by

$$I_\lambda(u) := \sum_{k=1}^{T+1} \left(\frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right) - \lambda \sum_{k=1}^{T} F(k, u(k)).$$

An easy computation ensures that I_λ turns out to be of class C^1 on W with

$$\begin{aligned} I'_\lambda(u)(v) : &= \sum_{k=1}^{T+1} \big(w(k-1)\phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) + q(k)\phi_{p(k)}(u(k))v(k) \big) \\ &- \sum_{k=1}^{T} \lambda f(k, u(k))v(k) \end{aligned}$$

for all $u, v \in W$.

Lemma 2.3. *The critical points of I_λ are exactly the solutions of problem (1).*

Proof. Let $\overline{u}$ be an arbitrary critical point of I_λ in W. Then $\overline{u}(0) = \overline{u}(T+1) = 0$ and for all $v \in W$, $I'_\lambda(\overline{u})(v) = 0$. Thus, for every $v \in W$, taking summation by parts into account, one has

$$\begin{aligned} 0 &= I'_\lambda(\overline{u})(v) \\ &= -\textstyle\sum_{k=1}^{T} \Big[\Delta(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))) - q(k)\phi_{p(k)}\,(\overline{u}(k)) + \lambda f(k, \overline{u}(k)) \Big] v(k). \end{aligned}$$

Since $v \in W$ is arbitrary, so one get

$$-\Delta(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1)) + q(k)\phi_{p(k)}(\overline{u}(k)) = \lambda f(k, \overline{u}(k))$$

for every $k \in [1,T]$. Therefore, $\overline{u}$ is a solution of problem (1). Since $\overline{u}$ is arbitrary, we conclude that every critical point of the functional I_λ in W is a solution of problem (1). □

Note also that every solution of problem (1) is a critical point to I_λ.

To describe the variational framework of problem (1) put Φ, Ψ as follows

$$\begin{aligned} \Phi(u) \;\; &: \;\; = \sum_{k=1}^{T+1} \left(\frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right), \\ \Psi(u) \;\; &: \;\; = \sum_{k=1}^{T} F(k, u(k)), \end{aligned}$$

for every $u \in W$. Then $I_\lambda = \Phi - \lambda\Psi$.
In the following, we prove that the functional I_λ satisfies the Palais–Smale condition.

Put

$$L_\infty := \min_{k\in[1,T]} \left(\liminf_{\xi\to\infty} \frac{F(k,\xi)}{|\xi|^{p^+}} \right)$$

and

$$\lambda^* := \frac{(2T)^{\frac{p^+-2}{2}} K^{p^+} (4\overline{w} + \overline{q})^{\frac{p^+}{2}}}{L_\infty p^-}.$$

Lemma 2.4. *Let $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, such that $f(k,x) \geq 0$ for all $x \leq 0$ and for all $k \in [1,T]$. If $L_\infty > 0$ then the functional I_λ satisfies the Palais–Smale condition and it is unbounded from below for all $\lambda \in]\lambda^*, +\infty[$.*

Proof. Let us fix $\lambda > \lambda^*$. Assume that a sequence $\{u_n\}$ is such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Since W is finite dimensional, it is sufficient to show that $\{u_n\}$ is bounded.

Put

$$u_n^+(k) := \max\{0, u_n(k)\} \text{ and } u_n^-(k) := \max\{0, -u_n(k)\}$$

for all $n \in \mathbb{N}$ and for all $k \in [0, T+1]$. By straightforward computation we can check that for all $n \in \mathbb{N}$ and for all $k \in [1, T+1]$ we have

$$\Delta u_n^+(k-1)\Delta u_n^-(k-1) \leq 0. \tag{10}$$

Indeed,

$$\Delta u_n^+(k-1)\Delta u_n^-(k-1) = (u_n^+(k) - u_n^+(k-1))(u_n^-(k) - u_n^-(k-1)) =$$

$$-(u_n^+(k)u_n^-(k-1) + u_n^+(k-1)u_n^-(k)) \leq 0.$$

Using (10) we have

$$-\sum_{k=1}^{T+1} w(k-1)|\Delta u_n(k-1)|^{p(k-1)-2}\Delta u_n(k-1)\Delta u_n^-(k-1)$$

$$\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u_n(k-1)|^{p(k-1)-2} \left(\Delta u_n^-(k-1)\right)^2$$

$$\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u_n^-(k-1)|^{p(k-1)}.$$

In the similar manner we get

$$-\sum_{k=1}^{T+1} q(k)|u_n(k)|^{p(k)-2}u_n(k)u_n^-(k) = \sum_{k=1}^{T+1} q(k)|u_n^-(k)|^{p(k)}.$$

By the above we obtain

$$-\Phi'(u_n)(u_n^-) = -\sum_{k=1}^{T+1}((w(k-1)\phi_{p(k-1)}(\Delta u_n(k-1))\Delta u_n^-(k-1)$$

$$+q(k)\phi_{p(k)}(u_n(k))u_n^-(k))$$

$$\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u_n^-(k-1)|^{p(k-1)} + q(k)|u_n^-(k)|^{p(k)} = \psi(u_n^-).$$

Moreover, by the assumption of f and by the definition of u_n^- we deduce that

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{T} f(k, u_n(k))u_n^-(k) \geq 0,$$

and in a consequence

$$0 \leq \psi(u_n^-) \leq -\Phi'(u_n)(u_n^-) \leq -\Phi'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-) = -I_\lambda'(u_n)(u_n^-),$$

for all $n \in \mathbb{N}$.

Since $I_\lambda'(u_n) \to 0$ as $n \to \infty$, hence $\psi(u_n^-) \to 0$ as $n \to \infty$. Thus, there exists $N > 0$ such that $\psi(u_n^-) < 1$ for any $n \geq N$. Hence $|u_n^-| < 1$ for any $n \geq N$ and then $0 \leq u_n^- < L$ for any $n \in \mathbb{N}$, where $L = \max\{1, u_1^-, u_2^-, u_3^-, ..., u_{N-1}^-\}$. This means that $\{u_n^-\}$ is bounded, and $u_n > -L$ for any $n \in \mathbb{N}$.

Now, arguing by a contradiction, we will show that $\{u_n\}$ is bounded. Suppose that $\{u_n\}$ is unbounded. We may assume that $\|u_n\| \to \infty$. Arguing similarly as in the proof of [21, Lemma 7] we get

$$\sum_{k=1}^{T+1} w(k-1)|\Delta u_n(k-1)|^{p(k-1)} \leq \sum_{k=1}^{T+1} w(k-1)|\Delta u_n(k-1)|^{p^+} + w^+(T+1).$$

In the same manner we get

$$\sum_{k=1}^{T+1} q(k)|u_n(k)|^{p(k)} \leq \sum_{k=1}^{T+1} q(k)|u_n(k)|^{p^+} + q^+(T+1).$$

Combining the above inequalities and bearing in mind (3) we obtain

$$\begin{aligned}\Phi(u_n) &\leq \frac{1}{p^-}\psi(u_n) \qquad (11)\\ &\leq \frac{1}{p^-}\left(\|u_n\|_{p^+}^{p^+} + (w^+ + q^+)(T+1)\right)\\ &\leq \frac{1}{p^-}\left(2^{\frac{p^+-2}{2}}K^{p^+}\|u_n\|^{p^+} + (w^+ + q^+)(T+1)\right).\end{aligned}$$

Now, notice that

$$\begin{aligned}\|u_n\|^2 &\leq \overline{w}\sum_{k=1}^{T+1}|\Delta u_n(k-1)|^2 + \overline{q}\sum_{k=1}^{T+1}|u_n(k)|^2\\ &\leq 4\overline{w}\sum_{k=1}^{T}|u_n(k)|^2 + \overline{q}\sum_{k=1}^{T}|u_n(k)|^2\\ &\leq (4\overline{w}+\overline{q})T^{\frac{p^+-2}{p^+}}\left(\sum_{k=1}^{T}|u_n(k)|^{p^+}\right)^{\frac{2}{p^+}}.\end{aligned}$$

Thus

$$\sum_{k=1}^{T}|u_n(k)|^{p^+} \geq \frac{\|u_n\|^{p^+}}{T^{\frac{p^+-2}{2}}(4\overline{w}+\overline{q})^{\frac{p^+}{2}}}. \qquad (12)$$

From $\liminf_{\xi\to\infty}\frac{F(k,\xi)}{|\xi|^{p^+}} \geq L_\infty$ for every $k \in [1,T]$ there exists $\delta_k > 0$ such that

$$F(k,\xi) > L_\infty|\xi|^{p^+} \text{ for all } \xi > \delta_k.$$

Moreover, for all $\xi \in [-L,\delta_k]$ we have

$$\begin{aligned}F(k,\xi) &\geq \min_{\xi\in[-L,\delta_k]}F(k,\xi) \geq \min_{\xi\in[-L,\delta_k]}F(k,\xi) + L_\infty\left(|\xi|^{p^+} - (\max\{\delta_k,L\})^{p^+}\right)\\ &\geq L_\infty|\xi|^{p^+} - \max\{L_\infty(\max\{\delta_k,L\})^{p^+} - \min_{\xi\in[-L,\delta_k]}F(k,\xi),0\}\\ &= L_\infty|\xi|^{p^+} - Q(k),\end{aligned}$$

where $Q(k) = \max\{L_\infty(\max\{\delta_k,L\})^{p^+} - \min_{\xi\in[-L,\delta_k]}F(k,\xi),0\}$ possesses only non-negative values for every $k \in [1,T]$.

Eventually,

$$F(k,\xi) \geq L_\infty|\xi|^{p^+} - Q(k),\ \forall\xi > -L \text{ and } \forall k \in [1,T].$$

Due to $u_n > -L$ for all $n \in \mathbb{N}$, we conclude that

$$F(k, u_n(k)) \geq L_\infty |u_n(k)|^{p^+} - Q(k), \quad \forall n \in \ \mathbb{N} \text{ and } \forall k \in [1, T]. \tag{13}$$

By (13) and (12) we get

$$\Psi(u_n(k)) = \sum_{k=1}^{T} F(k, u_n(k)) \geq L_\infty \frac{\|u\|^{p^+}}{T^{\frac{p^+-2}{2}} (4\overline{w} + \overline{q})^{\frac{p^+}{2}}} - \overline{Q}, \tag{14}$$

where

$$\overline{Q} = \sum_{k=1}^{T} Q(k).$$

By (11) and (14) we infer that

$$\begin{aligned}
I_\lambda(u_n) &= \Phi(u_n) - \lambda\Psi(u_n) \\
&\leq \frac{1}{p^-}\left(2^{\frac{p^+-2}{2}} K^{p^+} \|u_n\|^{p^+} + (w^+ + q^+)(T+1)\right) \\
&\quad -\lambda L_\infty \frac{\|u_n\|^{p^+}}{T^{\frac{p^+-2}{2}} (4\overline{w} + \overline{q})^{\frac{p^+}{2}}} + \lambda\overline{Q} \\
&= \left(\frac{2^{\frac{p^+-2}{2}} K^{p^+}}{p^-} - \lambda L_\infty \frac{1}{T^{\frac{p^+-2}{2}} (4\overline{w} + \overline{q})^{\frac{p^+}{2}}}\right) \|u_n\|^{p^+} \\
&\quad + \frac{1}{p^-}(w^+ + q^+)(T+1) + \lambda\overline{Q} \\
&= \frac{L_\infty(\lambda^* - \lambda)}{T^{\frac{p^+-2}{2}} (4\overline{w} + \overline{q})^{\frac{p^+}{2}}} \|u_n\|^{p^+} + \frac{1}{p^-}(w^+ + q^+)(T+1) + \lambda\overline{Q}.
\end{aligned}$$

Since $\|u_n\| \to \infty$ and $\lambda^* - \lambda < 0$, so $I_\lambda(u_n) \to -\infty$ and this is an absurd. Hence I_λ satisfies the Palais–Smale condition for all $\lambda \in]\lambda^*, \infty[$.

It remains to show that I_λ is unbounded from below. Let a sequence $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded and then $\|u_n\| \to \infty$. Arguing as before one has $I_\lambda(u_n) \to -\infty$ for all $\lambda \in]\lambda^*, +\infty[$ and the proof is complete. □

Now we give a lemma and the following notation. Put

$$A := \left(w(0) + w(T) + \sum_{k=1}^{T} q(k)\right),$$

$$L_0 := \min_{k\in[1,T]} \limsup_{t\to 0} \frac{F(k,t)}{|t|^{p^- - 1}}.$$

Lemma 2.5. *Let there exists $r > 0$, such that $u_\lambda \in W$ be a global minimum of the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r[)$ for some $\lambda \in]0, \infty[$. If $L_0 > 0$ then u_λ is nonzero.*

Proof. To this end, let us show that

$$\limsup_{\|u\|\to 0} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{15}$$

Due to our assumptions at zero, we can fix a sequence $\{\xi_n\}$ converging to zero such that

$$L_0 := \min_{k\in[1,T]} \limsup_{n\to\infty} \frac{F(k,\xi_n)}{|\xi_n|^{p^- -1}} > 0.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_n < 1$ for every $\nu < n$ and from $\limsup_{s\to 0} \frac{F(k,s)}{|s|^{p^- -1}} \geq L_0$, there exists $d \in (0,1)$, such that

$$\frac{F(k,\xi_n d)}{|\xi_n d|^{p^- -1}} \geq L_0, \qquad \forall \nu < n, k \in [1,T].$$

Put

$$v(k) := \begin{cases} d, & \text{for every } k \in [1,T]; \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

so

$$F(k,\xi_n v(k)) \geq L_0 |\xi_n v(k)|^{p^- -1}, \qquad \forall \nu < n, k \in [1,T].$$

Clearly $v \in W$ and for $r = \frac{1}{p^+}$ and for n sufficiently large,

$$\begin{aligned} \Phi(\xi_n v) &= \frac{w(0)}{p(0)}(\xi_n d)^{p(0)} + \frac{w(T)}{p(T)}(\xi_n d)^{p(T)} + \sum_{k=1}^{T} \frac{q(k)}{p(k)}(\xi_n d)^{p(k)} \\ &< \frac{(\xi_n d)^{p^-}}{p^-}\Big(w(0) + w(T) + \sum_{k=1}^{T} q(k)\Big) < \frac{1}{p^+} = r, \end{aligned}$$

hence $\xi_n v \in \Phi^{-1}(]-\infty, r[)$. For every $\nu < n$

$$\begin{aligned} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\sum_{k=1}^{T} F(k,\xi_n v)}{\frac{(\xi_n d)^{p^-}}{p^-}\big(w(0) + w(T) + \sum_{k=1}^{T} q(k)\big)} \\ &> \frac{T L_0 |\xi_n d|^{p^- -1}}{\frac{(\xi_n d)^{p^-}}{p^-} A} \to \infty, \quad n \to \infty, \end{aligned}$$

from which (15) clearly follows. Hence, there exists a sequence $\{w_n\} \subset W$ strongly converging to zero such that, for n sufficiently large, $w_n \in \Phi^{-1}(]-\infty, r[)$ and for any $\lambda > 0$, one can conclude that

$$\frac{1}{\lambda} < \frac{\Psi(w_n)}{\Phi(w_n)} \to \Phi(w_n) < \lambda\Psi(w_n) \to \Phi(w_n) - \lambda\Psi(w_n) < 0 \to I_\lambda(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(]-\infty, r[)$, we conclude that

$$I_\lambda(u_\lambda) \leq I_\lambda(w_n) < 0 = I_\lambda(0), \tag{17}$$

so that u_λ is not trivial. □

3 Existence of a solution

Our first aim is to show that problem (1) has at least one positive solution using Theorem 2.1. By a positive solution to problem (1) we mean such a function $u : [0, T+1] \to \mathbb{R}$ which satisfies the given equation on $[1, T]$, the boundary conditions and $u(k) > 0$ for all $k \in [1, T]$.

Put

$$c := \frac{1}{K}(2T+2)^{\frac{1}{2}}.$$

We state our main result as follows.

Theorem 3.1. *Let $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, such that $L_0 > 0$. Then for any*

$$\lambda \in \left]0, \frac{1}{p^+ \sum_{k=1}^{T} F(k, c)}\right[,$$

problem (1) has at least one positive solution $u_0 \in W$.

Proof. An easy computation ensures that Φ and Ψ turn out to be of class C^1 on W with

$$\Phi'(u)(v) = -\sum_{k=1}^{T} \left(\Delta(w(k-1)\phi_{p(k-1)}(\Delta u(k-1))v(k) - q(k)\phi_{p(k)}(u(k))v(k)\right),$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{T} f(k, u(k))v(k),$$

for all $u, v \in W$. By Lemma 2.3, the solutions of the equation $I'_\lambda = \Phi' - \lambda\Psi' = 0$ are exactly the solutions for problem (1), therefore to prove our result it is enough to apply Theorem 2.1.

The functional Φ is of class C^1 on the finite dimensional space W, so it is sequentially weakly lower semicontinuous. The functional Ψ is also of class C^1 on the finite dimensional space W, so it is sequentially weakly upper semicontinuous.

We will show that the functional Φ is coercive. Let $u \in W$ be fixed. Arguing similarly as in the proof of [21, Lemma 7], we get

$$\begin{aligned}
&\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p(k-1)} \\
&\geq \sum_{\{k\in[1,T+1]:|\Delta u(k-1)|<1\}} w(k-1)|\Delta u(k-1)|^{p^+} \\
&+ \sum_{\{k\in[1,T+1]:|\Delta u(k-1)|\geq 1\}} w(k-1)|\Delta u(k-1)|^2 \\
&= \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2 - \\
&+ \sum_{\{k\in[1,T+1]:|\Delta u(k-1)|<1\}} \Big(w(k-1)|\Delta u(k-1)|^2 - w(k-1)\Delta u(k-1)|^{p^+} \Big) \\
&\geq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2 - (T+1)w^+,
\end{aligned}$$

so

$$\sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p(k-1)} \geq \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^2 - w^+(T+1).$$

In the same manner we get

$$\sum_{k=1}^{T+1} q(k)|u(k)|^{p(k)} \geq \sum_{k=1}^{T+1} q(k)|u(k)|^2 - q^+(T+1).$$

Combining the above inequalities we obtain

$$\Phi(u) \geq \frac{1}{p^+}\|u\|^2 - (w^+ + q^+)(T+1).$$

Therefore Φ is coercive, i.e. $\Phi(u) \to \infty$ as $\|u\| \to \infty$.

Put

$$r := \frac{1}{p^+}.$$

For all $u \in W$ such that $\Phi(u) < r$, taking (6), (9) and (3) into account, one has

$$\frac{1}{p^+} > \Phi(u) \geq \frac{1}{p^+}\psi(u) \geq \frac{\|u\|_{p^+}^{p^+}}{p^+} \geq \frac{K^{p^+}\|u\|^{p^+}}{p^+},$$

so

$$\|u\| < \frac{1}{K}.$$

By (4) we obtain

$$\max_{k\in[1,T]} |u(k)| \leq (2T+2)^{\frac{1}{2}}\|u\| < (2T+2)^{\frac{1}{2}}\frac{1}{K} = c.$$

Therefore

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r[)} \sum_{k=1}^{T} F(k,u(k)) \leq \sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi) = \sum_{k=1}^{T} F(k,c).$$

Put

$$\phi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Taking into account the above computations and the definition of $\phi(r)$, since $0 \in \Phi^{-1}(]-\infty, r[)$ and $\Phi(0) = \Psi(0) = 0$ we have

$$\phi(r) \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq p^+ \sum_{k=1}^{T} F(k,c).$$

Therefore, owing to Theorem 2.1, for each

$$\lambda \in \left]0, \frac{1}{p^+ \sum_{k=1}^{T} F(k,c)}\right[\subset]0, \frac{1}{\phi(r)}[,$$

the functional I_λ admits one critical point $u_\lambda \in W$ such that $\Phi(u_\lambda) < r$.

Since f is non-negative we deduce that

$$\begin{cases} -\Delta(w(k-1)\phi_{p(k-1)}(\Delta u_\lambda(k-1))) + q(k)\phi_{p(k)}(u_\lambda(k)) \geq 0, & k \in [1,T], \\ u_\lambda(0) = u_\lambda(T+1) = 0, \end{cases}$$

By [26, Lemma 2.6] we have that either the ensured solution $u_\lambda(k)$ for every $k \in [1,T]$ is positive or $u \equiv 0$. Since $L_0 > 0$, by Lemma 2.5, we conclude that the obtained solution $u_\lambda \in W$ should be nonzero. Hence the ensured solution $u_\lambda \in W$ is positive. The proof is complete. □

Remark 3.2. *Theorem 1.1 is a special case of Theorem 3.1. Indeed,* $w(k) = 1$ *for all* $k \in [0, T]$, $q(k) = 1$ *for all* $k \in [1, T+1]$, $p(k) = k+3$ *for all* $k \in [0, T+1]$ *and* $K = (2T + 2)^{\frac{-2-T}{8+2T}}$.

Now we give an example to illustrate the above Theorem.

Example 3.3. *Let* $T = 10$, $p(k) = 3 + \frac{1}{11}k$ *for all* $k \in [0, 11]$, $w(k) = e^k$ *for all* $k \in [0, 10]$, $q(k) = 2^{k-1}$ *for all* $k \in [1, 11]$. *Thus* $p^+ = 4$, $p^- = 3$, $\overline{w} = \frac{e^{11}-1}{e-1} \simeq 34844.77384$, *and* $\overline{q} = 2^{11} - 1 = 2047$. *Hence*

$$K = (2\max\{\overline{w}, \overline{q}\})^{\frac{2-p^+}{2p^+}} = (2\frac{e^{11} - 1}{e - 1})^{-\frac{1}{4}} \simeq 0.06154717000,$$

and

$$c = (2T + 2)^{\frac{1}{2}} \frac{1}{K} \simeq 76.20847162.$$

Suppose

$$f(k, t) = \frac{\sqrt[3]{t^2}}{k(k + 1)}; \forall t \in \mathbb{R},\ k \in [1, 10].$$

Then

$$F(k, c) = \frac{3c^{\frac{5}{3}}}{5k(k + 1)},$$

$$L_0 := \min_{k\in[1,10]} \limsup_{t\to 0} \frac{F(k, t)}{|t|^2} = +\infty > 0,$$

and

$$\sum_{k=1}^{T} F(k, c) = \frac{3c^{\frac{5}{3}}}{5} \sum_{k=1}^{T} \frac{1}{k(k + 1)} = \frac{3c^{\frac{5}{3}}}{5} \sum_{k=1}^{T} \left(\frac{1}{k} - \frac{1}{k + 1}\right) = \frac{3c^{\frac{5}{3}}}{5} \frac{T}{T + 1}.$$

Eventually

$$\frac{1}{p^+ \sum_{k=1}^{T} F(k, c)} = 0.000334587.$$

Then for any $\lambda \in]0, 0.000334587[$ *the problem*

$$\begin{cases} -\Delta(e^{k-1}|\Delta u(k - 1)|^{1+\frac{1}{11}(k-1)}\Delta u(k - 1)) + 2^{k-1}|u(k)|^{1+\frac{1}{11}k}u(k) = \lambda\frac{\sqrt[3]{u^2(k)}}{k(k+1)}, \\ u(0) = u(11) = 0, \end{cases}$$

for any $k \in [1, T]$ *has at least one positive solution* $u_0 \in W$.

Next example illustrates Theorem 1.1.

Example 3.4. *Let $T \geq 2$ and $f(t) = \sqrt{t}$ such that $\lim_{t \to 0^+} \frac{\int_0^t f(s)ds}{t^2} = +\infty > 0$, Then for any*

$$\lambda \in \left] 0, \frac{3}{2T(T+4)(2T+2)^{\frac{3T+9}{T+4}}} \right[,$$

the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^k \Delta u(k-1)) + |u(k)|^{k+1} u(k) = \lambda \sqrt{u(k)}, \quad k \in [1, T], \\ u(0) = u(11) = 0, \end{cases}$$

has at least one positive solution $u_0 \in W$.

4 Multiple solutions

To show that problem (1) has multiple solutions, precisely it has of at least two positive solutions, we apply Theorem 2.2. Now, we provide the main result of this section.

Theorem 4.1. *Let $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, such that $f(k, x) = f(k, 0) = 0$ for all $x \leq 0$ and for all $k \in [1, T]$. Assume that there exist two positive constants c and d with*

$$d < \left(\frac{p^-}{p^+} \frac{(cK)^{p^+}}{(2T+2)^{\frac{p^+}{2}} A} \right)^{\frac{1}{p^-}} < \left(\frac{1}{p^+ A} \right)^{\frac{1}{p^-}}, \tag{18}$$

such that

$$\frac{\sum_{k=1}^T F(k, c)}{\frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}}} < \min \left\{ \frac{\sum_{k=1}^T F(k, d)}{\frac{d^{p^-} A}{p^-}}, \frac{1}{\lambda^*} \right\}, \tag{19}$$

Then for any

$$\lambda \in \Lambda^* = \left] \max \left\{ \frac{\frac{d^{p^-} A}{p^-}}{\sum_{k=1}^T F(k, d)}, \lambda^* \right\}, \frac{\frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}}}{\sum_{k=1}^T F(k, c)} \right[,$$

problem (1) has at least two positive solutions $u_{\lambda,1}, u_{\lambda,2} \in W$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. Let Φ and Ψ, be as mentioned before. It is well known that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.2. It is clear that

$\inf_W \Phi = \Phi(0) = \Psi(0) = 0$. It remains to verify condition (2) of Theorem 2.2. Fix $\lambda \in \Lambda^*$. Note that $L_\infty > 0$, thus by Lemma 2.4, the functional I_λ satisfies the Palais-Smale condition for each $\lambda > \lambda^*$ and it is unbounded from below. Let $\bar{v}(k)$ be in (16) and

$$r = \frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}},$$

and

$$\Phi(\bar{v}) = \frac{w(0)}{p(0)} d^{p(0)} + \frac{w(T)}{p(T)} d^{p(T)} + \sum_{k=1}^{T} \frac{q(k)}{p(k)} d^{p(k)}.$$

Since $d < \left\{\frac{1}{p^+ A}\right\}^{\frac{1}{p^-}} < 1$, so $\Phi(\bar{v}) < \frac{d^{p^-}}{p^-} A$. By the left hand inequality in (18),

$$\frac{d^{p^-} A}{p^-} < \frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}} = r.$$

Therefore $0 < \Phi(\bar{v}) < r$. Moreover

$$\Psi(\bar{v}) = \sum_{k=1}^{T} F(k, \bar{v}(k)) = \sum_{k=1}^{T} F(k, d).$$

On the other hand, for all $u \in W$ such that $\Phi(u) < r$, by the right hand inequality in (18), we have

$$\frac{1}{p^+}\psi(u) < \Phi(u) < r < \frac{1}{p^- p^+} < \frac{1}{p^+},$$

and then $\psi(u) < 1$. Taking (9) and (3) into account we get

$$rp^+ > \psi(u) > \|u\|_{p^+}^{p^+} > K^{p^+} \|u\|^{p^+}. \tag{20}$$

By (4) and (20) we obtain

$$|u(k)| \leq \|u\|_{\max} \leq (2T+2)^{\frac{1}{2}} \|u\| < (2T+2)^{\frac{1}{2}} \frac{(rp^+)^{\frac{1}{p^+}}}{K} = c, \quad \forall k \in [1, T].$$

From the definition of r, it follows that

$$\Phi^{-1}(]-\infty, r]) \subseteq \{u \in W : |u(k)| \leq c \ \text{ for all } k \in [1, T]\}.$$

Therefore, we get

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, r])} \Psi(u)}{r} \leq \frac{\sum_{k=1}^{T} \max_{|\xi| \leq c} F(k, \xi)}{\frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}}} = \frac{\sum_{k=1}^{T} F(k, c)}{\frac{(cK)^{p^+}}{p^+(2T+2)^{\frac{p^+}{2}}}}. \tag{21}$$

Moreover, one has

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} > \frac{\sum_{k=1}^{T} F(k,d)}{\frac{d^{p^-} A}{p^-}}. \tag{22}$$

Thus, from (19), (21) and (22), condition (2) of Theorem 2.2 follows.
Also by Lemma 2.4, for each

$$\lambda \in \Lambda^* \subset \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{\Phi^{-1}(]-\infty,r])} \Psi}\right[,$$

the functional I_λ satisfies the Palais–Smale condition and it is unbounded from below. Thus all assumptions in the Theorem 2.2 are fulfilled, hence Theorem 2.2 ensures that I_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in W$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$ for all $\lambda \in \Lambda^*$, and simultaneously, by the similar argument as in the proof of our main result from the previous section, they are positive solutions of (1). □

As an example, here, we point out the following special case of the above result.

Theorem 4.2. *Let $T \geq 2$ be a fixed positive integer and let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function, such that $f(0) = 0$ and where $L_\infty = \liminf_{\xi\to\infty} \frac{1}{\xi^{T+3}} \int_0^\xi f(t)dt$. Assume also that there exist two positive constants c and d with*

$$d < \left\{\frac{2c^{T+3}}{(T+3)(T+2)(2T+2)^{T+2}}\right\}^{\frac{1}{2}} < \left\{\frac{1}{(T+3)(T+2)}\right\}^{\frac{1}{2}}, \tag{23}$$

such that

$$T(T+3)(2T+2)^{T+2}\frac{\int_0^c f(t)dt}{c^{T+3}} < \min\left\{\frac{2T}{(T+2)}\frac{\int_0^d f(t)dt}{d^2}, \frac{2L_\infty}{5^{\frac{T+3}{2}}T^{\frac{T+1}{2}}(T+1)}\right\},$$

Then for any

$$\lambda \in \Lambda^* = \left]\max\left\{\frac{\frac{d^2(T+2)}{2}}{T\int_0^d f(t)dt}, \frac{5^{\frac{T+3}{2}}T^{\frac{T+1}{2}}(T+1)}{2L_\infty}\right\}, \frac{\frac{c^{T+3}}{(T+3)(2T+2)^{T+2}}}{T\int_0^c f(t)dt}\right[,$$

the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{k-1}\Delta u(k-1)) + |u(k)|^k u(k) = \lambda f(u(k)), \\ u(0) = u(T+1) = 0, \end{cases}$$

for any $k \in [1,T]$ has at least two positive solutions $u_{\lambda,1}, u_{\lambda,2} \in W$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

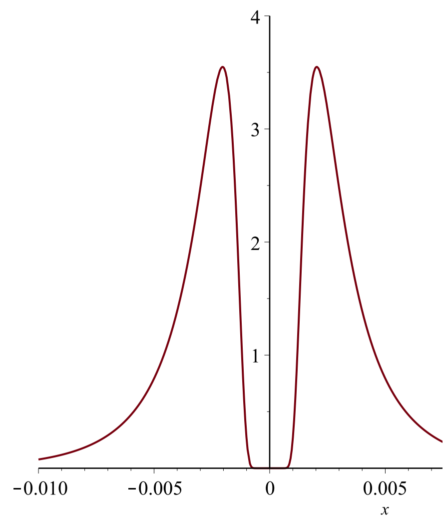

Figure 1: The graph of the function $f(x)$

Remark 4.3. *Theorem 4.2 is a special case of Theorem 4.1. Indeed,* $w(k) = 1$ *for all* $k \in [0, T]$, $q(k) = 1$ *for all* $k \in [1, T+1]$, $p(k) = k+2$ *for all* $k \in [0, T+1]$ *and* $K = (2T+2)^{\frac{-1-T}{6+2T}}$.

Next example illustrates Theorem 4.2.

Example 4.4. *Let for any* $x \in \mathbb{R}/\{0\}$ *(see FIGURE 1)*

$$f(x) = \frac{|\coth(200\,x)|}{(\sinh(200\,x))^2}\left(\cosh\left(\frac{x^4}{10000}\right)\right)^{10} \mathrm{e}^{-1/4\,(\sinh(200\,x))^{-2}}, \quad f(0) = 0,$$

Then for any $\lambda \in]0.008493226757, 0.01085069444[$ *the problem*

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{k-3}\Delta u(k-1)) + |u(k)|^{k-2}u(k) = \lambda f(u(k)), & k \in [1,3] \\ u(0) = u(4) = 0, \end{cases}$$

has at least two non-zero solutions. Indeed, $T = 3$ *and in this case the relation* (23) *convert to* $d < \sqrt{\frac{1}{491520}c^3} < \sqrt{1/30}$ *and by selecting* $c = 2$ *and* $d = 0.01$ *satisfying it. By using software Maple, one can calculate*

$$T(T+3)(2T+2)^{T+2}\frac{\int_0^c f(t)dt}{c^{T+3}} \simeq 92.16000000,$$

$$\frac{2T}{(T+2)}\frac{\int_0^d f(t)dt}{d^2} \simeq 117.7408809,$$

$$\frac{2}{5^{\frac{T+3}{2}}T^{\frac{T+1}{2}}(T+1)}\liminf_{\xi\to\infty}\frac{\int_0^\xi f(t)dt}{\xi^{T+3}} = \infty.$$

5 Uniqueness of a solution

In this section we prove under Lipschitzian continuous condition on f, that problem (1) has a unique solution. We start with providing the following lemma

Lemma 5.1. *([29]). Assume that* $p \geq 2$ *and* $c_p = \frac{2}{p(2^{p-1}-1)}$. *Then for any* $x, y \in \mathbb{R}$,

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq c_p|x-y|^p.$$

Put

$$c_{p(k)} := \frac{2}{p(k)(2^{p(k)-1}-1)},$$

for every $k \in [0, T+1]$ and put

$$c_{p^+} := \frac{2}{p^+(2^{p^+-1}-1)}.$$

Note that

$$c_{p^+} = \min_{k\in[0,T+1]} c_{p(k)}. \tag{24}$$

Theorem 5.2. *Let* $f : [1,T]\times\mathbb{R}\to\mathbb{R}$ *be a non-negative continuous function, such that* $f(k,0) = 0$ *for all* $k \in [1,T]$. *Assume that* f *is a positive Lipschitzian continuous function, that is, there exist a positive constant* ℓ *such that for any* $(k,x), (k,y) \in [1,T]\times\mathbb{R}$,

$$|f(k,x) - f(k,y)| \leq \ell|x-y|^{p^+-1},$$

where

$$\ell < \frac{\sum_{k=1}^T F(k,c)}{\frac{1}{p^+}}\frac{c_{p^+}}{L_2^{p^+}2^{\frac{p^+-2}{2}}(K)^{p^+}(2\frac{1}{KL_1})^{p^--p^+}}. \tag{25}$$

Then, for each

$$\lambda \in \left]0, \frac{1}{p^+\sum_{k=1}^T F(k,c)}\right[, \tag{26}$$

problem (1) has exactly one positive solution $u_0 \in W$.

Proof. The existence of at least one solution u to problem (1) immediately follows by Theorem 3.1. For proving the uniqueness part, we suppose that there exist two different functions u_1, u_2 satisfying problem (1). Then, by Lemma 2.3, $I'(u_1)(u_2) = I'(u_2)(u_1) = 0$, that is

$$\begin{aligned}&\textstyle\sum_{k=1}^{T+1}\Big[w(k-1)\phi_{p(k-1)}(\Delta u_1(k-1))\Delta(u_2-u_1)(k-1)\\ &+q(k)\phi_{p(k)}(u_1(k))(u_2-u_1)(k)\Big] = \lambda\textstyle\sum_{k=1}^{T} f(k,u_1(k))(u_2-u_1)(k),\end{aligned}\tag{27}$$

and

$$\begin{aligned}&\textstyle\sum_{k=1}^{T+1}\Big[w(k-1)\phi_{p(k-1)}(\Delta u_2(k-1))\Delta(u_2-u_1)(k-1)\\ &+q(k)\phi_{p(k)}(u_2(k))(u_2-u_1)(k)\Big] = \lambda\textstyle\sum_{k=1}^{T} f(k,u_2(k))(u_2-u_1)(k).\end{aligned}\tag{28}$$

Remaining part of the proof follows analogously as in [32], but we provide it in our setting. We consider two cases. In the first case $\|u_2-u_1\|_{p(.)} \geq 1$, by (8), (24), Lemma 5.1, (27), (28), the Lipschitzian condition, (3) and (5) we get

$$\begin{aligned}
& c_{p^+}\|u_2-u_1\|_{p(.)}^{p^-} \leq c_{p^+}\psi(u_2-u_1)\\
\leq\ & \sum_{k=1}^{T+1}\Big[c_{p(k-1)}w(k-1)|\Delta(u_2-u_1)(k-1)|^{p(k-1)} + c_{p(k)}q(k)|(u_2-u_1)(k)|^{p(k)}\Big]\\
\leq\ & \sum_{k=1}^{T+1}\Big[w(k-1)\left(\phi_{p(k-1)}(\Delta u_2(k-1)) - \phi_{p(k-1)}(\Delta u_1(k-1))\right)\\
& (\Delta u_2(k-1) - \Delta u_1(k-1))\\
+\ & q(k)\left(\phi_{p(k)}(u_2(k)) - \phi_{p(k)}(u_1(k))\right)(u_2(k)-u_1(k))\Big]\\
=\ & \lambda\sum_{k=1}^{T}\left(f(k,u_2(k)) - f(k,u_1(k))\right)(u_2-u_1)(k)\\
\leq\ & \lambda\sum_{k=1}^{T+1}\ell|u_2(k)-u_1(k)|^{p^+}\\
\leq\ & \lambda\ell\sum_{k=1}^{T+1}\Big[w(k-1)|\Delta(u_2-u_1)(k-1)|^{p^+} + q(k)|(u_2-u_1)(k)|^{p^+}\Big]\\
=\ & \lambda\ell\|u_2-u_1\|_{p^+}^{p^+} \leq \lambda\ell\,(KL_2)^{p^+}\,2^{\frac{p^+-2}{2}}\,\|u_2-u_1\|_{p(.)}^{p^+}.
\end{aligned}$$

Hence, in case $p^+ = p^-$ we obtain

$$1 \geq \frac{c_{p^+}}{\lambda \ell \left(KL_2\right)^{p^+} 2^{\frac{p^+-2}{2}}},$$

this contradicts (25). Whereas, in case $p^+ > p^-$ we have

$$\|u_2 - u_1\|_{p(.)} \geq \left(\frac{c_{p^+}}{\lambda \ell \left(KL_2\right)^{p^+} 2^{\frac{p^+-2}{2}}} \right)^{\frac{1}{p^+-p^-}} > \frac{2}{KL_1}.$$

By the proof of Theorem 3.1, since u_1, u_2 are solutions to problem (1), we have

$$\Phi(u_i) \ < \frac{1}{p^+}, \text{ for } \ i = 1, 2.$$

Taking (6), (9), (3) and (5) into account, one has

$$\Phi(u_i) > \frac{1}{p^+} \psi(u_i) > \frac{\|u_i\|_{p^+}^{p^+}}{p^+} > \frac{K^{p^+} \|u_i\|^{p^+}}{p^+} > \frac{K^{p^+} L_1^{p^+} \|u_i\|_{p(.)}^{p^+}}{p^+},$$

for $i = 1, 2$. Combining the above inequalities we obtain

$$\|u_i\|_{p(.)} < \frac{1}{KL_1}, \quad \text{for } i = 1, 2.$$

On the other hand

$$\frac{2}{KL_1} \geq \|u_1\|_{p(.)} + \|u_2\|_{p(.)} \geq \|u_1 - u_2\|_{p(.)} > \frac{2}{KL_1}.$$

It is a contradiction.
In the second case $\|u_2 - u_1\|_{p(.)} < 1$, by the similar preceding argument (using (7) in place of (8)) we get

$$c_{p^+} \|u_2 - u_1\|_{p(.)}^{p^+} \leq c_{p^+} \psi(u_2 - u_1) \leq \lambda \ell 2^{\frac{p^+-2}{2}} (KL_2)^{p^+} \|u_2 - u_1\|_{p(.)}^{p^+}.$$

Hence

$$(c_{p^+} - \lambda \ell 2^{\frac{p^+-2}{2}} (KL_2)^{p^+}) \|u_2 - u_1\|_{p(.)}^{p^+} \leq 0.$$

By (25), (26) and since $L_1 K < 2$, we have $(c_{p^+} - \lambda \ell 2^{\frac{p^+-2}{2}} (KL_2)^{p^+}) > 0$. Therefore $u_1 = u_2$. $\square$

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